# On the non-commutative neutrix product $\ln x_+ \circ x_+^{-s}$

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*Abstract.* The non-commutative neutrix product of the distributions $\ln x_+$ and $x_+^{-s}$ is proved to exist for $s = 1, 2, \ldots$ and is evaluated for $s = 1, 2$. The existence of the non-commutative neutrix product of the distributions $x_+^{-r}$ and $x_+^{-s}$ is then deduced for $r, s = 1, 2, \ldots$ and evaluated for $r = s = 1$.

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In the following, we let $N$ be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.

We now let $\varrho(x)$ be any infinitely differentiable function having the following properties:

(i) $\varrho(x) = 0$ for $|x| \geq 1$,
(ii) $\varrho(x) \geq 0$,
(iii) $\varrho(x) = \varrho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \varrho(x)\, dx = 1$.

Putting $\delta_n(x) = n\varrho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if $f$ is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2] or [3].

**Definition 1.** *Let $f$ and $g$ be distributions in $\mathcal{D}'$ for which on the interval $(a,b)$, $f$ is the $k$-th derivative of a locally summable function $F$ in $L^p(a,b)$ and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p+1/q=1$. Then the product $fg=gf$ of $f$ and $g$ is defined on the interval $(a,b)$ by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

**Definition 2.** *Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $g_n(x) = (g*\delta_n)(x)$. We say that the neutrix product $f \circ g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a,b)$ if*

$$\operatorname*{N-lim}_{n\to\infty} \langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

*for all functions $\phi$ in $\mathcal{D}$ with support contained in the interval $(a,b)$.*

Note that if

$$\lim_{n\to\infty} \langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle,$$

we simply say that the *product* $f.g$ exists and equals $h$, see [3].

It is obvious that if the product $f.g$ exists then the neutrix product $f\circ g$ exists and the two are equal. Further, it was proved in [3] that if the product $fg$ exists by Definition 1, then the product $f.g$ exists by Definition 2 and the two are equal. Note also that although the product defined in Definition 1 is always commutative, the neutrix product defined in Definition 2 is in general non-commutative.

The following theorem holds, see [7].

**Theorem 1.** *Let $f$ and $g$ be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f\circ g^{(i)}$ (or $f^{(i)}\circ g$) exist on the interval $(a,b)$ for $i=0,1,2,\ldots,r$. Then the neutrix products $f^{(k)}\circ g$ (or $f\circ g^{(k)}$) exist on the interval $(a,b)$ for $k=1,2,\ldots,r$ and*

$$f^{(k)}\circ g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f\circ g^{(i)}]^{(k-i)}$$

*or*

$$f\circ g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f^{(i)}\circ g]^{(k-i)}$$

*on the interval $(a,b)$ for $k=1,2,\ldots,r$.*

In the following two theorems, which were proved in [6] and [9] respectively, the distributions $x_+^{-r}$ and $x_-^{-r}$ are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)},$$

for $r = 1, 2, \ldots$ and is distinct from the definition given by Gel'fand and Shilov [8]. Further, the distribution $F(x_+, -r)\ln x_+$ is defined for an arbitrary $\phi$ in $\mathcal{D}$ by

$$\langle F(x_+, -r)\ln x_+, \phi(x)\rangle = \int_0^\infty x^{-r}\ln x\Big[\phi(x) - \sum_{k=0}^{r-2}\frac{x^k}{k!}\phi^{(k)}(0) + \\ - \frac{x^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\,dx,$$

for $r = 1, 2, \ldots$, where the sum is empty when $r = 1$, and $H$ denotes Heaviside's function. The distribution $x_+^{-r}\ln x_+$ is then defined by

$$x_+^{-r}\ln x_+ = F(x_+, -r)\ln x_+ + \frac{(-1)^r}{(r-1)!}\psi_1(r-1)\delta^{(r-1)}(x), \tag{1}$$

for $r = 1, 2, \ldots$, where

$$\psi_1(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r \frac{\psi(i)}{i}, & r \geq 1, \end{cases} \quad \psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r \frac{1}{i}, & r \geq 1. \end{cases}$$

It follows that

$$(\ln^2 x_+)' = 2x_+^{-1}\ln x_+, \quad (x_+^{-r}\ln x_+)' = -rx_+^{-r-1}\ln x_+ + x_+^{-r-1},$$

see [10].

