A note on regularly asymptotic points

Jiří Jelínek

Abstract. A condition of Schmets and Valdivia for a boundary point of a domain in the complex plane to be regularly asymptotic is ameliorated.

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Introduction

Using the notation by Schmets and Valdivia [2], we denote by Ω a non-void domain contained in the complex plane $\mathbb{C}$, by D a non-void subset of its boundary $\partial\Omega$. Throughout this paper we suppose that D is finite.

Definition. We say that a holomorphic function f on Ω has an asymptotic expansion at a boundary point $u \in \partial\Omega$ if for every $n = 0, 1, 2, \dots$ the limit

$$\lim_{\substack{z\in\Omega \\ z\to u}} f^{[n]}(z,u) = a_n \in \mathbb{C} \tag{1}$$

exists, where the functions $f^{[n]}$ are defined by induction

$$f^{[0]}(z,u) = f(z),$$

$$f^{[n+1]}(z,u) = \frac{f^{[n]}(z,u) - a_n}{z-u} . \tag{2}$$

So, in fact, we have

$$\lim_{\substack{z\in\Omega \\ z\to u}} \frac{f(z) - \sum_{j=0}^{n} a_j (z-u)^j}{(z-u)^{n+1}} = a_{n+1} \qquad (\forall\, n = 0, 1, 2, \dots).$$

We put $f^{[n]}(u) = a_n$. We say that the series $\sum\limits_{n=0}^{\infty} a_n (z-u)^n$ is the asymptotic expansion of f at u and write

$$f(z) \approx \sum_{n=0}^{\infty} a_n (z-u)^n \quad \text{at} \quad u.$$

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The set of all holomorphic functions on Ω having an asymptotic expansion at every point $u \in D$ is denoted by $\mathcal{A}(\Omega\,; D)$.

We say that D is regularly asymptotic for Ω if, for every family of complex numbers $\{\, a_{u,n}\,;\, u \in D,\, n = 0, 1, 2, \dots \}$, there is a function $f \in \mathcal{A}(\Omega\,; D)$ such that

$$f(z) \approx \sum_{n=0}^{\infty} a_{u,n}(z-u)^n \quad \text{at} \quad u$$

for every $u \in D$.

The aim of this paper is to generalize the following sufficient condition for D to be regularly asymptotic for Ω (Theorem 1). We give also a condition implying that a boundary point is not regularly asymptotic (Theorem 2).

Theorem ([2, Theorem 3.7]). *A finite set $D \subset \partial\Omega$ is regularly asymptotic for Ω if every point $u \in D$ has the following property:*
there are connected subsets $A_k \subset \mathbb{C} \setminus \Omega$ $(k = 1, 2, \dots)$ and $u \neq v_k \in A_k$ such that

$$\lim_{k\to\infty} v_k = u\,, \qquad \lim_{k\to\infty} \frac{\operatorname{diam} A_k}{|v_k - u|} = \infty.$$

As a consequence, a point $u \in \partial\Omega$ is regularly asymptotic for Ω if it belongs to a component of $\mathbb{C} \setminus \Omega$ containing more than one point.

Schmets and Valdivia [2] proved this theorem using the following

Proposition ([2, Proposition 3.6]). *A finite subset D of Ω is regularly asymptotic for Ω iff the following condition is satisfied: there is $r > 0$ such that for every compact subset $K \subset \Omega$ and $u \in D$, there is an integer $p \in \mathbb{N}$ such that, for every $h > 0$, there is a function $f \in \mathcal{A}(\Omega\,; D)$ verifying*

$$|f(z)| \leq 1 \quad \textit{for all} \quad z \in K \cup \left(\bigcup_{u' \in D} \{\, z' \in \Omega\,;\, |z' - u'| \leq r \} \right)$$

and

$$\left| f^{[p]}(u) \right| \;>\; h.$$

For proving the theorem, the authors applied the proposition with $p = 1$ and $f(z)$ equal to a multiple of a determination of $\sqrt{(z - v_k)(z - w_k)}$, $v_k, w_k \in A_k$. Using a higher p, we can generalize the cited result.

