Singlevaluedness of monotone operators on subspaces of GSG spaces

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Abstract. We extend Zajíček's theorem from [Za] about points of singlevaluedness of monotone operators on Asplund spaces. Namely we prove that every monotone operator on a subspace of a Banach space containing densely a continuous image of an Asplund space (these spaces are called GSG spaces) is singlevalued on the whole space except a σ-cone supported set.

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1. Introduction

In the sequel we consider only real Banach spaces. We denote by B_X the closed unit ball of the space X and by $B(x, \varepsilon)$ the open ball centered in x with radius ε. To shorten the notation we often write only $\sup \langle A, B\rangle$ instead of $\sup \{\langle a, b\rangle : a \in A, b \in B\}$.

A Banach space E is called *Asplund* if every continuous convex function on an open convex subset $G \subset E$ is Fréchet differentiable on a dense G_δ-subset of G. For more properties of Asplund spaces and convex functions, see e.g. [Ph].

A Banach space E is called a *GSG space*, if there exist an Asplund space V and a continuous linear mapping $T\colon V \to E$ such that $T(V)$ is dense in E. These spaces were introduced and studied by Stegall [St], see also [Fa, Sections 1.3, 1.4, 1.5].

Let E be a Banach space. A multivalued mapping $T\colon E \to E^*$ is called a *monotone operator* if for every x, $y \in E$, $x^* \in Tx$, and $y^* \in Ty$

$$\Big\langle x^* - y^*, x - y \Big\rangle \geq 0.$$

According to [Za], we introduce the following notions of "small" sets. Let E be a Banach space. If $e \in E$, $\|e\| = 1$, and $0 < c < 1$, define

$$A(e, c) = \{x \in E\colon x = \lambda e + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(e, c).$$

Definition. A set $M \subset E$ is said to be *cone supported* at $x \in M$ if there exist $R > 0$, $e \in X$, $\|e\| = 1$, and $0 < c < 1$ such that

$$M \cap B(x, R) \cap (x + A(e, c)) = \emptyset.$$

A subset of E is said to be *cone supported* if it is cone supported at all its points. A set is *σ-cone supported* if it can be written as a union of countably many cone supported sets.

In [Za], Zajíček proved the following theorem:

Theorem 1.1. *Let E be an Asplund space and let $T\colon E \to E^*$ be a locally bounded monotone operator with a domain $D(T) = \{x \in E\colon Tx \neq \emptyset\}$. Then there exists a σ-cone supported set $D \subset D(T)$ such that T is singlevalued at each point of $D(T) \setminus D$.*

We shall extend this result to the class of subspaces of GSG spaces. The proof is more transparent, when we divide it into two steps. In the first step we define a special property, *countable dentability*, and we prove that all subspaces of GSG spaces have this property. In the second step we prove that Zajíček's theorem is valid for all Banach spaces with this property.

Since a σ-cone supported set is of first category, our result is a strengthening of a theorem of Christensen and Kenderov [CK].

Unfortunately we do not know, whether the countable dentability is a characterization of subspaces of GSG spaces or not. We only know that the countable dentability implies fragmentability of the dual space.

2. Countable dentability

Definition. Let X be a Banach space. Let $A \subset X$ and $A^* \subset X^*$ be bounded sets, and let $\varepsilon > 0$. We say that (A, A^*) is an *ε-denting pair*, and we write $(A, A^*) \in \mathcal{D}(\varepsilon)$, if for every $\emptyset \neq M \subset A^*$ there exist $x \in X$ and $\alpha > 0$ such that

$$\sup \Big\langle S(M, x, \alpha) - S(M, x, \alpha), A \Big\rangle < \varepsilon,$$

where

$$S(M, x, \alpha) = \Big\{ x^* \in M\colon \Big\langle x^*, x \Big\rangle > \sup \Big\langle M, x \Big\rangle - \alpha \Big\}.$$

Lemma 2.1. *If $(A, B_{X^*}) \in \mathcal{D}(\varepsilon)$, and Y is a subspace of X, then $(A \cap Y, B_{Y^*}) \in \mathcal{D}(\varepsilon)$.*