**Theorem 2.** *The neutrix products $x_+^{-r} \circ x_-^{-s}$ and $x_-^{-s} \circ x_+^{-r}$ exist and*

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c_1}{(r+s-1)!}\delta^{(r+s-1)}(x),$$
$$x_-^{-s} \circ x_+^{-r} = \frac{(-1)^{r-1} c_1}{(r+s-1)!}\delta^{(r+s-1)}(x)$$

*for $r, s = 1, 2, \ldots$, where*

$$c_1(\varrho) = \int_0^1 \ln t\,\varrho(t)\,dt.$$

It was shown in [5] that with suitable choice of the function $\varrho$, $c_1(\varrho)$ can take any negative value.

**Theorem 3.** *The neutrix products $\ln x_+ \circ x_-^{-s}$ and $x_-^{-s} \circ \ln x_+$ exist and*

$$\ln x_+ \circ x_-^{-s} = \frac{1}{(s-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-1)}(x) + \\ - \sum_{i=1}^{s-1}\frac{(-1)^i c_1}{(s-i-1)!i!i}\delta^{(s-1)}(x),$$
$$= x_-^{-s} \circ \ln x_+$$
$$= (-1)^{s-1}\ln x_- \circ x_+^{-s} = (-1)^{s-1}x_+^{-s} \circ \ln x_-$$

*for* $s = 1, 2 \dots$ , *where*

$$c_2(\varrho) = \int_0^1 \ln^2 t \, \varrho(t) \, dt.$$

We now prove the following theorem:

**Theorem 4.** *The neutrix product* $\ln x_+ \circ x_+^{-s}$ *exists for* $s = 1, 2, \dots$ . *In particular,*

$$\ln x_+ \circ x_+^{-1} = x_+^{-1} \ln x_+, \tag{2}$$

$$\ln x_+ \circ x_+^{-2} = x_+^{-2} \ln x_+ + (c_1 - 1)\delta'(x). \tag{3}$$

Proof: We put

$$(x_+^{-s})_n = x_+^{-s} * \delta_n(x)$$

so that

$$(x_+^{-s})_n = \frac{(-1)^{s-1}}{(s-1)!} \int_{-1/n}^{x} \ln(x-t)\delta_n^{(s)}(t) \, dt$$

on the interval $[0, 1/n]$ and

$$(x_+^{-s})_n = \frac{(-1)^{s-1}}{(s-1)!} \int_{-1/n}^{1/n} \ln(x-t)\delta_n^{(s)}(t) \, dt = \int_{-1/n}^{1/n} (x-t)^{-s}\delta_n(t) \, dt$$

on the interval $[1/n, \infty)$.

Then

$$\begin{aligned}
(-1)^{s-1}(s-1)! &\int_0^1 x^k \ln x (x_+^{-s})_n \, dx \\
&= \int_0^1 x^k \ln x \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n^{(s)}(t) \, dt \, dx \\
&= \int_{-1/n}^{0} \delta_n^{(s)}(t) \int_0^1 x^k \ln x \ln(x-t) \, dx \, dt + \\
&\qquad + \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x-t) \, dx \, dt \\
&= (-1)^s \int_0^{1/n} \delta_n^{(s)}(t) \int_0^t x^k \ln x \ln(x+t) \, dx \, dt + \\
&\qquad + (-1)^s \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x+t) \, dx \, dt + \\
&\qquad + \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x-t) \, dx \, dt.
\end{aligned} \tag{4}$$

Now

$$\begin{aligned}
\int_0^t x^k \ln x \ln(x+t)\,dx &= \int_0^t x^k \ln x[\ln t + \ln(1+x/t)]\,dx \\
&= \frac{t^{k+1}\ln^2 t}{k+1} - \frac{t^{k+1}\ln t}{(k+1)^2} - \sum_{i=1}^{\infty} \frac{(-1)^i}{it^i}\int_0^t x^{k+i}\ln x\,dx \\
&= \frac{t^{k+1}\ln^2 t}{k+1} - \frac{t^{k+1}\ln t}{(k+1)^2} - \sum_{i=1}^{\infty}\Big[\frac{(-1)^i t^{k+1}\ln t}{i(k+i+1)} - \frac{(-1)^i t^{k+1}}{i(k+i+1)^2}\Big] \\
&= \alpha_{k1}t^{k+1}\ln^2 t + \beta_{k1}t^{k+1}\ln t + \gamma_{k1}t^{k+1}, \qquad (5)
\end{aligned}$$

