Some remarks to the compactness of steady compressible isentropic Navier-Stokes equations via the decomposition method

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Abstract. In [18]–[19], P.L. Lions studied (among others) the compactness and regularity of weak solutions to steady compressible Navier-Stokes equations in the isentropic regime with arbitrary large external data, in particular, in bounded domains. Here we investigate the same problem, combining his ideas with the method of decomposition proposed by Padula and myself in [29]. We find the compactness of the incompressible part u of the velocity field v and we give a new proof of the compactness of the "effective pressure" $\mathcal{P} = \varrho^\gamma - (2\mu_1 + \mu_2)\operatorname{div} v$. We derive some new estimates of these quantities in Hardy and Triebel-Lizorkin spaces.

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Classification: 76N, 35Q

1. Introduction

In 1933, J. Leray proved in [16]–[17] the existence of weak solutions for arbitrary large external data, of steady incompressible Navier-Stokes equations in several geometric situations (in particular for Ω bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, for Ω an exterior domain of $\mathbb{R}^3$ with infinite mass and prescribed zero or nonzero velocity at infinity, in $\Omega = \mathbb{R}^3, \dots$). Since that time, the incompressible Navier-Stokes equations have been extensively studied by many prominent mathematicians; hundreds of papers and several exhausting monographs have been devoted to the subject, see e.g. Galdi [9]–[10] and the references quoted there.

Considerably less is known for the (steady) compressible Navier-Stokes equations (sometimes called also Poisson-Stokes equations) in 2 or 3 space dimensions. We have a good knowledge of what happens near the equilibrium state (solutions in the subsonic regime, with "small" external forces or with "small" perturbations of arbitrary large potential forces and with "small" external data of the problem), see Beirao da Veiga [2], Farwig [7], Matsumura, Nishida [22]–[24], Nazarov, Novotný, Pileckas [28], Novotný [25]–[26], Novotný, Padula, Penel [34], Novotný, Penel [35]–[36], Novotný, Padula [29]–[33], Padula [37]–[41], Padula, Pileckas [45], Valli [55]–[56], Valli, Zajaczkowski [57], Pileckas, Zajaczkowski [46], Tani [51]–[52], Solonnikov [47], Solonnikov, Tani [48] and others.

For a long time the problem of the existence of weak solutions for arbitrary large external data seemed to be outside of the scope of current methods. The first attempt to solve it, in a very particular case of the nonstationary isothermal flows, is due to M. Padula [42]. However, the paper contains an error which has not been removed yet, cf. [43], [44]. The first positive step in the existence theory was done only recently: In 1993, P.L. Lions published two papers [18]–[19], where he announced and outlined the proofs of the existence of weak solutions for the steady and unsteady isentropic flows (in particular, in bounded domains), with arbitrary large external data.

In the steady case, the equations describing these flows, read:

$$
\begin{aligned}
-\mu_1\Delta v - (\mu_1+\mu_2)\nabla \operatorname{div} v + \nabla\varrho^\gamma &= \varrho f - \varrho v\cdot\nabla v \quad \text{in } \Omega,\\
\operatorname{div}(\varrho v) &= 0 \quad \text{in } \Omega,\\
\int_\Omega \varrho\, dx = m, \quad v|_{\partial\Omega} = 0, \quad \varrho &\geq 0 \quad \text{in } \Omega.
\end{aligned}
\tag{1.1}
$$

Here Ω is a smooth bounded N-dimensional domain ($N = 2,3$), ϱ and $v = (v_1,\dots v_N)$ are unknown functions (ϱ is the density and v is the velocity) while μ_1,μ_2 ($\mu_1 > 0$, $\mu_2 \geq -\frac{2}{N}\mu_1$) are given (constant) viscosities, $m > 0$ is the given total mass in the volume $|\Omega|$ ($|\cdot|$ denotes the Lebesgue measure of $\cdot$) and $f = (f_1,\dots f_N)$ is the prescribed density of external forces.

In the present paper we consider the same problem. Its main goal is to show, how the method of decomposition, introduced by Padula and myself in [29] for steady compressible flows near the equilibrium, can be applied to the study of different properties of weak solutions far from the equilibrium.

Besides several ideas which we took over from [18]–[19], the paper contains the following new approaches.

(1) <u>The supersonic version of the method of decomposition</u>: According to this approach, the compressible Navier-Stokes system (1.1) is split onto three simpler equations. A Stokes-like system, for the incompressible (solenoidal) part u of the velocity field and for the "effective pressure" $\mathcal{P} = \varrho^\gamma - (2\mu_1+\mu_2)\operatorname{div} v$; a Neumann-like problem for the compressible (irrotational) part $\nabla\phi$ of the velocity field; a (nonlinear) transport equation with the r.h.s. $\mathcal{P}$, for the density ϱ. This decomposition gives undoubtedly a different look on the original system.

(2) <u>The systematic use of Hardy spaces $h^p(\Omega)$ and Triebel-Lizorkin spaces $F^k_{p,2}$ with $0 < p \leq 1$</u>: For three-dimensional flows and for $3/2 < \gamma \leq 2$, the approach described above leads, in a natural way, to the estimates in Hardy and in Triebel-Lizorkin quasi-Banach spaces. These estimates are a consequence of the elliptic regularity of the Stokes problem in the decomposition. This procedure deserves certainly more investigation in the future.

The theorem proved by P.L. Lions reads ([18]–[20]):

Theorem 1.1. *Suppose* $\gamma > 1$ $(N = 2)$, $\gamma \geq 5/3$ $(N = 3)$ *and* $f \in L^\infty(\Omega)$. *Then* *there exists a couple $(\varrho, v) \in L^\gamma(\Omega) \times W^{1,2}(\Omega)$ a weak solution*[1] *to the problem* (1.1) *which is such that $\varrho \in L^q(\Omega)$ with $q = 2\gamma$ ($N = 3, \gamma \geq 3$ and $N = 2, \gamma > 1$) and $q = 3(\gamma - 1)$ ($N = 3, 5/3 \leq \gamma < 3$). Moreover, if $\gamma > 1$ ($N = 2$) and $\gamma > 3$ ($N = 3$), we have $\varrho \in L^\infty_{loc}(\Omega)$ and* $\mathrm{rot} v, \mathcal{P} = \varrho^\gamma - (2\mu_1 + \mu_2)\mathrm{div} v \in W^{1,p}_{loc}(\Omega)$, $1 < p < \infty$.

In order to explain the contribution of the method of decomposition to the investigation of problem (1.1), we describe briefly the three main steps in the proof of Theorem 1.1:

(1) The bounds for ϱ in $L^q(\Omega)$ and $v \in W^{1,2}(\Omega)$ obtained by an energy method (for $\gamma > 1$, $N = 2$ and $\gamma > 3/2$, $N = 3$).

(2) The bounds and the compactness for $\mathcal{P} = \varrho^\gamma - (2\mu_1 + \mu_2)\mathrm{div} v$ and for $\mathrm{rot} v$ (for $\gamma > 1$, $N = 2$ and $\gamma > 3/2$, $N = 3$).

(3) The passage to the limit in the term ϱ^γ (for $\gamma > 1$, $N = 2$ and $\gamma \geq 5/3$, $N = 3$), which is undoubtedly the most difficult part of the proof. Here one uses essentially (among others) the compactness of $\mathcal{P}$.

The method of decomposition concerns only the part (2) of the above description: it provides the compactness of $\mathcal{P}$ and ∇u (via the regularity of the Stokes problem). Moreover, this approach generates the whole scale of new estimates of u and $\mathcal{P}$ in the Triebel-Lizorkin and Hardy spaces. See Theorems 5.1 and 5.2 for details.

The paper is organized as follows

1. Introduction
2. The method of decomposition
3. Functional spaces (Preliminary results I)
4. Auxiliary linear problems (Preliminary results II)
5. Main theorems (Theorems 5.1 and 5.2)
6. Proof of the main theorems
7. Appendix (compactness and regularity of isentropic flows)

In Section 2, we introduce the "supersonic" method of decomposition; we give an equivalent formulation of system (1.1) which consists in the separation of the solenoidal and potential parts of the velocity field.

[1]We say that a couple (ϱ, v) is a weak solution of problem $(1.1)_{1-3}$ if it satisfies the integral identities

$$\mu_1 \int_\Omega \nabla v : \nabla \xi \, dx + (\mu_1 + \mu_2) \int_\Omega \mathrm{div} v \, \mathrm{div} \xi \, dx - \int_\Omega \varrho^\gamma \mathrm{div} \xi \, dx =$$
$$\int_\Omega \varrho f \cdot \xi \, dx + \int_\Omega \varrho v \otimes v : \nabla \xi \, dx, \forall \, \xi \in \mathcal{C}_0^\infty(\Omega),$$
$$\int_\Omega \varrho v \cdot \nabla \psi \, dx = 0, \forall \, \psi \in \mathcal{C}_0^\infty(\Omega)$$

and conditions $(1.1)_3$.

In Section 3, we recall definitions of functional spaces which are appropriate for further investigations [Lebesgue spaces L^p, Sobolev spaces $W^{k,p}$, Sobolev spaces of fractional derivatives (known also as the spaces of Bessel potentials) $H^{s,p}$, Hardy spaces $\mathcal{H}^p$, local Hardy spaces h^p, Triebel-Lizorkin spaces $F^{\Theta}_{p,q}$] along with some of their properties which will be needed in the proofs of Theorems 5.1 and 5.2.

In Section 4, we recall some well-known existence results and estimates for the auxiliary problems, which are needed in the sequel; in particular, Dirichlet problem for the Stokes operator (estimates in L^p spaces and in Triebel-Lizorkin spaces), for the operator div (estimates in L^p spaces) and for the Helmholtz decomposition (estimates in L^p spaces).

Section 5 is devoted to the statements of the main theorems and Section 6 to their proofs.

In the Appendix we formulate three theorems, all of them being particular cases of Theorem 1.1, due to P.L. Lions. Theorem 7.1 concerns the apriori estimates of weak solutions; it justifies the assumptions of Theorems 5.1 and 5.2. Theorems 7.2 and 7.3 illustrate the role of estimates of $\mathcal{P}$ and ∇u in the proofs of compactness of weak solutions (in particular, in the passage to the limit in the nonlinear term ϱ^γ) and of the regularity of weak solutions. In their proofs, we closely follow P.L. Lions ideas [18]–[19].

Acknowledgement. The paper would be never written without the existence of [18], [19]. I am thankful to P.L. Lions for the fruitful discussions I had with him during my short stay in Paris, in April 1993, and during the meeting "Analyse fonctionelle appliquée aux equations de Navier-Stokes et problèmes associés" in March 1994 in Toulon, where he was the principal lecturer. Just on this meeting he proposed to study the relation between his method and the method of decomposition. The present paper is the first fruit of such investigation.

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2. The method of decomposition

Let us look for the solution (ϱ, v) in the form

$$\varrho, \quad v = u + \nabla\phi, \tag{2.1}$$

where

$$\operatorname{div} u = 0 \ \text{ in } \ \Omega, \quad u \cdot \nu|_{\partial\Omega} = 0, \quad \frac{\partial\phi}{\partial\nu}|_{\partial\Omega} = 0 \tag{2.2}$$

(ν denotes an outwards normal to $\partial\Omega$). Then system (1.1) reads

$$\begin{aligned} -\mu_1\Delta u + \nabla\mathcal{P} &= \varrho f - \varrho v \cdot \nabla v \ \text{ in } \ \Omega, \\ \operatorname{div} u &= 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} &= -\nabla\phi|_{\partial\Omega}, \end{aligned} \tag{2.3}$$

where

$$\mathcal{P} = \varrho^\gamma - (2\mu_1 + \mu_2)\mathrm{div} v \tag{2.4}$$

which is equivalent to a nonlinear transport equation for ϱ

$$\varrho^\gamma + (2\mu_1 + \mu_2) v \cdot \nabla \ln \varrho = \mathcal{P}. \tag{2.5}$$

Finally, ϕ is governed by

$$\begin{aligned} \Delta\phi = -v \cdot \nabla \ln\varrho \quad x \in \Omega,\\ \frac{\partial \phi}{\partial \nu}|_{\partial\Omega} = 0. \end{aligned} \tag{2.6}$$

Let Ω' be an open subset (with sufficiently smooth boundary) of Ω such that $\overline{\Omega'} \subset \Omega$ (the superposed bar denotes the closure). Then there exists a cut-off function $\eta \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \le \eta \le 1$, $\eta(x) = 1$ if $x \in \Omega'$. Put

$$\tilde{u} = \eta u, \quad \tilde{\mathcal{P}} = \eta\mathcal{P}.$$

Then obviously

$$u(x) = \tilde{u}(x), \quad \tilde{\mathcal{P}}(x) = \mathcal{P}(x), \quad x \in \Omega'. \tag{2.7}$$

Moreover, $\tilde{u}$, $\tilde{\mathcal{P}}$ satisfy, in virtue of equation (2.3), the nonhomogeneous Stokes system

$$\begin{aligned} -\mu_1\Delta\tilde{u} + \nabla\tilde{\mathcal{P}} = \tilde{\mathcal{F}} + \tilde{\mathcal{G}} \;\text{ in }\; \Omega,\\ \mathrm{div}\tilde{u} = \tilde{g} \;\text{ in }\; \Omega,\\ \tilde{u}|_{\partial\Omega} = 0, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \tilde{g} = \nabla\eta \cdot u, \quad \tilde{\mathcal{F}} = \eta\varrho f - \eta\varrho v \cdot \nabla v,\\ \tilde{\mathcal{G}} = \nabla\eta\mathcal{P} - \mu_1(\Delta\eta u + 2\nabla\eta \cdot \nabla u). \end{aligned} \tag{2.9}$$

The decomposition (2.1)–(2.6) gives a different view on the equations. Especially:

(a) In the decomposed equations, the hyperbolic and elliptic aspects of the original system are separated.
(b) One of the equations of the new system is the Stokes equation. This allows us to use the great amount of known results of the elliptic regularity and apriori estimates in different functional spaces and in different geometrical situations.