Generalization

Theorem 1. *A finite set $D \subset \partial\Omega$ is regularly asymptotic for Ω if every point $u \in D$ has the following property:*
there are connected subsets A_k of $\mathbb{C} \setminus \Omega$ $(k = 1, 2, \dots)$, $u \neq v_k \in A_k$ and $q > 0$ such that

$$\lim_{k\to\infty} v_k = u, \tag{3}$$

and

$$\operatorname{diam} A_k > |v_k - u|^q. \tag{4}$$

Proof: Without loss of generality we can suppose that

$$|v_k - u| < \tfrac{1}{2} \tag{5}$$

and

$$q \geq 2. \tag{6}$$

If we replace A_k with a convenient connected closed subset of A_k, we can have, besides (4) and other hypotheses, in addition

$$\operatorname{diam} A_k < 2|v_k - u|^q. \tag{7}$$

This implies that $\operatorname{diam} A_k < |v_k - u|$, hence A_k does not contain the point u. As D is finite and $\lim \operatorname{diam} A_k = 0$, we have $D \cap A_k = \emptyset$ for k large enough. If we choose an integer

$$p \geq q + 1 \geq 3, \tag{8}$$

we have by (4), (5) and (8)

$$\operatorname{diam} A_k > |v_k - u|^{q-p-\frac{1}{4}} \cdot |v_k - u|^{p+\frac{1}{4}} > 2|v_k - u|^{p+\frac{1}{4}} .$$

As A_k is connected, it follows that we can choose a point $w_k \in A_k$ satisfying

$$|w_k - v_k| = |v_k - u|^{p+\frac{1}{4}} . \tag{9}$$

Thus, by (3) and (8) we have $\lim\limits_{k\to\infty} w_k = u$, moreover

$$\lim_{k\to\infty} \frac{w_k - u}{v_k - u} = \lim_{k\to\infty} \frac{(v_k - u) + (w_k - v_k)}{v_k - u} = 1. \tag{10}$$

Denote by g_k a determination of the analytic function $\sqrt{(\bullet - v_k)(\bullet - w_k)}$ defined on $\mathbb{C} \smallsetminus A_k$. Consequently, g_k is defined on Ω and belongs to $\mathcal{A}(\Omega\,; D)$ for k large enough. Evidently, for $k = 1, 2, \ldots$, the functions $|g_k|$ are bounded on the bounded set

$$K \cup \left(\bigcup_{u' \in D} \{\, z' \in \Omega\,;\, |z' - u'| \le r \} \right)$$

by a constant C independent on k. We will apply the cited proposition with the functions $f_k := \frac{g_k}{C}$ and with $2p$ instead of p. The function g_k, being holomorphic at the point u, has its asymptotic expansion equal to the Taylor expansion at u; so $f_k^{[2p]}(u) = \frac{1}{(2p)!} f_k^{(2p)}(u)$ and the result will follow from the Proposition if we prove

$$\lim_{k\to\infty} |g_k^{(2p)}(u)| = \infty. \tag{11}$$

To this end, fix an index k and denote

$$f_\alpha(z) := (z - v_k)^\alpha (z - w_k)^\alpha. \tag{12}$$

It can be verified by a direct calculation that

$$f_\alpha{}''(z) = \alpha(\alpha - 1) f_{\alpha-2}(z)(v_k - w_k)^2 + 2\alpha(2\alpha - 1) f_{\alpha-1}(z). \tag{13}$$

The meaning of this equality between multi-valued functions is as follows: if f_α in the formula (13) signifies a determination of (12), then (13) holds for

$$f_{\alpha-1}(z) = \frac{f_\alpha(z)}{(z - v_k)(z - w_k)}\,, \quad \text{and} \quad f_{\alpha-2}(z) = \frac{f_\alpha(z)}{(z - v_k)^2 (z - w_k)^2}\,.$$

For $\alpha = \frac{1}{2}$, the coefficient $2\alpha(2\alpha - 1)$ equals zero, but if we calculate higher derivatives of even order of the function $f_{\frac{1}{2}}$ using recurrence relation (13), we do not meet in (13) other zero coefficients. Thus

$$f_{\frac{1}{2}}{}''(z) = -\tfrac{1}{4} f_{-\frac{3}{2}}(z)(v_k - w_k)^2 \tag{14}$$

and from (13) follows by induction

$$f_{\frac{1}{2}}^{(2p)}(z) = \sum_{j=1}^{p} \alpha_j f_{\frac{1}{2}-p-j}(z)(v_k - w_k)^{2j} \tag{15}$$

with $\alpha_j \in \mathbb{R}$ depending only on j and p, $\alpha_1 \neq 0$. By (12) it follows

$$f_{\frac{1}{2}}^{(2p)}(u) = \sum_{j=1}^{p} \alpha_j (u - v_k)^{\frac{1}{2}-p-j} (u - w_k)^{\frac{1}{2}-p-j} (v_k - w_k)^{2j} = C_k \sum_{j=1}^{p} B_{k,j}\,,$$

where

$$C_k = \alpha_1 (u - v_k)^{-1-2p} \cdot (v_k - w_k)^2$$

and

$$B_{k,j} = \frac{\alpha_j}{\alpha_1} \cdot \frac{(u - w_k)^{\frac{1}{2}-p-j}}{(u - v_k)^{\frac{1}{2}-p-j}} \cdot \frac{(v_k - w_k)^{2j-2}}{(u - v_k)^{2j-2}} \ .$$