$$\begin{aligned}
\int_t^1 x^k \ln x \ln(x+t)\,dx &= \int_t^1 x^k \ln x[\ln x + \ln(1+t/x)]\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\qquad - \sum_{i=1}^{\infty}\frac{(-t)^i}{i}\int_t^1 x^{k-i}\ln x\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\quad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty}\Big[\frac{(-1)^i t^{k+1}\ln t}{i(k-i+1)} - \frac{(-1)^i t^{k+1}}{i(k-i+1)^2} + \frac{(-t)^i}{i(k-i+1)^2}\Big] + \\
&\qquad + \frac{(-t)^{k+1}}{2(k+1)}\ln^2 t \\
&= 2(k+1)^{-3} + \alpha_{k2}t^{k+1}\ln^2 t + \beta_{k2}t^{k+1}\ln t + \gamma_{k2}t^{k+1} + \\
&\qquad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty}\frac{(-t)^i}{i(k-i+1)^2}, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\int_t^1 x^k \ln x \ln(x-t)\,dx &= \int_t^1 x^k \ln x[\ln x + \ln(1-t/x)]\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\quad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty}\Big[\frac{t^{k+1}\ln t}{i(k-i+1)} - \frac{t^{k+1}}{i(k-i+1)^2} + \frac{t^i}{i(k-i+1)^2}\Big] \\
&= 2(k+1)^{-3} + \alpha_{k3}t^{k+1}\ln^2 t + \beta_{k3}t^{k+1}\ln t + \gamma_{k3}t^{k+1} + \\
&\qquad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty}\frac{t^i}{i(k-i+1)^2}, \qquad (7)
\end{aligned}$$

for $k = 0, 1, 2, \ldots$ .

Putting $nt = u$, we have

$$\int_0^{1/n} t^k \ln^i t \delta_n^{(s)}(t)\, dt = n^{s-k} \int_0^1 u^k (\ln u - \ln n)^i \varrho^{(s)}(u)\, du.$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^k \ln^i t\, \delta_n^{(s)}(t)\, dt = 0,$$

for $i = 0, 1, 2;\ k = 0, 1, 2 \ldots, s-1$ and $s = 1, 2, \ldots$ and

$$\lim_{n\to\infty} \int_0^{1/n} t^k \delta_n^{(s)}(t)\, dt = 0,$$

for $k = s+1, s+2, \ldots$ and $s = 1, 2, \ldots$ .

Further,

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln^i t\, \delta_n^{(s)}(t)\, dt = \int_0^1 u^s \ln^i u\, \varrho^{(s)}(u)\, du$$

and it follows easily by induction that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \delta_n^{(s)}(t)\, dt = (-1^s s! \int_0^1 \varrho(u)\, du = \tfrac{1}{2}(-1)^s s!, \tag{8}$$

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln t\, \delta_n^{(s)}(t)\, dt = (-1)^s s! c_1 + \tfrac{1}{2}(-1)^s s! \psi(s), \tag{9}$$

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln^2 t\, \delta_n^{(s)}(t)\, dt = (-1)^s s! c_2 + 2(-1)^s s! \psi(s) c_1 + \\ + (-1)^s s! \sum_{i=1}^{s-1} \frac{\psi(i)}{i+1},\end{aligned} \tag{10}$$

the sum being empty when $s = 1$.

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^1 x^k \ln x (x_+^{-s})_n\, dx = -(s-k-1)^{-2} \tag{11}$$

for $k = 0, 1, 2, \ldots s-2$ and $s = 1, 2 \ldots$ and with

$$\alpha_s = \alpha_{s-1,1} + \alpha_{s-1,2} + (-1)^s \alpha_{s-1,3}, \quad \beta_s = \beta_{s-1,1} + \beta_{s-1,2} + (-1)^s \beta_{s-1,3},$$
$$\gamma_s = \gamma_{s-1,1} + \gamma_{s-1,2} + (-1)^s \gamma_{s-1,3},$$

it follows from equations (4) to (10) that

$$\begin{aligned}\mathop{\mathrm{N\text{-}lim}}_{n\to\infty} \int_0^1 x^{s-1} \ln x (x_+^{-s})_n \, dx = & -(-1)^s s\alpha_s \Big[c_2 + 2\psi(s)c_1 + \sum_{i=1}^{s-1} \frac{\psi(i)}{i+1}\Big] + \\ & -(-1)^s s\beta_s [c_1 + \tfrac{1}{2}\psi(s)] - \tfrac{1}{2}(-1)^s s\gamma_s \\ = & \; \Lambda_s,\end{aligned} \tag{12}$$

for $s = 1, 2, \ldots$ .