3. Functional spaces (Preliminary results I)

Here we give a list of employed functional spaces. We also recall their definitions and basic properties needed in the sequel.

We denote by Ω a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; the outer normal to it is denoted ν. Denote by $\mathcal{S}$, as usual, the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^N$ and by $\mathcal{S}'$ its dual, the space of tempered distributions; by $\mathcal{D}(G)$ the space of infinitely differentiable functions with compact support in G and $\mathcal{D}'(G)$ the corresponding dual of distributions on G. Here and in the sequel, G stands for $\mathbb{R}^N$ or Ω.

Let $k = 1, 2, \ldots,\ 1 \le p \le \infty$. We denote by $L^p(G) = W^{0,p}(G)$ the usual Lebesgue space equipped with the norm $\|\cdot\|_{0,p,G}$ (or simply $\|\cdot\|_{0,p}$ if $G = \Omega$) and by $W^{k,p}(G)$ the usual Sobolev space with the norm $\|\cdot\|_{k,p,G} = \sum_{m=0}^{k} \|\nabla^m \cdot\|_{0,p,G}$ (or simply $\|\cdot\|_{k,p}$ if $G = \Omega$); $W_0^{k,p}(G)$ denotes the space of functions in $W^{k,p}(G)$ with zero traces. As usual, we denote by $W_{loc}^{k,p}(\Omega)$ a space of distributions belonging to $W^{k,p}(\Omega')$ for each domain Ω' such that $\overline{\Omega'} \subset \Omega$.

We wish to recall some well known properties of these spaces which we use currently in the proofs. The functions $u \in W_0^{k,p}(\Omega)$, or $u \in W^{k,p}(\Omega)$ such that $\int_\Omega u\,dx = 0$, satisfy the Poincaré inequality $\|u\|_{0,p} \le c\|\nabla u\|_{0,p}$, $1 \le p < \infty$. For $kp < N$, $1 \le p < \infty$, we have the Sobolev imbedding $W^{k,p}(\Omega) \subset L^s(\Omega)$, $s \in [1, \frac{Np}{(N-kp)}]$. If $s \in [1, \frac{Np}{(N-kp)})$ then this imbedding is compact. We also have an interpolation formula $\|u\|_{0,r}^r \le \|u\|_{0,q}^{(1-a)r} \|u\|_{0,p}^{ar}$, $a = \frac{p(r-q)}{r(p-q)}$, which holds for any $u \in L^q(G) \cap L^p(G)$, where $1 \le q < r \le p < \infty$.

Dual space to $W^{k,p'}(G)$ $(1/p' + 1/p = 1,\ 1 < p < \infty)$ is denoted by $(W^{k,p'}(G))^*$ and the corresponding duality norm is $\|\cdot\|_{k*,p,G}$ (or simply $\|\cdot\|_{*,p}$ if $G = \Omega$ and $k = 1$). Obviously $(W^{0,p'}(G))^* = L^p(G)$. Dual space to $W_0^{k,p'}(G)$ $(1/p' + 1/p = 1$, $1 < p < \infty)$ is denoted by $W^{-k,p}(G)$ and the corresponding duality norm is $\|\cdot\|_{-k,p,G}$ (or simply $\|\cdot\|_{-k,p}$ if $G = \Omega$).

We denote by $W^{k-1/p,p}(\partial\Omega)$ $(1 < p < \infty)$ the space of traces of functions from $W^{k,p}(\Omega)$ equipped with the natural norm

$$\|w\|_{W^{k-1/p,p},\partial\Omega} = \sup_{\{v \in W^{k,p}(\Omega),\, v|_{\partial\Omega} = w\}} \|v\|_{k,p}.$$

Consider the Banach space $E^p(\Omega) := \{b : b \in L^p(\Omega),\ \mathrm{div} b \in L^p(\Omega)\}$ with the norm $\|b\|_{E^p} := \|b\|_{0,p} + \|\mathrm{div} b\|_{0,p}$. Then we can still define a trace of the normal component of b, $b \cdot \nu|_{\partial\Omega} \in (W^{1-1/p',p'})(\partial\Omega)^*$, as it is clear from the identity $\int_{\partial\Omega} b \cdot \nu \varphi\, dS = \int_\Omega (\mathrm{div} b \varphi + b \cdot \nabla\varphi)\, dx$, $\forall\, \varphi \in W^{1,p'}(\Omega)$.

We will need also the spaces of Bessel potentials (Sobolev spaces with "fractional derivatives") $H^{s,p}(\mathbb{R}^N)$ and $H^{s,p}(\Omega)$: Let $1 \le p < \infty$, $-\infty < s < +\infty$ and let F be a Fourier transform, F^{-1} its inverse. Then put

$$H^{s,p}(\mathbb{R}^N) = \{u \in \mathcal{S}' : \|u\|_{s,p,\mathbb{R}^N} = \|F^{-1}((1 + |\xi|^2)^{s/2} F u)\|_{0,p,\mathbb{R}^N} < \infty\}.$$

It is a Banach space with norm $\|\cdot\|_{s,p,\mathbb{R}^N}$. When the domain is Ω, one defines

$$H^{s,p}(\Omega) = \{u : u = R_\Omega \bar{u}, \quad \bar{u} \in H^{s,p}(\mathbb{R}^N)\}.$$

a Banach space with norm

$$\|u\|_{s,p} = \|u\|_{s,p,\Omega} = \inf_{\{\bar{u}:\bar{u}\in H^{s,p}(\mathbb{R}^N), R_\Omega \bar{u}=u\}} \|\bar{u}\|_{s,p,\mathbb{R}^N}.$$

Here and in the sequel, R_Ω is the natural restriction from $\mathcal{S}'$ on $\mathcal{D}'(\Omega)$. From the various properties of these spaces recall the continuous imbedding $H^{s,p}(\Omega) \subset L^r(\Omega)$, $r \in [1, \frac{Np}{N-sp}]$ which holds provided that $0 < s < \infty$, $p \in [1, \infty)$, $sp < N$. If $r \in [1, \frac{Np}{N-sp})$, then the imbedding is compact.

We refer the reader who wish to have more details about all these spaces to [54], [15], [1], [53].

A particularly important role in the present paper is played by Hardy spaces and Triebel-Lizorkin quasi-Banach spaces. We start by recalling the definitions of Hardy spaces and local Hardy spaces in the whole $\mathbb{R}^N$ (see [49, p. 88–101], [54, p. 92]): Let $0 < p < \infty$, $\varphi \in \mathcal{S}$, $\int_{\mathbb{R}^N} \varphi\, dx \neq 0$, $\varphi_t(\cdot) = \frac{1}{t^N}\varphi(\frac{\cdot}{t})$. For $u \in \mathcal{S}'$, put

$$M_\varphi(u) = \sup_{t>0} |\varphi_t * u|$$

($*$ denotes the convolution). Then Hardy space $\mathcal{H}^p(\mathbb{R}^N)$ is defined as

$$\mathcal{H}^p(\mathbb{R}^N) = \{u \in \mathcal{S}' : M_\varphi(u) \in L^p(\mathbb{R}^N)\}.$$

It is a quasi-Banach space equipped with the quasinorm $\|u\|_{\mathcal{H}^p,\varphi,\mathbb{R}^N} = \|M_\varphi(u)\|_{0,p,\mathbb{R}^N}$. (For $0 < p < 1$, we have denoted by $L^p(\mathbb{R}^N)$ a (quasi-Banach) space of all measurable functions on $\mathbb{R}^N$ with finite (quasinorm) $\|\cdot\|_{0,p} = (\int_{\mathbb{R}^N} |\cdot|^p\, dx)^{1/p}$.) Similarly put

$$\bar{M}_\varphi(u) = \sup_{t\in(0,1)} |F^{-1}(\varphi^t F u)|$$

where $\varphi^t(x) = \varphi(tx)$. Then we define a local Hardy space $h^p(\mathbb{R}^N)$:

$$h^p(\mathbb{R}^N) = \{u \in \mathcal{S}' : \bar{M}_\varphi(u) \in L^p(\mathbb{R}^N)\}$$

a quasi-Banach space equipped with quasinorm $\|u\|_{h^p,\varphi,\mathbb{R}^N} = \|\bar{M}_\varphi(u)\|_{0,p,\mathbb{R}^N}$.

With this definition at hand it is natural to define the Hardy spaces $\mathcal{H}^p(\Omega)$ and the local Hardy spaces $h^p(\Omega)$, as follows (see [54, p. 192–193]):

$$\mathcal{H}^p(\Omega) = \{u : u = R_\Omega \bar{u}, \quad \bar{u} \in \mathcal{H}^p(\mathbb{R}^N)\}$$

and

$$h^p(\Omega) = \{u : u = R_\Omega \bar{u}, \quad \bar{u} \in h^p(\mathbb{R}^N)\}.$$

They are quasi-Banach spaces when equipped with quasinorms $\|u\|_{\mathcal{H}^p} = \|u\|_{\mathcal{H}^p,\Omega} = \inf_{\{\bar{u}:\bar{u}\in\mathcal{H}^p(\mathbb{R}^N), R_\Omega\bar{u}=u\}} \|\bar{u}\|_{\mathcal{H}^p,\mathbb{R}^N}$ and $\|u\|_{h^p} = \|u\|_{h^p,\Omega} = \inf_{\{\bar{u}:\bar{u}\in h^p(\mathbb{R}^N), R_\Omega\bar{u}=u\}} \|\bar{u}\|_{h^p,\mathbb{R}^N}$, respectively. They obey the following relation:

Lemma 3.1. *$\mathcal{H}^p(\Omega) \subset h^p(\Omega)$ and $\|u\|_{h^p} \le c\|u\|_{\mathcal{H}^p}$.*

Proof: Due to the definition of $\mathcal{H}^p(\Omega)$ and $h^p(\Omega)$, it is sufficient to prove that $\mathcal{H}^p(\mathbb{R}^N) \subset h^p(\mathbb{R}^N)$ and $\|u\|_{h^p,\mathbb{R}^N} \le c\|u\|_{\mathcal{H}^p,\mathbb{R}^N}$. To this end we calculate (using the fact that the Fourier transform of convolution is equal to the product of Fourier transforms)

$$F^{-1}(\varphi^t \cdot Fu) = F^{-1}F(F^{-1}\varphi^t * u) = F^{-1}\varphi^t * u$$

and

$$\begin{aligned} F^{-1}\varphi^t(x-y) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i(x-y)\xi}\varphi(t\xi)\,d\xi = \\ \frac{1}{(2\pi)^N}\frac{1}{t^N}\int_{\mathbb{R}^N} e^{-i\frac{x-y}{t}z}\varphi(z)\,dz = \\ \frac{1}{t^N}\psi(\frac{x-y}{t}) = \psi_t(x-y) \end{aligned}$$

with

$$\psi(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ixz}\varphi(z)\,dz \in \mathcal{S}.$$