Proof: Take any $\emptyset \neq M \subset B_{Y^*}$. Without loss of generality we can assume that M is convex and w*-closed. The mapping $Q\colon X^* \to Y^*$ defined as $x^* \mapsto {y^*}_{|Y}$ is linear and w*-to-w* continuous. Therefore the set $Q^{-1}(M) \cap B_{X^*}$ is w*-compact and convex. Moreover, as an easy consequence of Hahn-Banach theorem we get that $Q(Q^{-1}(M) \cap B_{X^*}) = M$. Assume that $\mathcal{M}$ is a nonempty linearly ordered system of w*-compact and convex subsets of B_{X^*}, such that every $N \in \mathcal{M}$ satisfies $Q(N) = M$. Let $N_0 = \bigcap \mathcal{M}$. Clearly N_0 is a w*-compact convex set and $Q(N_0) \subset M$. We will show that $Q(N_0) = M$. Take any $m \in M$ and define the system $\mathcal{M}_m = \{N \cap Q^{-1}(m) \colon N \in \mathcal{M}\}$ of nonempty w*-compact subsets of X^*. A compactness argument guarantees that the intersection $\bigcap \mathcal{M}_m$ is nonempty; take $n \in \bigcap \mathcal{M}_m$. Thus $n \in N_0$ and $Q(n) = m$. We have just verified the assumptions of Zorn lemma. Therefore there exists a minimal convex and w*-compact set $\tilde{M} \subset B_{X^*}$, which satisfies the following condition: $Q(\tilde{M}) = M$. From the assumptions there exist $x \in X$ and $\alpha > 0$ such that

$$\sup \left\langle S(\tilde{M}, x, \alpha) - S(\tilde{M}, x, \alpha), A \right\rangle < \varepsilon.$$

Define $M_1 = Q(\tilde{M} \setminus S(\tilde{M}, x, \alpha))$; it is a nonempty w*-compact convex set. The minimality of $\tilde{M}$ gives us that M_1 is a proper subset of M and therefore, because M_1 is w*-compact and convex, there exist $y \in Y$ and $\beta > 0$ such that $S(M, y, \beta) \cap M_1 = \emptyset$. Now

$$\sup \left\langle S(M, y, \beta) - S(M, y, \beta), A \cap Y \right\rangle \le \sup \left\langle S(\tilde{M}, x, \alpha) - S(\tilde{M}, x, \alpha), A \right\rangle < \varepsilon.$$

□

Definition. Let E be a Banach space. We say that E is *countably dentable* (CD) if there exists a sequence of bounded sets $\{A_n\}_{n \in \mathbb{N}}$ from E with the following property: for every $x \in E$ and every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $x \in A_n$ and (A_n, B_{E^*}) is an ε-denting pair.

An Asplund V space is an easy example of a countably dentable space. Indeed, take $A_n = n \cdot B_V$ and use the w*-dentability property of duals to Asplund spaces.

Proposition 2.2. *Let E be countably dentable and M be its closed subspace. Then M is also a countably dentable space.*

Proof: Let $\{A_n\}$ be the subsets of E from the definition of CD property. Define $B_n = A_n \cap M$, for $n \in \mathbb{N}$. Fix $x \in M$ and $\varepsilon > 0$. We know that there exists $n \in \mathbb{N}$ such that $x \in A_n$ and $(A_n, B_{E^*}) \in \mathcal{D}(\varepsilon)$ and therefore from Lemma 2.1 it follows that $(B_n, B_{M^*}) \in \mathcal{D}(\varepsilon)$. □

Proposition 2.3. *Let E be countably dentable, let X be a Banach space, and let $T \colon E \to X$ be a linear and continuous mapping such that $\overline{TE} = X$. Then X is a countably dentable space.*

Proof: Without loss of generality we may assume that $\|T\| \le 1$. Let $\{A_n\}$ be the subsets of E from the definition of CD property. Define

$$B_{p,n} = T(A_n) + \frac{1}{p} B_X$$

for $p, n \in \mathbb{N}$. The sets $B_{p,n}$ are clearly bounded.