Now let $\phi$ be an arbitrary function in $\mathcal{D}$. Then with $2n^{-1} < \eta < 1$,

$$\begin{aligned}\langle \ln x_+ (x_+^{-s})_n, \phi(x)\rangle = & \int_0^\infty \ln x (x_+^{-s})_n \phi(x)\, dx \\ = & \int_0^\eta \ln x (x_+^{-s})_n \Big[\phi(x) - \sum_{k=0}^{s-1} \frac{x^k}{k!}\phi^{(k)}(0)\Big] dx + \\ & + \int_\eta^\infty \ln x (x_+^{-s})_n \Big[\phi(x) - \sum_{k=0}^{s-2} \frac{x^k}{k!}\phi^{(k)}(0) - \frac{x^{s-1}}{(s-1)!}\phi^{(s-1)}(0)H(1-x)\Big] dx + \\ & + \sum_{k=0}^{s-1} \frac{\phi^{(k)}(0)}{k!} \int_0^1 x^k \ln x (x_+^{-s})_n \, dx + \sum_{k=0}^{s-2} \frac{\phi^{(k)}(0)}{k!} \int_1^\infty x^k \ln x (x_+^{-s})_n \, dx.\end{aligned}$$

Since $(x_+^{-s})_n$ converges uniformly to the function $x^{-s}$ on the interval $[\eta, \infty)$, it follows that

$$\begin{aligned}\lim_{n\to\infty} & \int_\eta^\infty \ln x (x_+^{-s})_n \Big[\phi(x) - \sum_{k=0}^{s-2} \frac{x^k}{k!}\phi^{(k)}(0) - \frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big] dx \\ & = \int_\eta^\infty x^{-s} \ln x \Big[\phi(x) - \sum_{k=0}^{s-2} \frac{x^k}{k!}\phi^{(k)}(0) - \frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big] dx \\ & = \int_0^\infty x^{-s} \ln x \Big[\phi(x) - \sum_{k=0}^{s-2} \frac{x^k}{k!}\phi^{(k)}(0) - \frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big] dx + \\ & \qquad + O(\eta \ln \eta),\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=0}^{s-2} \frac{\phi^{(k)}(0)}{k!} \int_1^\infty x^k \ln x (x_+^{-s})_n \, dx = & \sum_{k=0}^{s-2} \frac{\phi^{(k)}(0)}{k!} \int_1^\infty x^{k-s} \ln x \, dx \\ = & \sum_{k=0}^{s-2} \frac{\phi^{(k)}(0)}{k!(s-k-1)^2},\end{aligned}$$

and on using equations (11) and (12), we have

$$\mathop{\mathrm{N\text{-}lim}}_{n\to\infty} \sum_{k=0}^{s-1} \frac{\phi^{(k)}(0)}{k!} \int_0^1 x^k \ln x (x_+^{-s})_n \, dx = \sum_{k=0}^{s-2} \frac{\phi^{(k)}(0)}{k!(s-k-1)^2} + \frac{\Lambda_s \phi^{(s-1)}(0)}{(s-1)!}.$$

Further,

$$\int_0^\eta \ln x(x_+^{-s})_n\Big[\phi(x)-\sum_{k=0}^{s-1}\frac{x^k}{k!}\phi^{(k)}(0)\Big]\,dx = \int_0^{2/n} x^s\ln x(x_+^{-s})_n\phi^{(s)}(\xi x)\,dx+ \\ +\int_{2/n}^{\eta} x^s\ln x(x_+^{-s})_n\phi^{(s)}(\xi x)\,dx,$$

where $0<\xi<1$. Now on the interval $[0,2/n]$, it is easily seen that

$$(x_+^{-s})_n=\frac{(-1)^{s-1}}{(s-1)!}\int_{-1/n}^{1/n}\ln(x-t)_+\delta_n^{(s)}(t)\,dt=O(n^s\ln n)$$