Hence

$$\begin{aligned} \sup_{t\in(0,1)}|F^{-1}(\varphi^t Fu)| \le \sup_{t\in(0,\infty)}|F^{-1}(\varphi^t Fu)| \\ \le \sup_{t\in(0,\infty)}|\psi_t * u|, \end{aligned}$$

which yields the required statement (take e.g. $\phi(x) = e^{-|x|^2}$; then $\psi(x) = \frac{1}{2^N\pi^{N/2}}e^{-|x|^2/4}$, i.e. the condition $\int_{\mathbb{R}^N}\psi(x)\,dx \ne 0$ is automatically satisfied). Lemma 3.1 is thus proved. □

Next we wish to discuss the Div-curl lemma in Hardy spaces. Let us consider the scalar product

$$d \cdot b \tag{3.1}$$

of two vector fields d, b that satisfy

$$\mathrm{rot}\,d = 0, \quad \mathrm{div}\,b = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N). \tag{3.2}$$

If $d \in L^q_{loc}(\mathbb{R}^N)$ and $b \in L^{q'}_{loc}(\mathbb{R}^N)$ $(1 < q < \infty, 1/q + 1/q' = 1)$, then $d \cdot b \in L^1_{loc}(\mathbb{R}^N)$; thus the product (3.1) is well defined. This is not the case in the following situation

$$d \in L^p(\mathbb{R}^N) \quad (1 < p < N), \quad b \in \mathcal{H}^q(\mathbb{R}^N) \quad (0 < q < \infty)$$

where

$$1/p + 1/q < 1 + 1/N. \tag{3.3}$$

However, we can still define (3.1) as shown in [5]: Since $\mathrm{rot}\, d = 0$, then in virtue of the Stokes formula, there exists $\pi \in L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, $\nabla\pi \in L^p(\mathbb{R}^N)$ such that $d = \nabla\pi$. Then we put

$$\langle d \cdot b, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)} = \langle \pi b, \nabla\varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)} \tag{3.4}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\mathbb{R}^N)}$ means the duality in $\mathcal{D}'(\mathbb{R}^N)$. (Notice that $\mathrm{div}(\pi b)$ is formally equal to $\nabla\pi \cdot b + \pi \mathrm{div} b = d \cdot b$.) Now, under the above conditions on p, q, πb makes sense at least in $L^1_{loc}(\mathbb{R}^N)$. Indeed, the reader easily verifies that $\frac{N-p}{Np} + 1/q < 1$. The following version of div-curl lemma is due to [5, Theorem II.3]:

Lemma 3.2. *Let*

$$d \in L^p(\mathbb{R}^N) \quad (1 < p < N), \quad b \in \mathcal{H}^q(\mathbb{R}^N) \quad (0 < q < \infty) \tag{3.5}$$

such that (3.2)–(3.3) *hold. Then* $d \cdot b \in \mathcal{H}^r(\mathbb{R}^N)$, $1/r = 1/p + 1/q$ *and*

$$\|d \cdot b\|_{\mathcal{H}^r,\mathbb{R}^N} \le \|d\|_{0,p,\mathbb{R}^N} \|b\|_{\mathcal{H}^q,\mathbb{R}^N}. \tag{3.6}$$

This statement implies directly a similar statement in Ω.

Lemma 3.3. *Let* $1 < p < N$, $1 < q < \infty$, *satisfy* (3.3) *and*

$$d \in L^p(\Omega), \quad b \in L^q(\Omega),$$

$$\mathrm{rot}\, d = 0, \quad \mathrm{div} b = 0, \quad b \cdot \nu|_{\partial\Omega} = 0.$$

Then

$$d \cdot b \in \mathcal{H}^r(\Omega), \quad 1/r = 1/p + 1/q \tag{3.7}$$

and

$$\|d \cdot b\|_{\mathcal{H}^r} \le \|d\|_{0,p} \|b\|_{0,q}. \tag{3.8}$$

Proof: First, using the Stokes formula, we write

$$\nabla\pi = d$$

where

$$\pi \in W^{1,p}(\Omega) \cap L^{\frac{Np}{N-p}}(\Omega).$$

We can thus define, similarly as in Lemma 3.2

$$\langle d \cdot b, \varphi \rangle_{\mathcal{D}'(\Omega)} = \langle \pi b, \nabla \varphi \rangle_{\mathcal{D}'(\Omega)}.$$

Put

$$\tilde{\pi}(x) = E\pi(x)$$

where E is a continuous extension of $W^{1,p}(\Omega)$ onto $W^{1,p}(\mathbb{R}^N)$ and

$$\tilde{b} = b \ \text{ in } \ \Omega, \quad \tilde{b} = 0 \ \text{ otherwise.}$$

Put $\tilde{d} = \nabla \tilde{\pi}$. Then certainly

$$\tilde{\pi} \in W^{1,p}(\mathbb{R}^N) \cap L^{\frac{Np}{N-p}}(\mathbb{R}^N), \quad \tilde{b} \in L^q(\mathbb{R}^N)$$

and

$$\mathrm{rot}\tilde{d} = 0, \quad \mathrm{div}\tilde{b} = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N).$$

We define

$$\langle \tilde{d} \cdot \tilde{b}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)} := \langle \tilde{\pi}\tilde{b}, \nabla \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).$$

This definition is reasonable, provided the condition $b \cdot \nu|_{\partial\Omega}$ is satisfied. In virtue of the definition of the distribution $d \cdot b$, we have

$$\langle d \cdot b, \varphi \rangle_{\mathcal{D}'(\Omega)} = \langle \tilde{d} \cdot \tilde{b}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)} = \langle \tilde{\pi}\tilde{b}, \nabla \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N)}, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

i.e. $d \cdot b = \tilde{d} \cdot \tilde{b}$ in $\mathcal{D}'(\Omega)$. We thus obtain by Lemma 3.2

$$\|d \cdot b\|_{\mathcal{H}^r} \leq \|\tilde{d} \cdot \tilde{b}\|_{\mathcal{H}^r, \mathbb{R}^N} \leq \|\tilde{d}\|_{0,p} \|\tilde{b}\|_{0,q},$$

which completes the proof. □

The last spaces to be recalled are the Triebel-Lizorkin spaces. The reader is referred to [54] for all details. For the sake of completeness, we firstly give their definition (which is rather complicated). However, this definition as well as the definition of the interpolated Triebel-Lizorkin spaces in the next paragraph, are not absolutely necessary for the understanding of their properties (see Lemmas 3.4–3.7) needed in the sequel. The reader can therefore skip them at the first reading.

Let $\Psi(\mathbb{R}^N)$ be a family of all systems $\eta = \{\eta_j(x)\}_{j=0}^{\infty}$ such that (1) $\eta_j \in \mathcal{D}(\mathbb{R}^N)$; (2) $\mathrm{supp}\eta_0 \subset \{x : |x| \leq 2\}$; (3) $\mathrm{supp}\eta_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$; (4) for any multiindex δ there exists a positive constant c_δ such that $2^{j|\delta|}|\nabla^\delta \eta_j(x)| \leq c_\delta$, $j = 0, 1, \ldots$, $x \in \mathbb{R}^N$; (5) $\sum_{j=1}^{\infty} \eta_j(x) = 1$, $x \in \mathbb{R}^N$.

Let $-\infty < s < \infty$, $0 < q < \infty$, $0 < p < \infty$. Then we define a Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^N)$:

$$F^s_{p,q}(\mathbb{R}^N) =$$
$$= \{u : u \in \mathcal{S}', \|u\|_{F^s_{p,q},\mathbb{R}^N} := \|\{\sum_{j=0}^{\infty} |2^{sj} F^{-1}(\eta_j F u)|^q\}^{1/q}\|_{0,p,\mathbb{R}^N} < \infty\}.$$

It is a quasi-Banach space (a Banach space if $p, q \geq 1$) equipped with the quasinorm $\|\cdot\|_{F^s_{p,q},\mathbb{R}^N}$.

The Triebel-Lizorkin spaces can be the subject of the complex interpolation: Let A be a strip in the complex plane $A = \{z : 0 < \mathrm{Re} z < 1\}$. Take $f(z)$ such that: (i) $f(z) \in \mathcal{S}'$ for any $z \in \bar{A}$ (closure of A); (ii) for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $G(\xi, z) := [F^{-1}(\varphi F f(z)](\xi)$ is a uniformly continuous and bounded function on $\mathbb{R}^N \times \bar{A}$; (iii) for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and fixed $\xi \in \mathbb{R}^N$, $G(\xi, z)$ is an analytic function on A; (iv) for any $t \in \mathbb{R}^1$, $f(it) \in F^{s_0}_{p_0,q_0}(\mathbb{R}^N)$, $f(1+it) \in F^{s_1}_{p_1,q_1}(\mathbb{R}^N)$ and $\max_{l=0,1} \sup_{t\in\mathbb{R}^1} \|f(l+it)\|_{F^{s_l}_{p_l,q_l}} < \infty$. Let $-\infty < s_0, s_1 < +\infty$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 < \infty$, $0 < \Theta < 1$. Then the interpolation space $[F^{s_0}_{p_0,q_0}(\mathbb{R}^N), F^{s_1}_{p_1,q_1}(\mathbb{R}^N)]_\Theta$ is defined as follows

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^N), F^{s_1}_{p_1,q_1}(\mathbb{R}^N)]_\Theta :=$$
$$\{g : \exists f(z) \quad \text{(satisfying (i)–(iv)) such that} \quad g = f(\Theta)\}.$$

It is a quasi-Banach space with the quasinorm

$$\|g\|_{[F^{s_0}_{p_0,q_0}(\mathbb{R}^N), F^{s_1}_{p_1,q_1}(\mathbb{R}^N)]_\Theta} = \inf \max_{l=0,1} \sup_{t\in\mathbb{R}^1} \|f(l+it)\|_{F^{s_l}_{p_l,q_l}}$$

where the infimum is taken over all admissible functions $f(z)$ in the sense of (i)–(iv).

The natural definition of the Tribel-Lizorkin spaces in Ω is the following:

$$F^s_{p,q}(\Omega) = \{u : u = R_\Omega \bar{u}, \quad \bar{u} \in F^s_{p,q}(\mathbb{R}^N)\}.$$

It is a Banach space with the quasi-norm

$$\|u\|_{F^s_{p,q}} = \|u\|_{F^s_{p,q},\Omega} := \inf_{\{\bar{u}:\bar{u}\in F^s_{p,q}(\mathbb{R}^N), R_\Omega \bar{u} = u\}} \|\bar{u}\|_{F^s_{p,q},\mathbb{R}^N}.$$

Similarly the interpolation space $[F^{s_0}_{p_0,q_0}(\Omega), F^{s_1}_{p_1,q_1}(\Omega)]_\Theta$ is defined by

$$[F^{s_0}_{p_0,q_0}(\Omega), F^{s_1}_{p_1,q_1}(\Omega)]_\Theta := R_\Omega [F^{s_0}_{p_0,q_0}(\mathbb{R}^N), F^{s_1}_{p_1,q_1}(\mathbb{R}^N)]_\Theta.$$

Next we recall several theorems on Triebel-Lizorkin spaces needed in the sequel. As far as the imbedding are concerned, we have (see [54, p. 196–197]):

Lemma 3.4.
(i) *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 < \infty$ *and* $s_0 - N/p_0 \geq s_1 - N/p_1$. *Then* $F^{s_0}_{p_0,q_0}(\Omega) \subset F^{s_1}_{p_1,q_1}(\Omega)$ *(continuous imbedding).*
(ii) *Let* $-\infty < s < \infty$, $0 < p_1 \leq p_0 < \infty$, $0 < q_0 < \infty$. *Then*
$$F^s_{p_0,q_0}(\Omega) \subset F^s_{p_1,q_0}(\Omega).$$

The interpolation spaces are characterized as follows ([54, p. 203–206], [14, p. 18–20]):

Lemma 3.5. *Let*
$$-\infty < s_0, s_1 < +\infty, \quad 0 < q_0, q_1 < \infty, \quad 0 < p_0, p_1 < \infty, \quad 0 \leq \Theta \leq 1$$
and
$$s = (1-\Theta)s_0 + \Theta s_1, \quad \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$
Then
$$[F^{s_0}_{p_0,q_0}(\Omega), F^{s_1}_{p_1,q_1}(\Omega)]_\Theta = F^s_{p,q}(\Omega)$$
algebraically and topologically.