Now we pass to the limit. By (9) and (3) we have

$$\lim_{k\to\infty} |C_k| = \lim_{k\to\infty} \alpha_1 |v_k - u|^{-1-2p+2p+\frac{1}{2}} = \infty$$

and by (10), (9), (3) and (8), we have

$$\lim_{k\to\infty} B_{k,1} = 1, \quad \lim_{k\to\infty} B_{k,j} = 0 \ \text{ for } \ j \geq 2.$$

This proves the relation (11) and consequently the theorem. □

Now we will consider a domain Ω of the form

$$\Omega = \widetilde{\Omega} \setminus \left(\{u\} \cup \bigcup_{k=1}^{\infty} A_k \right) \tag{16}$$

where $\widetilde{\Omega}$ is a domain including the point u and A_k are disjoints closed subsets of $\widetilde{\Omega} \setminus \{u\}$ with $\lim\limits_{k\to\infty} \operatorname{dist}(A_k, u) = 0$.

Theorem 2. *Suppose that there are points $v_k \in A_k$ with $\lim v_k = u$ and numbers $R_k > \operatorname{diam} A_k$ for which the set*

$$G = \bigcup_{k=1}^{\infty} \{z\,;\, |z - v_k| < R_k\} \cup \{u\}$$

is not neighbourhood of the point u and

$$\sum_{k=1}^{\infty} \frac{\operatorname{diam} A_k}{R_k^q} < \infty \quad \text{for every} \quad q \geq 0. \tag{17}$$

Then the point u is not regularly asymptotic for the domain Ω.

Proof: At first, we need some preparation and auxiliary claims. As the set G is not neighbourhood of zero, there are points $z_m \in \Omega$ $(m \in \mathbb{N})$ with

$$z_m \neq u, \ \lim z_m = u \quad \text{and} \quad |v_k - z_m| \geq R_k \tag{18}$$

for all $m, k \in \mathbb{N}$. Consequently,

$$|v_k - u| \geq R_k \tag{19}$$

and thanks to $\lim v_k = u$ we obtain by reindexation

$$R_k \searrow 0. \tag{20}$$

Let us put $d_k = \operatorname{diam} A_k + e^{-\frac{k}{R_k}}$, denote by D_k the disk $\{z\,;\, |z - v_k| \leq d_k\}$ and by ∂D_k its boundary circle $\{z\,;\, |z - v_k| = d_k\}$ counter-clockwise oriented. Then

$$A_k \subset \operatorname{int} D_k$$

and by (17)

$$\sum_{k=1}^{\infty} \frac{d_k}{R_k^q} < \infty \tag{21}$$

for all $q \geq 0$. We can suppose

$$\sum_{k=1}^{\infty} \frac{d_k}{R_k} < \frac{1}{4}\ ; \tag{22}$$

otherwise we replace $\widetilde{\Omega}$ with $\widetilde{\Omega} \setminus \bigcup_{k=1}^{l} A_k$ for a convenient l. Then by (19) and (22) the distance of D_k from the point u is

$$|v_k - u| - d_k \geq R_k - d_k \geq R_k - \tfrac{1}{4} R_k = \tfrac{3}{4} R_k\,. \tag{23}$$

Claim 1. *For any* $R > 0$ *there is a circle*

$$\kappa_\varrho := \{\, z\,;\, |z - u| = \varrho\} \ \subset\ \widetilde{\Omega} \setminus \bigcup_{k=1}^{\infty} D_k$$

with $0 < \varrho < R$.

Let us observe that only relations (19), (20), (22) are needed for the proof of this claim.