Fix $x \in X$ and $\varepsilon > 0$. Let $p \in \mathbb{N}$ be such that $\frac{3}{p} < \varepsilon$. There exists $e \in E$ such that $\|Te - x\| < \frac{1}{p}$. Take $n \in \mathbb{N}$ such that $e \in A_n$ and $(A_n, B_{E^*}) \in \mathcal{D}\left(\frac{1}{p}\right)$. Then

$$x = (x - Te) + Te \in \frac{1}{p} B_X + T(A_n) = B_{p,n}.$$

Now let $\emptyset \neq M \subset B_{X^*}$. Since $(A_n, B_{E^*}) \in \mathcal{D}\left(\frac{1}{p}\right)$, there exist $w \in E$ and $\alpha > 0$ such that

$$\sup \left\langle S(T^*(M), w, \alpha) - S(T^*(M), w, \alpha), A_n \right\rangle < \frac{1}{p}.$$

Then

$$\begin{aligned}
&\sup \left\langle S(M, Tw, \alpha) - S(M, Tw, \alpha), B_{p,n} \right\rangle \\
&\quad = \sup \left\langle S(M, Tw, \alpha) - S(M, Tw, \alpha), T(A_n) + \frac{1}{p} B_X \right\rangle \\
&\quad \leq \sup \left\langle S(M, Tw, \alpha) - S(M, Tw, \alpha), T(A_n) \right\rangle \\
&\qquad + \sup \left\langle S(M, Tw, \alpha) - S(M, Tw, \alpha), \frac{1}{p} B_X \right\rangle \\
&\quad = \sup \left\langle S(T^*(M), w, \alpha) - S(T^*(M), w, \alpha), A_n \right\rangle + \frac{1}{p} \cdot \operatorname{diam} S(M, Tw, \alpha) \\
&\quad \leq \frac{1}{p} + \frac{2}{p} < \varepsilon
\end{aligned}$$

Thus the space X is countably dentable. $\square$

Corollary 2.4. *Every subspace of a GSG space is countably dentable. In particular, subspaces of WCG spaces are countably dentable.*

Proof: We already know that Asplund spaces are CD. Proposition 2.2 and Proposition 2.3 extend this property to all subspaces of GSG spaces. The second statement follows immediately from the well-known interpolation theorem [Di, Chapter 5, Section 4, Theorem 3]. $\square$

3. Singlevaluedness of monotone operators

Lemma 3.1 ([Za, Lemma 2]). *Let X be a Banach space and let $T\colon X \to X^*$ be a monotone operator. Let $H \subset \{x \in X : Tx \neq \emptyset\}$, $x \in H$, $v \in X$, $\|v\| = 1$, $c \in \mathbb{R}$, $\varepsilon > 0$, $K > 0$, $x^* \in Tx$,*

(i) $\langle x^*, v \rangle > c + \varepsilon$, *and*
(ii) $\lim\limits_{\delta \to 0+} \operatorname{diam} T(B(x, \delta) \cap H) < K$.

Then there exists $\varrho > 0$ *such that for every*

$$\tilde{x} \in B(x, \varrho) \cap H \cap (x + A(v, \varepsilon/K))$$

and every $\tilde{x}^* \in T\tilde{x}$ *the inequality* $\langle \tilde{x}^*, v\rangle > c$ *holds.*

Lemma 3.2 ([Za, Corollary 1]). *Let* X *be a Banach space. Suppose that* $M \subset X$ *is not* σ*-cone supported. Then there exists* $\emptyset \neq N \subset M$ *such that* N *is cone supported at no point of* N*.*

Now we are ready to extend Zajíček's theorem mentioned in the introduction.

Theorem 3.3. *Let* X *be a countably dentable space and let* $T\colon X \to X^*$ *be a monotone operator with a domain* $D(T) = \{x \in X : Tx \neq \emptyset\}$*. Then there exists a* σ*-cone supported set* $D \subset D(T)$ *such that* T *is singlevalued at each point of* $D(T) \setminus D$*.*

Proof: At first assume that Tx is bounded for every $x \in D(T)$. Then we can write $D(T)$ as the union of the countable system of sets $\Delta_k = \{x \in D(T) : \sup \|Tx\| < k\}$, $k \in \mathbb{N}$. It is easy to see that if the statement of our theorem is true for every $T_{|\Delta_k}$ it is also true for T. Thus without loss of generality we may assume that T is globally bounded by some constant $K > 0$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be the system of subsets of X from the definition of CD spaces. Suppose on the contrary that

$$D = \{x \in D(T)\colon T \text{ is not singlevalued at } x\}$$

is not a σ-cone supported set. Define

$$D_{n,m} = \left\{x \in D(T)\colon \sup\left\langle Tx - Tx, A_n\right\rangle > \frac{K}{m}\right\}.$$