Lemma 3.6.
$$-\infty < s_0, s_1 < +\infty, \quad 0 < q_0, q_1 < \infty, \quad 0 < p_0, p_1 < \infty, \quad 0 < \Theta < 1$$
and $u \in X \cap Y$ *where* X, Y *stands for* $F^{s_0}_{p_0,q_0}(\Omega)$, $F^{s_1}_{p_1,q_1}(\Omega)$, *respectively. Then* $u \in [X, Y]_\Theta$ *and*
$$\|u\|_{[X,Y]_\Theta} \leq \|u\|^{1-\Theta}_X \|u\|^\Theta_Y.$$

We have several useful isometric isomorphisms (see [54]):

Lemma 3.7.
$$F^0_{p,2}(\Omega) = h^p(\Omega), \quad 0 < p \leq 1,$$
$$F^s_{p,2}(\Omega) = H^{s,p}(\Omega), \quad 1 < p < \infty, \quad s \in \mathbb{R}^1,$$
$$F^m_{p,2}(\Omega) = W^{m,p}(\Omega), \quad m = 0, 1, \dots, \quad 1 < p < \infty.$$

Finally recall a sufficient condition for the pointwise multipliers for the local Hardy spaces, in its simplest form (the proof can be found in [54, p. 197]):

Lemma 3.8. *Let* $p \in (\frac{N}{N+1}, \infty)$, $\eta \in \mathcal{C}^\infty_0(\Omega)$ *and* $a \in h^p(\Omega)$. *Then* $\eta a \in h^p(\Omega)$ *and*
$$\|\eta a\|_{h^p} \leq \|\eta\|_{C^N} \|a\|_{h^p}.$$

We conclude this section by several remarks on the notation:

If not introduced above otherwise, a norm in a Banach space X is denoted $\|\cdot\|_X$. All norms refer to Ω; if a norm refers to another domain (say G), then we indicate it as another index at the norm; e.g. $\|\cdot\|_{k,p}$ means a norm in $W^{k,p}(\Omega)$ while $\|\cdot\|_{k,p,G}$ or $\|\cdot\|_{k,p,\partial\Omega}$ are norms in $W^{k,p}(G)$ or in $W^{k,p}(\partial\Omega)$. If not stated explicitly otherwise, we do not distinguish between the spaces of vector and scalar valued functions; e.g. both $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega; \mathbb{R}^m)$, $m \in \mathbb{N}$ are denoted $W^{k,p}(\Omega)$. The difference is always clear from the context.

4. Auxiliary linear problems (Preliminary results II)

In the proofs, we often use various properties of the Dirichlet problem for the Stokes operator, of the Dirichlet problem for the divergence operator and of the Helmholtz decomposition. These results are nowadays considered as the mathematical folklore; although some of them were proved only very recently (as e.g. the estimates for the Stokes problem in Triebel-Lizorkin spaces).

We start with the Stokes problem:

$$\begin{aligned} -\mu_1 \Delta u + \nabla \mathcal{P} &= \mathcal{F} \text{ in } \Omega, \\ \operatorname{div} u = g, \quad x &\in \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{4.1}$$

The following two theorems trace back to Cattabrigga [4] (see also Galdi [9] for different variants of it):

Lemma 4.1 (Stokes problem in Sobolev spaces, weak solutions).
Let $\Omega \in \mathcal{C}^2$ be a bounded domain of $\mathbb{R}^N$ and $\mathcal{F} \in W^{-1,p}(\Omega)$, $g \in L^p(\Omega)$, $\int_\Omega g\,dx = 0$, $1 < p < \infty$. Then the problem (4.1) *possesses just one solution* $(u, \mathcal{P})$

$$u \in W_0^{1,p}(\Omega), \quad \mathcal{P} \in L^p(\Omega), \quad \int_\Omega \mathcal{P}\,dx = 0$$

which satisfies the estimate

$$\|u\|_{1,p} + \|\mathcal{P}\|_{0,p} \le c(\|\mathcal{F}\|_{-1,p} + \|g\|_{0,p}). \tag{4.2}$$

If $(\bar{u}, \bar{\mathcal{P}})$ is another solution in the class $W_0^{1,p}(\Omega) \times L^p(\Omega)$, then $\bar{u} = u$, $\bar{\mathcal{P}} = \mathcal{P} + c$, $c \in \mathbb{R}^1$.

Lemma 4.2 (Stokes problem in Sobolev spaces, regularity).
Let $k = 0, 1, \ldots,\ 1 < p < \infty$. Let $\Omega \in \mathcal{C}^{k+2}$ be a bounded domain of $\mathbb{R}^N$ and $\mathcal{F} \in W^{k,p}(\Omega)$, $g \in W^{k+1,p}(\Omega)$, $\int_\Omega g\,dx = 0$. Then the problem (4.1) *possesses just one solution* $(u, \mathcal{P})$

$$u \in W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad \mathcal{P} \in W^{k+1,p}(\Omega), \quad \int_\Omega \mathcal{P}\,dx = 0$$

which satisfies the estimate

$$\|u\|_{k+2,p} + \|\mathcal{P}\|_{k+1,p} \le c(\|\mathcal{F}\|_{k,p} + \|g\|_{k+1,p}). \tag{4.3}$$

If $(\bar{u}, \bar{\mathcal{P}})$ is another solution in the class $(W_0^{1,p}(\Omega) \cap W^{k+2,p}(\Omega)) \times W^{k+1,p}(\Omega)$, then $\bar{u} = u$, $\bar{\mathcal{P}} = \mathcal{P} + c$, $c \in \mathbb{R}^1$.

Next theorem is a consequence of general theory of pseudodifferential operators, see e.g. [54] or [13]. The statement as formulated here is proved in [14]. It reads:

Lemma 4.3 (Stokes problem in Triebel-Lizorkin spaces).
Let $\Omega \subset \mathbb{R}^N$ *be a smooth bounded open set,* $0 < p, q < \infty$, $-\infty < s < \infty$ *such that*

$$s + 2 > \max(1/p, N/p - N + 1) \tag{4.4}$$

and

$$\mathcal{F} \in F^s_{p,q}(\Omega), \quad g \in F^{s+1}_{p,q}(\Omega), \quad \int_\Omega g\,dx = 0.$$

Then there exists just one solution of the problem (4.1)

$$u \in F^{s+2}_{p,q}(\Omega), \quad \mathcal{P} \in F^{s+1}_{p,q}(\Omega), \quad \int_\Omega \mathcal{P}\,dx = 0,$$

which satisfies the estimate

$$\|u\|_{F^{s+2}_{p,q}} + \|\mathcal{P}\|_{F^{s+1}_{p,q}} \le c(\|\mathcal{F}\|_{F^s_{p,q}} + \|g\|_{F^{s+1}_{p,q}}). \tag{4.5}$$

If $(\bar{u}, \bar{\mathcal{P}}) \in F^{s+2}_{p,q}(\Omega) \times F^{s+1}_{p,q}(\Omega)$ *is another solution of* (4.1), *then* $\bar{u} = u$ *and* $\bar{\mathcal{P}} = \mathcal{P} + c$, $c \in \mathbb{R}^1$.

Remark 4.1.
(1) In the present paper, Lemma 4.3 will be used with $s = 0$, $N = 3$; this means in particular, in virtue of (4.4), $p > 3/4$.

(2) The existence of $u, \mathcal{P}$ is proved in Johnsen [14, Theorem 5.2.1]; the estimate (4.5) is proved also in [14], see Theorem 4.3.2 and the first paragraph in Section 5.2. As was already mentioned, the proof uses the theory of pseudodifferential operators in the general context described in [13] and [12].

□

Next we investigate the divergence equation:

$$\begin{aligned} &\mathrm{div}\omega = g, \quad x \in \Omega, \\ &\omega|_{\partial\Omega} = 0. \end{aligned} \tag{4.6}$$

The following theorem can be found in Bogovskij [3, Theorem 1].

Lemma 4.4 (Div equation in Sobolev spaces).
Let $k = 0, 1, \dots$, $1 < p < \infty$ *and* $\Omega \in \mathcal{C}^{k+1} \subset \mathbb{R}^N$ *be a bounded domain. Let*

$$g \in W^{k,p}_0(\Omega), \quad \int_\Omega g\,dx = 0$$

(*i.e., in particular,* $g \in L^p(\Omega)$, *if* $k = 0$). *Then there exists at least one solution*

$$\omega \in W^{k+1,p}_0(\Omega)$$

of the problem (4.6) *which is such that*

$$\|\omega\|_{k+1,p} \le c\|g\|_{k,p}. \tag{4.7}$$

The last problem of this subsection is the Helmholtz decomposition, i.e. the problem to find u (a vector field) and ϕ (a scalar function) such that a given vector field v satisfies

$$v = u + \nabla\phi, \quad \mathrm{div} u = 0. \tag{4.8}$$

A survey of the results concerning the Helmholtz decomposition of the Lebesgue spaces and Sobolev spaces is in [9, Chapter 3]. Here we need the following results:

Lemma 4.5 (Helmholtz decomposition).

(a) *Let* $1 < p < \infty$, $\Omega \in \mathcal{C}^2$ *and* $v \in L^p(\Omega)$. *Then there exists just one* (u, ϕ)

$$u \in L^p(\Omega), \quad \mathrm{div} u = 0 \;\; \mathit{in} \;\; \mathcal{D}'(\Omega) \tag{4.9}$$

and

$$\phi \in W^{1,p}(\Omega), \quad \int_\Omega \phi\, dx = 0 \tag{4.10}$$

such that (4.8) *holds. Moreover, we have*

$$\|u\|_{0,p} + \|\phi\|_{1,p} \le \|v\|_{0,p}. \tag{4.11}$$

(b) *Let* $v \in W^{k,p}(\Omega), \quad k = 1, 2, \dots$. *Then* (u, ϕ) (*see* (4.10)–(4.11)) *satisfies*

$$u \in W^{k,p}(\Omega), \quad \mathrm{div} u = 0, \quad u \cdot \nu|_{\partial\Omega} = 0 \tag{4.12}$$

and

$$\phi \in W^{k+1,p}(\Omega), \quad \frac{\partial \phi}{\partial \nu}|_{\partial\Omega} = 0. \tag{4.13}$$

Moreover, we have the estimate

$$\|u\|_{k,p} + \|\phi\|_{k+1,p} \le \|v\|_{k,p}. \tag{4.14}$$

5. Main theorems

It is known (see P.L. Lions [18]–[19] or Theorem 7.1 in the Appendix) that any weak solution (ϱ, v) to the problem (1.1) with $f \in L^\infty(\Omega)$ is bounded in $L^q(\Omega) \times W_0^{1,2}(\Omega)$ where $q = 3(\gamma - 1)$ (if $N = 3$ and $3/2 < \gamma < 3$) and $q = 2\gamma$ (if $N = 2$, $\gamma > 1$ or if $N = 3$ and $\gamma \geq 3$). We will show that this property is sufficient for the compactness of ∇u and $\mathcal{P}$: both ∇u and $\mathcal{P}$ are bounded in a convenient Sobolev spaces with positive fractional derivatives (the spaces of Bessel potentials). Moreover for "small" γ's, one gets a "subtle" estimates in Hardy and Triebel-Lizorkin spaces.

The proof relies essentially on the regularity properties of the Stokes problem (2.8) in different functional settings. We formulate these results in two theorems. Theorem 5.1 concerns the cases $\gamma > 1$ $(N = 2)$ and $\gamma > 2$ $(N = 3)$: in this situation, the r.h.s.

$$(\tilde{\mathcal{F}} + \tilde{\mathcal{G}}, \tilde{g}) \tag{5.1}$$

of the system (2.8) is in a space $L^s(\Omega) \times W^{1,s}(\Omega)$ with $s > 1$ and one applies the usual theory of the regularity of elliptic operators in Sobolev spaces to get the corresponding estimates. Theorem 5.2 concerns the case $3/2 < \gamma \leq 2$ and $N = 3$. In such case, $\tilde{\mathcal{F}}$ does not belong to a Lebesgue space with $s > 1$. However, we show that it belongs to a convenient Hardy spaces. Then, using the very recent results of the elliptic regularity to Stokes problem in Triebel-Lizorkin quasi-Banach spaces we get, similarly as in the previous case, the corresponding estimates of $\mathcal{P}$ and ∇u.

Similar results were proved by P.L. Lions using a different method, see [18]–[19]. He also introduced an approach allowing to deduce from the compactness of $\mathcal{P}$ the strong convergence of ϱ, i.e. to prove the compactness of the weak solutions to the problem (1.1) and their regularity. We explain the essence of this procedure, in the light of the decomposition, in the Appendix.