and so

$$\lim_{n\to\infty}\int_0^{2/n}x^s\ln x(x_+^{-s})_n\,dx=0.$$

Putting $K=\sup\{|\phi^{(s)}(x)|\}$, we have

$$\left|\int_{2/n}^{\eta}x^s\ln x(x_+^{-s})_n\phi(\xi x)\,dx\right|\le -K\int_{-1/n}^{1/n}\delta_n(t)\int_{2/n}^{\eta}x^s(x-t)^{-s}\ln x\,dx\,dt,$$

where

$$\begin{aligned}\int_{2/n}^{\eta}x^s(x-t)^{-s}\ln x\,dx &= \sum_{k=0}^{\infty}\int_{2/n}^{\eta}\binom{-s}{k}\frac{(-t)^k}{x^k}\ln x\,dx\\ &=\eta\ln\eta-\eta-2n^{-1}\ln(2/n)+2n^{-1}+\tfrac12\,st[\ln^2\eta-\ln^2(2\nu)]+\\ &\quad+\sum_{k=2}^{\infty}(-t)^k\binom{-s}{k}\left[\frac{x^{1-k}\ln x}{1-k}-\frac{x^{1-k}}{(1-k)^2}\right]_{2/n}^{\eta}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\int_{2/n}^{\eta}x^s\ln x(x_+^{-s})_n\phi(\xi x)\,dx=O(\eta\ln\eta).$$

Since we also have

$$\int_0^\eta x^{-s}\Big[\phi(x)-\sum_{k=0}^{s-1}\frac{x^k}{k!}\phi^{(k)}(0)\Big]\,dx=O(\eta\ln\eta),$$

we see that

$$\operatorname*{N-lim}_{n\to\infty}\langle\ln x_+(x_+^{-s})_n,\phi(x)\rangle=\langle F(x_+,-s)\ln x_+,\phi(x)\rangle+\frac{\Lambda_s\phi^{(s-1)}(0)}{(s-1)!}.$$

This proves the existence of the neutrix product $\ln x_+ \circ x_+^{-s}$ and in fact

$$\begin{aligned}\ln x_+ \circ x_+^{-s} &= F(x_+, -s)\ln x_+ - \frac{(-1)^s \Lambda_s}{(s-1)!}\delta^{(s-1)}(x) \\ &= x_+^{-s}\ln x_+ - \frac{\Lambda_s + \psi_1(s-1)}{(s-1)!}(-1)^s\delta^{(s-1)}(x),\end{aligned}$$

on using equation (1). It can be shown that

$$\alpha_s = 0, \quad \beta_s = \frac{(-1)^s}{s}\psi(s-1), \quad \gamma_s = \frac{(-1)^{s+1}}{s^2}[\psi(s-1) + s\chi(s-1)],$$
$$\Lambda_s = -c_1\psi(s-1) + \frac{1}{2}\big[\chi(s-1) - \psi^2(s-1)\big]$$

for $s = 1, 2, \dots$ , where

$$\chi(s) = \begin{cases} 0, & s = 0, \\ \sum_{i=1}^{s} 1/i^2, & s \geq 1 \end{cases}$$

so that in particular,

$$\Lambda_1 = 0, \quad \Lambda_2 = -c_1$$

and equations (2) and (3) follow. This completes the proof of the theorem. □

**Corollary 1.** *The neutrix products $x_+^{-s} \circ \ln x_+$ and $x_+^{-r} \circ x_+^{-s}$ exist for $r, s = 1, 2, \dots$ . In particular,*

$$x_+^{-1} \circ \ln x_+ = x_+^{-1}\ln x_+ + (c_2 + 2c_1)\delta(x), \tag{13}$$
$$x_+^{-2} \circ \ln x_+ = x_+^{-2}\ln x_+ - (c_2 + \tfrac{1}{2})\delta'(x), \tag{14}$$
$$x_+^{-1} \circ x_+^{-1} = x_+^{-2} + (2c_1 - \tfrac{1}{2})\delta'(x). \tag{15}$$

Proof: The existence of the product $x_+^{-r} \circ x_+^{-s}$ follows immediately from Theorems 1 and 4 for $r, s = 1, 2, \dots$ .