Take any $x \in D$. Since T is not singlevalued at x, there exist $v \in X$, $\|v\| = 1$, and $m \in \mathbb{N}$, such that

$$\sup\left\langle Tx - Tx, v\right\rangle > \frac{K}{m}.$$

Now there exists $n \in \mathbb{N}$ so that $v \in A_n$ and $(A_n, B_{X^*}) \in \mathcal{D}\left(\frac{1}{m}\right)$, and therefore $x \in D_{n,m}$. Thus

$$D = \bigcup\left\{D_{n,m}\colon n, m \in \mathbb{N}, (A_n, B_{X^*}) \in \mathcal{D}\left(\frac{1}{m}\right)\right\}.$$

Consequently there are $m, n \in \mathbb{N}$ such that (A_n, B_{X^*}) is an $\frac{1}{m}$-denting pair, and $D_{n,m}$ is not a σ-cone supported set. By Lemma 3.2 there exists a set $\emptyset \neq N \subset D_{n,m}$, which is cone supported at no point of this set. Moreover $T(N) \subset K \cdot B_{X^*}$.

Since $(A_n, B_{X^*}) \in \mathcal{D}\left(\frac{1}{m}\right)$ there exist $v \in X$, $\|v\| = 1$, and $c > 0$ such that the weak* slice

$$S = \left\{x^* \in T(N)\colon \left\langle x^*, v\right\rangle > c\right\}$$

of $T(N)$ is nonempty and

$$\sup \left\langle \frac{1}{K}S - \frac{1}{K}S, A_n \right\rangle < \frac{1}{m}.$$

As $S \neq \emptyset$, we can choose $x \in N$ and $x^* \in S \cap T(x)$. Choose $\varepsilon > 0$ such that $\langle x^*, v\rangle > c + \varepsilon$. Since $\operatorname{diam} T(N) < 3K$, by Lemma 3.1 there exists $\varrho > 0$ such that for each $\tilde{x} \in B(x, \varrho) \cap N \cap \left(x + A(v, \frac{\varepsilon}{3K})\right)$ and each $\tilde{x}^* \in T\tilde{x}$ the inequality $\langle \tilde{x}^*, v\rangle > c$ holds. Since N is not cone supported at x, the intersection $B(x, \varrho) \cap N \cap \left(x + A(v, \frac{\varepsilon}{3K})\right)$ is a nonempty set and we can choose $\tilde{x}$ from this set. Since $N \subset D_{n,m}$ we have

$$\sup \left\langle T\tilde{x} - T\tilde{x}, A_n \right\rangle > \frac{K}{m}.$$

But $T\tilde{x} \subset S$ and so

$$\sup \left\langle T\tilde{x} - T\tilde{x}, A_n \right\rangle < \frac{K}{m},$$

a contradiction.

Assume now that there are some $x \in D(T)$ such that Tx is unbounded. Then we can construct a monotone operator $\tilde{T}$ in the following way: if the cardinality of Tx is 0 or 1, let $\tilde{T}x = Tx$; if the cardinality of Tx is more than 2, define $\tilde{T}x$ to be equal to arbitrary two points of them. Clearly $\tilde{T}$ is a monotone operator and every Tx is bounded. Moreover $D(\tilde{T}) = D(T)$, and the set where $\tilde{T}$ is not singlevalued coincides with the set where T is not singlevalued. We already know that the statement of this theorem is true for $\tilde{T}$ and therefore it is true also for T. □

Theorem 3.4. *Let E be a subspace of a GSG space and let $T\colon E \to E^*$ be a monotone operator with a domain $D(T) = \{x \in E\colon Tx \neq \emptyset\}$. Then there exists a σ-cone supported set $D \subset D(T)$ such that T is singlevalued at each point of $D(T) \setminus D$.*

Proof: Put together Corollary 2.4 and Theorem 3.3. □

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