Here and in the sequel K denotes a generic positive constant which is, in particular, dependent of $\|f\|_{0,\infty}$, m and eventually of Ω'.

Theorem 5.1. *Let $\gamma > 1$ $(N = 2)$, $\gamma > 2$ $(N = 3)$ and $m > 0$, $f \in L^\infty(\Omega)$. Put*

$$\begin{gathered} q = 2\gamma \quad (N = 2), \\ q = 3(\gamma - 1) \quad (N = 3, \gamma < 3), \quad q = 2\gamma \quad (N = 3, \gamma \geq 3) \end{gathered} \tag{5.2}$$

and

$$\theta_0 = \frac{Nq}{(N-1)q + N}, \quad \theta = \theta_0 \quad (N = 3) \quad 1 < \theta < \theta_0 \quad (N = 2). \tag{5.3}$$

Let (ϱ, v)

$$\varrho \in L^q(\Omega), \quad v \in W_0^{1,2}(\Omega) \tag{5.4}$$

be a weak solution of problem (1.1)*. Consider the quantities* $\mathcal{P}$*,* u *and* ϕ *where* $\mathcal{P}$ *is defined by* (2.4) *and* u, ϕ *comes from the Helmholtz decomposition of* v *(see Lemma* 4.5*). Then*

$$\mathcal{P} \in W^{1,\theta}_{loc}(\Omega), \quad u \in W^{2,\theta}_{loc}(\Omega) \cap W^{1,2}_0(\Omega), \quad \phi \in W^{2,2}(\Omega). \tag{5.5}$$

Moreover, if (ϱ, v) *satisfies estimate*

$$\|\varrho\|_{0,q} + \|v\|_{1,2} \le K(\|f\|_{0,\infty}, m) \tag{5.6}$$

then

$$\|\phi\|_{2,2} + \|\mathcal{P}\|_{1,\theta,\Omega'} + \|u\|_{2,\theta,\Omega'} \le K(\|f\|_{0,\infty}, m, \Omega') \tag{5.7}$$

for any smooth domain Ω' *such that* $\overline{\Omega'} \subset \Omega$*.*

Remark 5.1. The statement of Theorem 5.1 can be reformulated as follows: Let $\{\varrho_n, v_n)\}_{n=1}^{\infty}$ be a sequence of weak solutions to system (1.1) satisfying the estimate (5.6) uniformly with respect to n (see Theorem 7.1). Take the sequences

$$\mathcal{P}^n = (\varrho^n)^\gamma - (2\mu_1 + \mu_2)\mathrm{div} v^n, \quad u^n, \phi^n \tag{5.8}$$

where u^n, ϕ^n is a Helmholtz decomposition of v^n (see Lemma 4.5). Then there exists a subsequence $\{(\mathcal{P}^{n'}, u^{n'}, \phi^{n'})\}_{n' \in \mathbb{N}}$ and a triplet $(\mathcal{P}, u, \phi)$ satisfying (5.5) such that

$$\begin{aligned} \mathcal{P}^{n'} \to \mathcal{P} \quad \text{weakly in} \quad W^{1,\theta}(\Omega') \quad \text{and strongly in} \quad L^{r_3}(\Omega'), \\ 1 \le r_3 < \frac{N\theta}{N-\theta} \quad (N = 2,3), \\ u^{n'} \to u \quad \text{weakly in} \quad W^{1,2}_0(\Omega), \\ u^{n'} \to u \quad \text{weakly in} \quad W^{2,\theta}(\Omega') \quad \text{and strongly in} \quad W^{1,r_3}(\Omega'), \\ 1 \le r_3 < \frac{N\theta}{N-\theta} \quad (N = 2,3), \\ \nabla\phi^{n'} \to \nabla\phi \quad \text{weakly in} \quad W^{1,2}_0(\Omega) \\ \phi^{n'} \to \phi \quad \text{weakly in} \quad W^{2,2}(\Omega) \quad \text{and strongly in} \quad W^{1,r_2}(\Omega), \\ 1 \le r_2 < \infty \quad (N=2), \quad 1 \le r_2 < 6 \quad (N=3). \end{aligned} \tag{5.9}$$

Moreover

$$v = u + \nabla\phi, \quad \mathrm{div} u = 0 \tag{5.10}$$

and the estimate (5.7) holds.

Remark 5.2.

- The formula (5.5) for u implies, in particular

$$\mathrm{rot}\, v \in W^{1,\theta}(\Omega') \tag{5.11}$$

  and the estimate (5.7) yields, in particular

$$\|\mathrm{rot}\, v\|_{1,\theta,\Omega'} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{5.12}$$

- The precise values of θ_0.
  If $N = 3$, $2 < \gamma < 3$ then $\theta_0 = \frac{3(\gamma-1)}{2\gamma-1} \in (1, 6/5)$. If $N = 3$, $\gamma \ge 3$ then $\theta_0 \in (\frac{3(\gamma+1)}{2\gamma+5}, \frac{6\gamma}{4\gamma+3})$ and $\frac{3(\gamma+1)}{2\gamma+5} \in [12/11, 3/2)$, $\frac{6\gamma}{4\gamma+3} \in [6/5, 3/2)$. If $N = 2$, $\gamma > 1$ then $\theta_0 \in (\frac{2\gamma+2}{\gamma+3}, \frac{2\gamma}{\gamma+1})$ and $\frac{2\gamma+2}{\gamma+3} \in (1, 2)$, $\frac{2\gamma}{\gamma+1} \in (1, 2)$.
- The precise bounds for r_3.
  If $N = 3$, $2 < \gamma < 3$ then $r_3 \in [1, q/\gamma)$; if $N = 3$, $\gamma \ge 3$, then q can be chosen such that $r_3 \in [1, 6\gamma/(2\gamma+3))$; if $N = 2, \gamma > 1$ then q can be chosen such that $r_3 \in [1, 2\gamma)$.
- In any case $\mathcal{P} \in W^{1,\theta}(\Omega')$ implies $\mathcal{P} \in L^{q/\gamma}(\Omega')$. The same is true for ∇u. However, in the case $N = 3$, $\gamma > 3$, Theorem 5.1 gives an improvement, in particular, for the summability of $\mathcal{P}$. Indeed, according to (5.4), $\mathcal{P} \in L^{q/\gamma}(\Omega')$ while (5.7) yields, by the Sobolev type imbeddings for $H^{s,\theta}$-spaces, $\mathcal{P} \in L^{r_3}(\Omega')$ with a $r_3 > q/\gamma$. The proof of Theorem 7.3 (see Appendix) is based just on this fact.

Theorem 5.2. *Let*

$$N = 3, \quad 3/2 < \gamma \le 2 \tag{5.13}$$

and $m > 0$, $f \in L^\infty(\Omega)$. *Let* Ω' *be an open subset of* Ω *with smooth boundary and such that* $\overline{\Omega'} \subset \Omega$. *Put*

$$\begin{gathered} q = 3(\gamma-1), \quad b_0 = \frac{3(\gamma-1)}{2\gamma-1}, \quad b_1 = \frac{3(\gamma-1)}{\gamma}, \\ a = \frac{3(\gamma-1)}{\gamma(1+\Theta)-\Theta}, \quad \Theta \in [0,1]. \end{gathered} \tag{5.14}$$

(a) *Let* (ϱ, v) *satisfying* (5.4) *be a weak solution of problem* (1.1). *Take functions* $\mathcal{P}$ *(see* (2.4)*) and* u, ϕ *(see* (2.1) *and Lemma* 4.5*). Then*

$$\begin{aligned} u &\in F^2_{b_0,2}(\Omega') \cap F^1_{b_1,2}(\Omega') \cap F^{1+\Theta}_{a,2}(\Omega'), \\ \mathcal{P} &\in F^1_{b_0,2}(\Omega') \cap F^0_{b_1,2}(\Omega') \cap F^{\Theta}_{a,2}(\Omega'), \\ \phi &\in W^{2,2}(\Omega). \end{aligned} \tag{5.15}$$

Moreover, if (ϱ, v) *satisfies* (5.6) *then we have*

$$\|\phi\|_{2,2} + \|u\|_{F^2_{b_0,2},\Omega'} + \|u\|_{F^1_{b_1,2},\Omega'} + \|u\|_{F^{1+\Theta}_{a,2},\Omega'} + \\ + \|\mathcal{P}\|_{F^1_{b_0,2},\Omega'} + \|\mathcal{P}\|_{F^0_{b_1,2},\Omega'} + \|\mathcal{P}\|_{F^{\Theta}_{a,2},\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega'). \tag{5.16}$$

(b) *If*

$$\Theta \in (0, \frac{2\gamma - 3}{\gamma - 1}) \tag{5.17}$$

then $a > 1$ *and*

$$u \in H^{1+\Theta,a}(\Omega'), \quad \mathcal{P} \in H^{\Theta,a}(\Omega'). \tag{5.18}$$

Moreover, if (ϱ, v) *satisfies* (5.6) *then*

$$\|u\|_{1+\Theta,a,\Omega'} + \|\mathcal{P}\|_{\Theta,a,\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega'). \tag{5.19}$$

Remark 5.3. Let $\{\varrho^n, v^n\}_{n=1}^{\infty}$ be a sequence of weak solutions to the problem (1.1) satisfying the estimate (5.6) uniformly with respect to n and let $\mathcal{P}^n$, u^n, ϕ^n be given by (5.8). Then there exists a subsequence $\{(\varrho^{n'}, v^{n'}, \mathcal{P}^{n'}, u^{n'}, \phi^{n'})\}_{n' \in \mathbb{N}}$ and $(\varrho, v, \mathcal{P}, u, \phi)$ with ϱ, v, $\mathcal{P}$, u, ϕ belonging to (5.4), (5.15) such that

$$\begin{aligned} \varrho^{n'} \to \varrho \quad \text{weakly in} \quad L^q(\Omega); \\ v^{n'} \to v \quad \text{weakly in} \quad W_0^{1,2}(\Omega) \quad \text{and strongly in} \quad L^{r_2}(\Omega), \\ 1 \leq r_2 < 6; \\ \mathcal{P}^{n'} \to \mathcal{P} \quad \text{weakly in} \quad H^{\Theta,a}(\Omega') \quad \text{and strongly in} \quad L^{r_4}(\Omega'), \\ 1 \leq r_4 < b_1; \\ u^{n'} \to u \quad \text{weakly in} \quad W_0^{1,2}(\Omega); \\ u^{n'} \to u \quad \text{weakly in} \quad H^{1+\Theta,a}(\Omega') \quad \text{and strongly in} \quad W^{1,r_4}(\Omega'), \\ 1 \leq r_4 < b_1; \\ \phi^{n'} \to \phi \quad \text{weakly in} \quad W^{2,2}(\Omega) \quad \text{and strongly in} \quad W^{1,r_2}(\Omega), \\ 1 \leq r_2 < 6 \quad . \end{aligned} \tag{5.20}$$

Moreover the identities (5.10) hold.

6. Proof of Theorems 5.1 and 5.2

Proof of Theorem 5.1:
Clearly, the couple $(\tilde{\mathcal{P}}, \tilde{u})$, see Section 2, satisfies the Stokes system (2.8). Lemma 4.5 yields in particular $u \in W^{1,2}(\Omega)$ and the identity (2.4) furnishes $\mathcal{P} \in L^{q/\gamma}(\Omega)$. Further we have

$$\begin{aligned}\|\varrho f\|_{0,\theta} \le c\|\varrho\|_{0,\theta}\|f\|_{0,\infty} \le c\|\varrho\|_{0,q}\|f\|_{0,\infty} \le c \quad (N=2,3),\\ \|\varrho v\cdot\nabla v\|_{0,\theta} \le c\|\varrho\|_{0,q}\|\nabla v\|_{0,2}\|v\|_{0,6} \le c \quad (N=3),\\ \|\varrho v\cdot\nabla v\|_{0,\theta} \le c\|\varrho\|_{0,q}\|\nabla v\|_{0,2}^2 \le c \quad (N=2).\end{aligned}$$

Therefore, the r.h.s. $(\tilde{\mathcal{F}}+\tilde{\mathcal{G}}, \tilde{g})$ of the equation (2.8) belongs to $L^\theta(\Omega)\times W^{1,\theta}(\Omega)$ and we have, in virtue of (5.6), the estimate

$$\|\tilde{\mathcal{F}}\|_{0,\theta} + \|\tilde{\mathcal{G}}\|_{0,\theta} + \|\tilde{g}\|_{1,\theta} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.1}$$

Let

$$\pi = \frac{1}{|\Omega|}\int_\Omega \mathcal{P}\, dx (\in \mathbb{R}^1). \tag{6.2}$$

Obviously, due to (2.4)

$$\|\pi\|_{1,\theta} \le c\|\mathcal{P}\|_{0,1} \le c(\|\varrho\|_{0,\gamma} + \|v\|_{1,2}) \le K(\|f\|_{0,\infty}, m).$$

Thus, Lemma 4.2 applied to the equation (2.8) completes the proof of (5.5) and of the estimate (5.7) for u, $\mathcal{P}$. The bounds for ϕ follow directly from Lemma 4.5. Theorem 5.1 is thus proved. $\square$

Proof of Theorem 5.2:
We divide the proof in several steps.