The product of the locally summable function $\ln x_+$ by itself exists by Definition 1 and is equal to the locally summable function $\ln^2 x_+$. Differentiating the equation

$$\ln x_+ \ln x_+ = \ln x_+ \circ \ln x_+ = \ln^2 x_+,$$

we get

$$x_+^{-1} \circ \ln x_+ + \ln x_+ \circ x_+^{-1} = 2x_+^{-1}\ln x_+.$$

The existence of the neutrix product $x_+^{-1} \circ \ln x_+$ and equation (13) follows from equation (2). The existence of $x_+^{-s} \circ \ln x_+$ now follows from this result, the existence of $x_+^{-r} \circ x_+^{-s}$ and Theorem 1.

Differentiating equation (2), we get

$$x_+^{-1} \circ x_+^{-1} - \ln x_+ \circ x_+^{-2} = x_+^{-2} - x_+^{-2} \ln x_+$$

and equation (15) follows on using equation (3).

Differentiating equation (13), we get

$$x_+^{-1} \circ x_+^{-1} - x_+^{-2} \circ \ln x_+ = x_+^{-2} - x_+^{-2} \ln x_+$$

and equation (14) follows on using equation (15). □

**Corollary 2.** *The neutrix products* $\ln x_- \circ x_-^{-s}$, $x_-^{-s} \circ \ln x_-$ *and* $x_-^{-r} \circ x_-^{-s}$ *exist for* $r, s = 1, 2, \dots$ *. In particular,*

$$\begin{aligned}
\ln x_- \circ x_-^{-1} &= x_-^{-1} \ln x_-, \\
x_-^{-1} \circ \ln x_- &= x_-^{-1} \ln x_-, \\
\ln x_- \circ x_-^{-2} &= x_-^{-2} \ln x_- - (c_1 - 1)\delta'(x), \\
x_-^{-2} \circ \ln x_- &= x_-^{-2} \ln x_- - (c_1 - 1)\delta'(x), \\
x_-^{-1} \circ x_-^{-1} &= x_-^{-2} - (c_1 - 1)\delta'(x).
\end{aligned}$$

Proof: Replacing $x$ by $-x$ in $\ln x_+$, $x_+^{-s}$ and $\delta^{(s)}(x)$ gives us $\ln x_-$, $x_-^{-s}$ and $(-1)^s \delta^{(s)}(x)$ respectively. The results now follow immediately from Theorem 4 and Corollary 1. □

**Corollary 3.** *The neutrix products* $\ln |x| \circ x^{-s}$, $x^{-s} \circ \ln |x|$ *and* $x^{-r} \circ x^{-s}$ *exist and*

$$\begin{aligned}
\ln |x| \circ x^{-s} &= x^{-s} \ln |x| = x^{-s} \circ \ln |x|, \\
x^{-r} \circ x^{-s} &= x^{-r-s},
\end{aligned}$$

*for* $r, s = 1, 2, \dots$ *.*

Proof: Since the products $\ln x_+ \circ x_+^{-s}$, $x_+^{-s} \circ \ln x_+$ and $x_+^{-r} \circ x_+^{-s}$ are of the form

$$\begin{aligned}
\ln x_+ \circ x_+^{-s} &= x_+^{-s} \ln x_+ + M_s \delta^{(s-1)}(x), \\
x_+^{-s} \circ \ln x_+ &= x_+^{-s} \ln x_+ + M_s' \delta^{(s-1)}(x), \\
x_+^{-r} \circ x_+^{-s} &= x_+^{-r-s} + M_{rs} \delta^{(r+s-1)}(x),
\end{aligned}$$

for some constants $M_s, M_s'$ and $M_{rs}$, it follows that we then have

$$\begin{aligned}
\ln x_- \circ x_-^{-s} &= x_-^{-s} \ln x_- - (-1)^s M_s \delta^{(s-1)}(x), \\
x_-^{-s} \circ \ln x_- &= x_-^{-s} \ln x_- - (-1)^s M_s' \delta^{(s-1)}(x), \\
x_-^{-r} \circ x_-^{-s} &= x_-^{-r-s} - (-1)^{r+s} M_{rs} \delta^{(r+s-1)}(x).
\end{aligned}$$

Noting that the neutrix product is clearly distributive with respect to addition and that

$$x^{-s} \ln |x| = x_+^{-s} \ln x_+ + (-1)^s x_-^{-s} \ln x_-, \quad x^{-s} = x_+^{-s} + (-1)^s x_-^{-s},$$

for $s = 1, 2, \dots$ , the results follow from these equations and Theorems 2 and 3.

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