Proof: Choose a k' for which

$$R_{k'} < R. \tag{24}$$

By (23) (deduced from (19) and (22)) and (20), for $k \leq k'$, we have

$$|v_k - u| - d_k \geq \tfrac{3}{4} R_{k'}\,.$$

Consequently, the disks D_k $(k = 1, 2, \dots, k')$ do not meet the disk $\{z\,;\, |z-u| \le \frac{1}{2} R_{k'}\}$. On the other hand, for $k > k'$ the disk D_k is contained in the annulus

$$\{z\,;\, |v_k - u| - d_k \le |z-u| \le |v_k - u| + d_k\} \tag{25}$$

of the width $2d_k$. By (20) and (22), the sum of the widths is

$$\sum_{k=k'+1}^{\infty} 2d_k \;\le\; R_{k'} \sum \frac{2d_k}{R_k} \;<\; \frac{1}{2} R_{k'}\,,$$

hence the sets (25) cannot cover the set $\{z\,;\, 0 < |z-u| \le \frac{1}{2} R_{k'}\}$ and the claim is proved. □

Let f be a holomorphic function on Ω having an asymptotic expansion at the point u with coefficients a_n $(n = 0, 1, \dots)$. We will prove that u is not regularly asymptotic showing that the coefficients cannot be (cf. (21))

$$a_n \;=\; n^n + 4^{n+1} \cdot \sum_{k=1}^{\infty} \frac{d_k}{R_k^{n+1}}\;. \tag{26}$$

Due to Claim 1, choose circles κ_{ϱ_j} $(j = 1, 2, \dots)$ contained in Ω and disjoints with disks D_k (for each $k, j \in \mathbb{N}$),

$$\varrho_j \searrow 0,\;\; \varrho_j > \varrho_{j+1}. \tag{27}$$

As the limit $\lim\limits_{z \to u,\, z \in \Omega} f(z) = a_0$ exists, we can suppose that ϱ_1 is so small that for some b we have

$$|f(z)| \le b \;\text{ whenever }\; z \in \Omega\,,\, |z-u| \le \varrho_1 \tag{28}$$

and that

$$\{z\,;\, |z-u| \le \varrho_1\,\} \subset \widetilde{\Omega}.$$

Let N_j be the set of the indexes $k \in \mathbb{N}$ for which

$$D_k \subset \{z\,;\, \varrho_{j+1} < |z-u| < \varrho_j\}\,.$$

Then N_j is finite; denote by γ_j the boundary cycle of the set $\bigcup_{k \in N_j} D_k$ directed so that the interior of $\bigcup_{k \in N_j} D_k$ lies to the left of γ_j. γ_j is the sum of arcs of the circles ∂D_k, is situated in Ω and satisfies

$$\{z\,;\, \operatorname{ind}_{\gamma_j} z = 1\} = \operatorname{int} \bigcup_{k \in N_j} D_k.$$

Hence the cycle $\kappa_1 - \gamma_1 - \cdots - \gamma_J - \kappa_{J+1}$ ($J \in \mathbb{N}$) is homologous with zero in Ω, so we can use the Cauchy formula below. Namely, by (18) and (22) the point z_m does not belong to any disk D_k. For m large enough we have $|z_m - u| < \varrho_1$, then for J large enough we have $\varrho_{J+1} < |z_m - u|$ and thus

$$f(z_m) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} - \sum_{j=1}^{J} \int_{\gamma_j} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} - \int_{\kappa_{J+1}} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} \right].$$

Thanks to (27) and (28), we have $\lim\limits_{J\to\infty} \int_{\kappa_{J+1}} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} = 0$, so

$$f(z_m) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_m} \right]. \tag{29}$$

Claim 2. *If m is as large as $|z_m - u| < \varrho_1$, then for $n = 0, 1, 2, \ldots$, we have*

$$f^{[n]}(z_m, u) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{(\zeta - z_m)(\zeta - u)^n} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)d\zeta}{(\zeta - z_m)(\zeta - u)^n} \right] \tag{30}$$

and

$$a_n = \lim_{m\to\infty} f^{[n]}(z_m, u) = \frac{1}{2\pi i} \cdot \left[\int_{\kappa_1} \frac{f(\zeta)d\zeta}{(\zeta - u)^{n+1}} - \sum_{j=1}^{\infty} \int_{\gamma_j} \frac{f(\zeta)d\zeta}{(\zeta - u)^{n+1}} \right]. \tag{31}$$

Proof: We shall proceed by induction. First we deduce the formula (31) from (30) using Lebesgue majorization theorem. As any point ζ of a cycle γ_j belongs to ∂D_k for some k, we have by (28), (18), definition of ∂D_k, (19) and (23)

$$\begin{aligned} \left| \frac{f(\zeta)}{(\zeta - z_m)(\zeta - u)^n} \right| &= \left| \frac{f(\zeta)}{(\zeta - v_k - (z_m - v_k))(\zeta - u)^n} \right| \\ \le \frac{b}{(R_k - d_k)(|v_k - u| - d_k)^n} &\le \frac{b}{(R_k - d_k)^{n+1}} \le \left(\frac{4}{3}\right)^{n+1} \frac{b}{R_k^{n+1}}. \end{aligned}$$