<u>*First step* – Estimates of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$, $\tilde{g}$ in Triebel-Lizorkin spaces</u>

Firstly we prove, that $\tilde{\mathcal{F}}$ (see (2.9)) belongs to certain Hardy space. We find by Holder's inequality

$$\|\varrho v\|_{0,\frac{6(\gamma-1)}{\gamma+1}} \le c\|\varrho\|_{0,3(\gamma-1)}\|v\|_{0,6} \le K(\|f\|_{0,\infty}, m), \tag{6.3}$$

i.e.,

$$\varrho v \in L^{\frac{6(\gamma-1)}{\gamma+1}}(\Omega). \tag{6.4}$$

Put $b=\varrho v_i$, $d=\nabla v_i$ $(i=1,\dots,N)$. Then all assumptions in Lemma 3.3 are satisfied provided that $\frac{1+\gamma}{6(\gamma-1)}+\frac{1}{2}<\frac{4}{3}$, i.e. if

$$\gamma > 3/2. \tag{6.5}$$

As a consequence, we have for $i = 1, \dots, N$

$$\varrho v \cdot \nabla v_i \in \mathcal{H}^r(\Omega), \quad r = \frac{3(\gamma-1)}{2\gamma-1} \tag{6.6}$$

and the estimate

$$\|\varrho v \cdot \nabla v\|_{\mathcal{H}^r} \leq K(\|f\|_{0,\infty}, m). \tag{6.7}$$

Lemma 3.1 yields the same bound for $h^r(\Omega)$-quasinorm of $\varrho v \cdot \nabla v$; this quasinorm is equivalent, in virtue of Lemma 3.7, to the $F^0_{r,2}(\Omega)$-quasinorm. Therefore

$$\|\varrho v \cdot \nabla v\|_{F^0_{r,2}} \leq K(\|f\|_{0,\infty}, m). \tag{6.8}$$

Using Lemma 3.4 (ii) with $s = 0$, $p_1 = r$, $q_0 = 2$, $p_0 = q$ (recall that $\varrho f \in L^q(\Omega)$) and Lemma 3.7, we obtain

$$\|\varrho f\|_{F^0_{r,2}} \leq \|\varrho f\|_{F^0_{q,2}} \leq \|\varrho\|_{0,q}\|f\|_{0,\infty} \leq K(\|f\|_{0,\infty}, m). \tag{6.9}$$

Taking into account (6.8), (6.9) and Lemma 3.8, we see

$$\|\tilde{\mathcal{F}}\|_{F^0_{r,2}} \leq K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.10}$$

Finally, we recall that by Lemma 4.5, $u \in W^{1,2}(\Omega)$ and, in virtue of the identity (2.4), $\mathcal{P} \in L^{q/\gamma}(\Omega)$. Therefore, Lemma 3.4 gives immediately (see (2.9))

$$\|\tilde{g}\|_{F^1_{r,2}} + \|\tilde{\mathcal{G}}\|_{F^0_{r,2}} \leq K(\|f\|_{0,\infty}, m, \Omega').$$

<u>*Second step* – Estimates of $(\tilde{u}, \tilde{\mathcal{P}})$ in Triebel-Lizorkin spaces</u>

We apply Lemma 4.3 to the Stokes problem (2.8) with $(\tilde{\mathcal{F}} + \tilde{\mathcal{G}}, \tilde{g}) \in F^0_{r,2}(\Omega) \times F^1_{r,2}(\Omega)$. Its assumptions are satisfied provided that $\max(1/r, 3/r - 2) < 2$, which yields by (6.6) the condition

$$\gamma > 3/2. \tag{6.11}$$

For such γ's, we thus get the estimate

$$\|\overline{\tilde{\mathcal{P}}}\|_{F^1_{r,2}} + \|\tilde{u}\|_{F^2_{r,2}} \leq \|\tilde{\mathcal{F}}\|_{F^0_{r,2}} + \|\tilde{\mathcal{G}}\|_{F^0_{r,2}} + \|\tilde{g}\|_{F^1_{r,2}} \tag{6.12}$$

with

$$\overline{\tilde{\mathcal{P}}} = \tilde{\mathcal{P}} - \frac{1}{|\Omega|}\int_\Omega \tilde{\mathcal{P}}\, dx. \tag{6.13}$$

Further, we have

$$|\int_\Omega \tilde{\mathcal{P}}\,dx| \le \|\tilde{\mathcal{P}}\|_{F^1_{r,2}} \tag{6.14}$$

(cf. equation (2.8), imbedding $F^1_{r,2}(\Omega) \subset L^{q/\gamma}(\Omega)$ which follows from Lemma 3.4 (i) with $s_0 = 1$, $p_0 = r$, $q_0 = 2$, $s_1 = 0$, $p_1 = q/\gamma, q_1 = 2$, and bounds (5.6) for ϱ, v). The inequalities (6.14) and (6.12) thus give

$$\|\tilde{\mathcal{P}}\|_{F^1_{r,2}} + \|\tilde{u}\|_{F^2_{r,2}} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.15}$$

<u>*Third step* – Estimates of $\tilde{u}, \tilde{\mathcal{P}}$ in Sobolev spaces</u>

On the other hand we have the estimates ($\gamma > 3/2$):

$$\|\varrho v \otimes v\|_{0,\frac{3(\gamma-1)}{\gamma}} \le c\|\varrho\|_{0,3(\gamma-1)}\|v\|^2_{0,6} \le K(\|f\|_{0,\infty}, m), \tag{6.16}$$

and

$$\|\varrho f\|_{0,\frac{3(\gamma-1)}{\gamma}} \le c\|\varrho\|_{0,3(\gamma-1)}\|f\|_{0,\infty} \le K(\|f\|_{0,\infty}, m). \tag{6.17}$$

Hence

$$\tilde{\mathcal{G}},\ \tilde{\mathcal{F}} \in W^{-1,\frac{3(\gamma-1)}{\gamma}}(\Omega), \quad \tilde{g} \in L^{\frac{3(\gamma-1)}{\gamma}}(\Omega) \tag{6.18}$$

and

$$\|\tilde{\mathcal{F}}\|_{-1,\frac{3(\gamma-1)}{\gamma}} + \|\tilde{\mathcal{G}}\|_{-1,\frac{3(\gamma-1)}{\gamma}} + \|\tilde{g}\|_{0,\frac{3(\gamma-1)}{\gamma}} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.19}$$

Applying Lemma 4.1 to (2.8), we get immediately

$$\|\overline{\tilde{\mathcal{P}}}\|_{0,\frac{3(\gamma-1)}{\gamma}} + \|\tilde{u}\|_{1,\frac{3(\gamma-1)}{\gamma}} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.20}$$

The same arguments as those in (6.12)–(6.14) lead finally to

$$\|\tilde{\mathcal{P}}\|_{0,\frac{3(\gamma-1)}{\gamma}} + \|\tilde{u}\|_{1,\frac{3(\gamma-1)}{\gamma}} \le K(\|f\|_{0,\infty}, m, \Omega'). \tag{6.21}$$

<u>*Fourth step* – Estimates of fractional derivatives</u>

We have by (6.15), (6.21) and Lemma 3.7

$$\begin{aligned} &\tilde{\mathcal{P}} \in F^0_{\frac{3(\gamma-1)}{\gamma},2}(\Omega) \cap F^1_{\frac{3(\gamma-1)}{2\gamma-1},2}(\Omega), \\ &\tilde{u} \in F^1_{\frac{3(\gamma-1)}{\gamma},2}(\Omega) \cap F^2_{\frac{3(\gamma-1)}{2\gamma-1},2}(\Omega). \end{aligned} \tag{6.22}$$

The interpolation, see Lemmas 3.5 and 3.6, yields

$$\tag{6.23} \begin{gathered} \tilde{\mathcal{P}} \in F^{\Theta}_{a,2}(\Omega), \quad \tilde{u} \in F^{1+\Theta}_{a,2}(\Omega), \\ a = \frac{3(\gamma-1)}{\gamma(1+\Theta)-\Theta}, \quad \Theta \in [0,1] \end{gathered}$$

and the estimates

$$\tag{6.24} \begin{gathered} \|\tilde{\mathcal{P}}\|_{F^{\Theta}_{a,2}} \le \|\tilde{\mathcal{P}}\|^{1-\Theta}_{F^{0}_{\frac{3(\gamma-1)}{\gamma},2}} \|\tilde{\mathcal{P}}\|^{\Theta}_{F^{1}_{\frac{3(\gamma-1)}{2\gamma-1},2}}, \\ \|\tilde{u}\|_{F^{1+\Theta}_{a,2}} \le \|\tilde{u}\|^{1-\Theta}_{F^{1}_{\frac{3(\gamma-1)}{\gamma},2}} \|\tilde{u}\|^{\Theta}_{F^{2}_{\frac{3(\gamma-1)}{2\gamma-1},2}}. \end{gathered}$$

The last inequalities, (6.15) and (6.21) yield the estimate (5.16) for u and $\mathcal{P}$, in virtue of the identities (2.7). The estimate of the $W^{2,2}$-norm of ϕ follows from Lemma 4.5. Theorem 5.2 (a) is thus proved.

Fifth step – Regularity of $\tilde{\mathcal{P}}$

Let a be as defined in (5.14). We compute that

$$\tag{6.25} a > 1$$

if and only if

$$\tag{6.26} 0 < \Theta < \frac{2\gamma-3}{\gamma-1}$$

(notice $\frac{2\gamma-3}{\gamma-1} \le 1$ provided $\gamma \in (3/2, 2]$). In this light, the statement (b) is a particular case of the statement (a), in virtue of Lemma 3.7. This completes the proof of Theorem 5.2. □

7. Appendix – the compactness of weak solutions

The main goal of the Appendix is to illustrate how the estimates of $\mathcal{P}$ imply the compactness and the regularity of weak solutions. To this end we formulate three theorems; each of them is a particular case of Lions' theorems (see [19, Theorems 2 and 3]): Theorem 7.1 deals with apriori estimates for weak solutions. Theorem 7.2 concerns the passage to the limit in the nonlinear term ϱ^γ and Theorem 7.3 provides the regularity of weak solutions.

Theorem 7.1. *Let $\gamma > 1$ $(N = 2)$, $\gamma > 5/3$ $(N = 3)$, $m > 0$, $f \in L^\infty(\Omega)$ and let q satisfy conditions* (5.2)*. Let $(\varrho, v) \in L^q(\Omega) \times W^{1,2}_0(\Omega)$ be a weak solution to the problem* (1.1)*. Then*

$$\tag{7.1} \|\varrho\|_{0,q} + \|v\|_{1,2} \le K(\|f\|_{0,\infty}, m).$$

Proof of Theorem 7.1:

In the sequel, c, c' are positive constants dependent of m, Ω and q. Multiplying equation $(1.1)_1$ scalarly by v, we get, after some calculations (using among others the continuity equation $(1.1)_2$):

$$\|\nabla v\|_{0,2}^2 \le c\left|\int_\Omega \varrho f \cdot v \, dx\right|. \tag{7.2}$$

If $N = 3$, the r.h.s. of (7.2) can be estimated by using the Holder inequality, the Sobolev imbedding and the interpolation of Lebesgue spaces, as follows

$$\begin{aligned} \left|\int_\Omega \varrho f \cdot v \, dx\right| &\le \|f\|_{0,\infty}\|v\|_{0,6}\|\varrho\|_{0,6/5} \le \\ & c\|f\|_{0,\infty}\|\nabla v\|_{0,2}\|\varrho\|_{0,1}^{1-\lambda}\|\varrho\|_{0,q}^{\lambda} \le \\ & c'\|f\|_{0,\infty}\|\nabla v\|_{0,2}\|\varrho\|_{0,q}^{\lambda}, \end{aligned} \tag{7.3}$$

where

$$q > 6/5, \quad \lambda = \frac{q}{6(q-1)}\,. \tag{7.4}$$

If $N = 2$, then we have similarly

$$\begin{aligned} \left|\int_\Omega \varrho f \cdot v \, dx\right| &\le \|f\|_{0,\infty}\|v\|_{0,r'}\|\varrho\|_{0,r} \le \\ & c\|f\|_{0,\infty}\|\nabla v\|_{0,2}\|\varrho\|_{0,1}^{1-\lambda}\|\varrho\|_{0,q}^{\lambda} \le \\ & c'\|f\|_{0,\infty}\|\nabla v\|_{0,2}\|\varrho\|_{0,q}^{\lambda}, \end{aligned} \tag{7.5}$$

where

$$1 < r < q < \infty, \quad \lambda = \frac{q(r-1)}{r(q-1)}\,. \tag{7.6}$$

Notice that for any $q > 1$ fixed, r can be chosen in such a way that $\lambda \to 0$; of course, the coefficient c' depends of r and tends to ∞ as $r \to 1$. The estimates (7.2)–(7.6) yield, in both cases $N = 2, 3$

$$\|\nabla v\|_{0,2} \le c\|f\|_{0,\infty}\|\varrho\|_{0,q}^{\lambda}. \tag{7.7}$$

Suppose

$$q > \gamma \tag{7.8}$$

and consider it in the form

$$q = \gamma + \alpha, \quad \alpha > 0. \tag{7.9}$$

Let $\max(1, 1/\alpha) < s < \infty$. According to Lemma 4.4, there exists a solution $\omega \in W_0^{1,s}(\Omega)$ of problem

$$\begin{aligned} \operatorname{div}\omega = \varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\,dx, \\ \omega|_{\partial\Omega} = 0, \end{aligned} \tag{7.10}$$

which satisfies estimate

$$\|\omega\|_{1,s} \le c\|\varrho\|^\alpha_{0,\alpha s}. \tag{7.11}$$

Multiplying equation $(1.1)_1$ scalarly by ω (more precisely, by using the duality in $W_0^{1,2}(\Omega)$ in each term of equation $(1.1)_1$), we get

$$\begin{aligned} \|\varrho\|^{\gamma+\alpha}_{0,\gamma+\alpha} = \frac{1}{|\Omega|}\|\varrho\|^\alpha_{0,\alpha}\|\varrho\|^\gamma_{0,\gamma} + \\ \mu_1\int_\Omega \nabla v : \nabla\omega\,dx + (\mu_1+\mu_2)\int_\Omega \operatorname{div}v\operatorname{div}\omega\,dx - \\ \int_\Omega \varrho f\cdot\omega\,dx - \int_\Omega(\varrho v\otimes v) : \nabla\omega\,dx. \end{aligned} \tag{7.12}$$

Now, in order to estimate the terms at the right hand side, we use the Holder inequality, the estimate (7.11) and the interpolation between $L^1(\Omega)$ and $L^q(\Omega)$. The first term at the r.h.s. gives, for $N = 2, 3$:

$$\begin{aligned} \|\varrho\|^\alpha_{0,\alpha}\|\varrho\|^\gamma_{0,\gamma} \le c\|\varrho\|_{0,1}^{\frac{q}{q-1}}\|\varrho\|_{0,q}^{\frac{q(q-2)}{q-1}} \le c'\|\varrho\|_{0,q}^{\frac{q(q-2)}{q-1}} \quad (\alpha \ge 1), \\ \|\varrho\|^\alpha_{0,\alpha}\|\varrho\|^\gamma_{0,\gamma} \le c\|\varrho\|^\gamma_{0,q} \quad (0 < \alpha < 1) \end{aligned} \tag{7.13}$$

The second and the third terms yield the estimate, for $N = 2, 3$:

$$\begin{aligned} \mu_1\int_\Omega \nabla v : \nabla\omega\,dx + (\mu_1+\mu_2)\int_\Omega \operatorname{div}v\operatorname{div}\omega\,dx \le \\ \le c\|\nabla v\|_{0,2}\|\nabla\omega\|_{0,2} \le \|\nabla v\|_{0,2}\|\varrho\|^\alpha_{2\alpha} \le c\|f\|_{0,\infty}\|\varrho\|^{\alpha+\lambda}_{0,q}. \end{aligned} \tag{7.14}$$

The last inequality holds provided that

$$\gamma \ge \alpha. \tag{7.15}$$

The fourth term gives the estimate ($N = 2, 3$):

$$|\int_\Omega \varrho f\cdot\omega\,dx| \le \|f\|_{0,\infty}\|\varrho\|_{0,\gamma+\alpha}\|\omega\|_{\frac{\gamma+\alpha}{\alpha}} \le c\|\varrho\|^{1+\alpha}_{0,q}. \tag{7.16}$$

The last term in (7.12) furnishes, for $N = 3$:

$$
\begin{aligned}
(7.17)\qquad |\int_\Omega (\varrho v \otimes v) : \nabla\omega\, dx| &\le \|\varrho\|_{0,\gamma+\alpha}\|v\|_{0,6}^2\|\nabla\omega\|_{0,\frac{\gamma+\alpha}{\alpha}} \le \\
&\le c\|\varrho\|_{0,q}\|\nabla v\|_{0,2}^2\|\varrho\|_{0,q}^\alpha \le c'\|f\|_{0,\infty}^2\|\varrho\|_{0,q}^{1+\alpha+2\lambda}.
\end{aligned}
$$

The above inequality holds provided $1/3 + (1+\alpha)/q \le 1$, i.e. if

$$
(7.18)\qquad \gamma \ge \frac{3+\alpha}{2}\,.
$$

Finally, it yields, for $N = 2$:

$$
\begin{aligned}
(7.19)\qquad |\int_\Omega (\varrho v \otimes v) : \nabla\omega\, dx| &\le \|\varrho\|_{0,\gamma+\alpha}\|v\|_{0,r}^2\|\nabla\omega\|_{0,\frac{\gamma+\alpha}{\alpha}} \le \\
&\le c\|\varrho\|_{0,q}\|\nabla v\|_{0,2}^2\|\varrho\|_{0,q}^\alpha \le c'\|f\|_{0,\infty}^2\|\varrho\|_{0,q}^{1+\alpha+2\lambda}
\end{aligned}
$$

for appropriately chosen $r \in (1,\infty)$. The estimates (7.12)–(7.19) yield

$$
(7.20)\qquad \|\varrho\|_{0,q} \le K(\|f\|_{0,\infty}, m)
$$

provided q satisfies conditions (5.2) and $\gamma > 1$ $(N = 2)$ or $\gamma > 5/3$ $(N = 3)$. Moreover, we have, in virtue of (7.7)

$$
(7.21)\qquad \|\nabla v\|_{0,2} \le K(\|f\|_{0,\infty}, m).
$$

Theorem 7.1 is thus proved. □

Theorem 7.2. *Let $\gamma > 1$ $(N = 2)$, $\gamma > 2$ $(N = 3)$, $m > 0$, $f \in L^\infty(\Omega)$ and let q satisfy the condition* (5.2). *Let $\{(\varrho^n, v^n)\}_{n=1}^\infty$, $(\varrho^n, v^n) \in L^q(\Omega) \times W_0^{1,2}(\Omega)$, be a sequence of weak solutions to the problem* (1.1). *Then there exists a subsequence $\{(\varrho^{n'}, v^{n'})\}_{n'\in\mathbb{N}}$ and a couple (ϱ, v)*

$$
(7.22)\qquad \varrho \in L^q(\Omega), \quad v \in W_0^{1,2}(\Omega)
$$

such that

$$
(7.23)\qquad
\begin{aligned}
\varrho^{n'} \to \varrho \quad &\text{weakly in} \quad L^q(\Omega) \quad \text{and strongly in} \quad L^{r_1}(\Omega) \\
&\qquad\qquad\qquad\qquad (1 \le r_1 < q), \\
v^{n'} \to v \quad &\text{weakly in} \quad W_0^{1,2}(\Omega) \quad \text{and strongly in} \quad L^{r_2}(\Omega), \\
&1 \le r_2 < \infty \quad (N = 2), \quad 1 \le r_2 < 6 \quad (N = 3).
\end{aligned}
$$

Moreover (ϱ, v) satisfies (1.1) *and we have estimate*

$$
(7.24)\qquad \|\varrho\|_{0,q} + \|v\|_{1,2} \le K(\|f\|_{0,\infty}, m).
$$

PROOF OF THEOREM 7.2:

The proof relies on three fundamental facts which we recall in the sequel.

Denote $\mathcal{C}_0^0(\Omega)$ a Banach space of continuous functions with compact support in Ω (equipped with the norm $\max_{x\in\bar{\Omega}} |u(x)|$) and by $\mathcal{M}(\Omega)$ its dual. The chain of imbeddings ($1 < p < \infty$)

$$L^p(\Omega) \subset L^1(\Omega) \subset \mathcal{M}(\Omega)$$

gives a clear sense to the statement "a sequence u_n of functions from $L^1(\Omega)$ converges *-weakly in $\mathcal{M}(\Omega)$ to a function $u \in L^1(\Omega)$". We have the following classical statement (see e.g. [50]):

Corollary 7.1. *Let $m \in \mathbb{N}$ and $\mathcal{K}$ be a domain of $\mathbb{R}^N$. Let $\mathcal{G} : \mathbb{R}^m \to \mathbb{R}^1$ be a lower continuous convex function. Let the sequence of functions $u_n \in L^1(\mathcal{K};\mathbb{R}^m)$ converges weakly * in $\mathcal{M}(\mathcal{K})$ to a function $u \in L^1(\mathcal{K};\mathbb{R}^m)$. Then*

$$\mathcal{G}(u) \le \liminf_{n\to\infty} \int_{\mathcal{K}} \mathcal{G}(u_n)\, dx. \tag{7.25}$$

The second fact is the theorem about the range of the monotone operators (see e.g. J.L. Lions [21]).

Corollary 7.2. *Let X be a reflexive, separable Banach space (norm $\|\cdot\|_X$) with dual X^* (norm $\|\cdot\|_{X^*}$) and $M : X \to X^*$ be a bounded, monotone operator, such that*

(i) *for any $u, v, w \in X$, $< M(u + \cdot v), w >_{X^*}$ is a continuous function from $\mathbb{R}^1 \to \mathbb{R}^1$,*

(ii)
$$\frac{< Mv, v >_{X^*}}{\|v\|_X} \to 0 \;\; \text{as} \;\; \|v\|_X \to \infty.$$

Then M is a surjective operator, i.e. to any $z \in X^*$ there exists $v \in X$ such that $z = Mv$.

The third fact is an equivalent formulation of the continuity equation $(1.1)_2$ used in [18]–[19]: Let ϱ, v be sufficiently smooth satisfying equation $(1.1)_2$. Then it obviously satisfies the identities

$$\varrho \mathrm{div} v = -\mathrm{div}(\varrho \ln \varrho v) \tag{7.26}$$

and

$$\varrho^\delta \mathrm{div} v = -\frac{1}{\delta - 1}\mathrm{div}(\varrho^\delta v), \quad \delta \ne 1. \tag{7.27}$$

The same statement holds for $\varrho \in L^q(\Omega)$ ($1 < q < \infty$) and $v \in W_0^{1,r}(\Omega)$ ($1 < r < \infty$) satisfying $(1.1)_2$ in the sense of distributions, provided $r > \frac{q}{q-1}$ and

$0 < \delta \leq \frac{q(r-1)}{r}, \delta \neq 1$; it is a consequence of Lemma II.1 in Di Perna and Lions [6].