Hence the function g defined by $g(\zeta) = \left(\frac{4}{3}\right)^{n+1} \frac{b}{R_k^{n+1}}$ for $\zeta \in \partial D_k \setminus \bigcup_{k'=1}^{k-1} \partial D_{k'}$ is a majorant. Thanks to (21), it is integrable even on the set

$$\bigcup_{k=1}^{\infty} \partial D_k \supset \bigcup_{j=1}^{\infty} \gamma_j \tag{32}$$

with respect to the length measure. Hence the implication (30) $\Rightarrow$ (31) is proved.

Induction: If we put $n = 0$, the formula (30) turns into the Cauchy formula (29). Using the recurrent definition (cf. (2))

$$f^{[n+1]}(z_m, u) = \frac{f^{[n]}(z_m, u) - a_n}{z_m - u} ,$$

we deduce easily the formula (30) for $n+1$ from (30) and (31) and the claim is proved. $\square$

Now we complete the proof of the theorem. Integrating in (31) along $\bigcup_{k=1}^{\infty} \partial D_k$ instead of $\bigcup_{j=1}^{\infty} \gamma_j$, we obtain by (32), (28) and (23)

$$\begin{aligned} |a_n| \;&\le\; \frac{1}{2\pi} \cdot \left[2\pi \varrho_1 \frac{b}{\varrho_1^{n+1}} + \sum_{k=1}^{\infty} 2\pi d_k \frac{b}{(|v_k - u| - d_k)^{n+1}} \right] \\ &\le \frac{b}{\varrho_1^n} + \left(\frac{4}{3}\right)^{n+1} b \cdot \sum_{k=1}^{\infty} \frac{d_k}{R_k^{n+1}} , \end{aligned}$$

which cannot be true for all n together with (26).

Corollary. *Suppose the domain Ω to be of the form* (16) *with*

$$\sum (\operatorname{dist}(A_k, u))^p < \infty \tag{33}$$

for some $p > 0$. If, for every $q \ge 0$,

$$\operatorname{diam} A_k \le (\operatorname{dist}(A_k, u))^q \tag{34}$$

except a finite number (depending on q) of indexes k, then the point u is not regularly asymptotic.

PROOF: Choose points $v_k \in A_k$ so that $\operatorname{dist}(A_k, u) = |v_k - u|$. Hence, except a finite number of indexes k,

$$\operatorname{diam} A_k \le |v_k - u|^q. \tag{35}$$

Thanks to (33), we can suppose without loss of generality that

$$\sum_{k=1}^{\infty} |v_k - u|^p < \frac{1}{4} . \tag{36}$$

So, putting for a moment $d_k = |v_k - u|^{p+1}$ and $R_k = |v_k - u|$, we have

$$\sum_{k=1}^{\infty} \frac{d_k}{R_k} < \frac{1}{4} ,$$

which is the relation (22). Also the relation (20) can be satisfied by reindexation and we can apply Claim 1 affirming that there are circles κ_ϱ with arbitrarily small ϱ, disjoint with disks $\{z\,;\,|z-v_k|\le|v_k-u|^{p+1}\}$. Now we change the notation putting $R_k=|v_k-u|^{p+1}$. By this way we see that, for any $R>0$, there is a circle κ_ϱ, $0<\varrho<R$ disjoint with $\{z\,;\,|z-v_k|\le R_k\}$. It verifies the hypothesis of Theorem 2 that G is not neighbourhood of the point u. Now, choose a $q\ge 0$. By (35) we have

$$\operatorname{diam} A_k \le |v_k-u|^{q(p+1)+p}$$

except a finite number of indexes k. It follows by the last definition of R_k and by (36) that

$$\sum_{k=1}^{\infty}\frac{\operatorname{diam} A_k}{R_k^q}<\infty$$

and Theorem 2 gives the result. □

Remark. Suppose that for the domain Ω of the form (16) the hypothesis (33) of the preceding corollary is satisfied. If in addition the sets A_k are connected, the preceding corollary with Theorem 1 show that the relation (34) characterizes that the point u is not regularly asymptotic. Indeed, if for some q the relation (34) is not satisfied for an infinite number of indexes k, we obtain the hypothesis (4) of Theorem 1 for a suitable subsequence of $\{A_k\}$.

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

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