With these results at hands and with the estimates (5.7) and (7.1), we can proceed to the limit process. From the estimates (5.7), (7.1), we have immediately the following limits (at least for the chosen subsequences of $\{(\varrho^n, v^n)\}_{n=1}^{n=\infty}$):

$$\begin{aligned}
\varrho^n \to \varrho \quad &\text{weakly in} \quad L^q(\Omega);\\
\varrho^n \ln\varrho^n \to \overline{\varrho\ln\varrho} \quad &\text{weakly in} \quad L^{\bar q}(\Omega), \quad 2 < \bar q < q;\\
(\varrho^n)^\gamma \to \overline{\varrho^\gamma} \quad &\text{weakly in} \quad L^{q/\gamma}(\Omega);\\
(\varrho^n)^{\gamma+1} \to \overline{\varrho^{\gamma+1}} \quad &\text{weakly in} \quad L^{q/(\gamma+1)}(\Omega);\\
v^n \to v \quad \text{weakly in} \quad W_0^{1,2}(\Omega) \quad &\text{and strongly in} \quad L^{r_2}(\Omega),\\
1 \leq r_2 < \infty \quad (N=2), \quad &1 \leq r_2 < 6 \quad (N=3);\\
\mathcal{P}^n \to \mathcal{P} \quad \text{weakly in} \quad W^{1,\theta}(\Omega') \quad &\text{and strongly in} \quad L^{r_3}(\Omega'),\\
\forall \quad \Omega' : \quad \overline{\Omega'} \subset \Omega, \quad &1 \leq r_3 < \frac{N\theta}{N-\theta} \quad (N=2,3)\\
u^n \to u \quad &\text{weakly in} \quad W_0^{1,2}(\Omega),\\
u^n \to u \quad \text{weakly in} \quad W^{2,\theta}(\Omega') \quad &\text{and strongly in} \quad W^{1,r_3}(\Omega'),\\
\forall \quad \Omega' : \quad \overline{\Omega'} \subset \Omega, \quad &1 \leq r_3 < \frac{N\theta}{N-\theta} \quad (N=2,3),\\
\nabla\phi^{n'} \to \nabla\phi \quad &\text{weakly in} \quad W_0^{1,2}(\Omega)\\
\phi^{n'} \to \phi \quad \text{weakly in} \quad W^{2,2}(\Omega) \quad &\text{and strongly in} \quad W^{1,r_2}(\Omega),\\
1 \leq r_2 < \infty \quad (N=2), \quad &1 \leq r_2 < 6 \quad (N=3).
\end{aligned} \tag{7.28}$$

Moreover (7.28) implies immediately

$$\begin{aligned}
&\mathcal{P}^n\varrho^n \to \mathcal{P}\varrho, \quad \varrho^n v^n \otimes v^n \to \varrho v \otimes v, \quad \varrho^n v^n \to \varrho v,\\
&\varrho^n \ln\varrho^n v^n \to \overline{\varrho\ln\varrho} v \quad \text{at least in the sense of distributions;}\\
&\varrho^n \operatorname{div} v^n \to -\operatorname{div}(\overline{\varrho\ln\varrho} v) \quad \text{weakly in} \quad L^{\frac{2q}{2+q}}(\Omega).
\end{aligned} \tag{7.29}$$

The only thing which remains open is the strong convergence of ϱ^n. We again closely follow [18]–[19]. The identity (2.8) with ϱ^n, $\mathcal{P}^n$, v^n yields, when $n \to \infty$,

$$\mathcal{P} = \overline{\varrho^\gamma} - (2\mu_1 + \mu_2)\operatorname{div} v. \tag{7.30}$$

Multiplying (7.30) by ϱ, we get

$$\mathcal{P}\varrho = \overline{\varrho^\gamma}\varrho - (2\mu_1 + \mu_2)\varrho \operatorname{div} v. \tag{7.31}$$

The third limit in (7.29) implies in particular

$$\operatorname{div}(\varrho v) = 0$$

in the sense of distributions, hence the identity (7.26) yields

$$\mathcal{P}\varrho = \overline{\varrho^\gamma}\varrho + (2\mu_1 + \mu_2)\operatorname{div}(\varrho\ln\varrho v). \tag{7.32}$$

Integrating the last equation over Ω we get, after the integration by parts in the second term at the r.h.s.:

$$\int_\Omega \mathcal{P}\varrho\, dx = \int_\Omega \overline{\varrho^\gamma}\varrho\, dx. \tag{7.33}$$

On the other hand, multiplying the identity (2.8) with ϱ^n, $\mathcal{P}^n$, v^n, by ϱ^n, and then using the identity (7.26), we get

$$\mathcal{P}^n\varrho^n = (\varrho^n)^{\gamma+1} - (2\mu_1 + \mu_2)\operatorname{div}(\varrho^n\ln\varrho^n v^n). \tag{7.34}$$

The last equation yields, when $n \to \infty$, the identity:

$$\int_\Omega \mathcal{P}\varrho\psi\, dx = \int_\Omega \overline{\varrho^{\gamma+1}}\psi\, dx + \int_\Omega \operatorname{div}(\overline{\varrho\ln\varrho}v)\psi\, dx, \quad \forall\,\psi \in \mathcal{C}_0^\infty(\Omega).$$

From here, by the density argument

$$\int_\Omega \mathcal{P}\varrho\, dx = \int_\Omega \overline{\varrho^{\gamma+1}}\, dx + \int_\Omega \operatorname{div}(\overline{\varrho\ln\varrho}v)\, dx.$$

The second term at the r.h.s. disappears when integrating by parts. The last equality therefore furnishes

$$\int_\Omega \mathcal{P}\varrho\, dx = \int_\Omega \overline{\varrho^{\gamma+1}}\, dx. \tag{7.35}$$

By (7.33), (7.35), we finally get the identity

$$\int_\Omega (\overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma}\varrho)\, dx = 0. \tag{7.36}$$

The further procedure is the following: We use Corollary 7.1 with $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^1$, $\mathcal{G}(t,z) = |t||z|$ and with $t_n = (\varrho^n)^\gamma$, $z_n = \varrho^n$. We thus get, for any open set $\mathcal{K} \subset \Omega$,

$$\int_\mathcal{K} (\overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma}\varrho)\, dx \geq 0. \tag{7.37}$$

The formulas (7.36) and (7.37) yield the identity

$$\overline{\varrho^\gamma}\varrho = \overline{\varrho^{\gamma+1}} \quad \text{a.e. in} \quad \Omega. \tag{7.38}$$

Now, define

$$Mz = z^\gamma \quad (z \geq 0), \qquad Mz = -|z|^\gamma \quad (z < 0). \tag{7.39}$$

Put in Corollary 7.2, $X = L^{\gamma+1}(\Omega)$, i.e. $X^* = L^{\frac{\gamma+1}{\gamma}}(\Omega)$ and apply it to the operator M (the reader easily sees that it satisfies all assumptions). The surjectivity thus yields

$$\overline{\varrho^\gamma} = \varrho^\gamma \quad \text{a.e. in} \quad \Omega. \tag{7.40}$$

The last equality yields the strong convergence of $\varrho^n \to \varrho$ in $L^\gamma(\Omega)$. Moreover, since ϱ^n is bounded in $L^q(\Omega)$, we have also the strong convergence $\varrho^n \to \varrho$ in $L^{r_1}(\Omega)$, $\quad 1 \leq r_1 < q$. Theorem 7.2 is thus proved. □

P.L. Lions showed me recently, how it is possible to prove Theorem 7.2 for $5/3 < \gamma \leq 2$. This case is not contained in the present proof; the arguments (7.30)–(7.40) have to be essentially modified.

Theorem 7.3. *Let $\gamma > 1$ $(N = 2)$, $\gamma > 3$ $(N = 3)$ and $m > 0$, $f \in L^\infty(\Omega)$. Let $\{(\varrho^n, v^n)\}_{n=1}^\infty$ be a sequence of weak solutions to the problem* (1.1) *such that*

$$\varrho^n \in L^\infty(\Omega), \qquad v^n \in W_0^{1,\infty}(\Omega). \tag{7.41}$$

(a) *Then the couple (ϱ, v) (a weak limit of a chosen subsequence $\{(\varrho^{n'}, v^{n'})\}_{n' \in \mathbb{N}}$ – see* (7.22)–(7.23) *in Theorem* 7.2) *is such that*

$$\varrho \in L^p_{loc}(\Omega), \qquad v \in W^{1,p}_{loc}(\Omega), \qquad 2\gamma \leq p < \infty \tag{7.42}$$

and satisfies the estimate

$$\|\varrho\|_{0,p,\Omega'} + \|v\|_{1,p,\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega') \tag{7.43}$$

for any smooth domain Ω' such that $\overline{\Omega'} \subset \Omega$.

(b) *Consider the quantities $\mathcal{P}$, u, ϕ (see* (2.1)–(2.2), (2.4) *and Lemma* 4.5). *Then*

$$\mathcal{P} \in W^{1,p}_{loc}(\Omega), \qquad u \in W^{2,p}_{loc}(\Omega), \qquad \phi \in W^{2,p}_{loc}(\Omega) \tag{7.44}$$

and we have the estimate

$$\|\mathcal{P}\|_{1,p,\Omega'} + \|u\|_{2,p,\Omega'} + \|\phi\|_{2,p,\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega') \tag{7.45}$$

for any smooth domain Ω' such that $\overline{\Omega'} \subset \Omega$.

PROOF OF THEOREM 7.3:

Firstly, we prove that for $N = 3$, $\gamma > 3$, $q = 2\gamma$, $2\gamma \le p < \infty$, $m > 0$, $f \in L^\infty(\Omega)$ and $\{(\varrho^n, v^n)\}_{n=1}^\infty$ a sequence of weak solutions to problem (1.1) satisfying (7.41) and the estimate (7.1), it holds

$$\|\varrho^n\|_{0,\bar q,\Omega'} + \|v^n\|_{0,p,\Omega'} + \|v^n\|_{1,2} \le K(\|f\|_{0,\infty}, m, \Omega'), \tag{7.46}$$

with a $\bar q > 3\gamma$.

For the sake of simplicity, we omit the indexes n at (ϱ, v); we write simply (ϱ, v) instead of (ϱ^n, v^n). Multiplying the equation $(1.1)_1$ scalarly by $v|v|^{s-2}$, $s > 2$ and integrating over Ω, we get

$$\begin{aligned}\mu_1\|\nabla v|v|^{\frac{s-2}{2}}\|_{0,2}^2 + \mu_1\|\nabla(|v|^{\frac{s}{2}})\|_{0,2}^2 + \frac{\mu_1+\mu_2}{2}\|\mathrm{div} v|v|^{\frac{s-2}{2}}\|_{0,2}^2 \le \\ \le c\{|\int_\Omega \varrho f\cdot v|v|^{s-2}\,dx| + |\int_\Omega \varrho^\gamma \mathrm{div}(v|v|^{s-2})\,dx|\}.\end{aligned} \tag{7.47}$$

Notice that the contributions due to the convective term are identically zero as the consequence of the continuity equation $(1.1)_2$. By using the Sobolev imbeddings, the estimate (7.47) can be rewritten in the form:

$$\begin{aligned}\|\nabla v|v|^{\frac{s-2}{2}}\|_{0,2}^2 + \|v\|_{0,3s}^s \le \\ \le c\{|\int_\Omega \varrho f\cdot v|v|^{s-2}\,dx| + |\int_\Omega \varrho^\gamma \mathrm{div}(v|v|^{s-2})\,dx|\}.\end{aligned} \tag{7.48}$$

The first term at the r.h.s. yields the estimate

$$|\int_\Omega \varrho f\cdot v|v|^{s-2}\,dx| \le \|f\|_{0,\infty}\|\varrho\|_{0,\frac{3s}{2s+1}}\|v\|_{0,3s}^{s-1}. \tag{7.49}$$

For $s \in (2,\infty)$, $\frac{3s}{2s+1} \le 2\gamma$. Hence, by Theorem 7.1

$$|\int_\Omega \varrho f\cdot v|v|^{s-2}\,dx| \le c\|f\|_{0,\infty}\|v\|_{0,3s}^{s-1}. \tag{7.50}$$

The second term is estimated as follows:

$$\begin{aligned}|\int_\Omega \varrho^\gamma \mathrm{div}(v|v|^{s-2})\,dx| \le \|\nabla v|v|^{\frac{s-2}{2}}\|_{0,2}\|\varrho^\gamma|v|^{\frac{s-2}{2}}\|_{0,2} \le \\ \le \mu_1\|\nabla v|v|^{\frac{s-2}{2}}\|_{0,2}^2 + \frac{c}{\mu_1}\|\varrho^\gamma|v|^{\frac{s-2}{2}}\|_{0,2}^2.\end{aligned} \tag{7.51}$$

Further we have

$$\|\varrho^\gamma|v|^{\frac{s-2}{2}}\|_{0,2}^2 \le \|v\|_{0,3s}^{s-2}\|\varrho\|_{0,\frac{3\gamma s}{s+1}}^{2\gamma}. \tag{7.52}$$

The estimates (7.48)–(7.52) yield in particular

$$\|v\|_{0,3s} \leq c(\|\varrho\|^{\gamma}_{0,\frac{3\gamma s}{s+1}} + K(\|f\|_{0,\infty}, m)). \tag{7.53}$$

Let η, Ω', $\tilde{u}$, $\tilde{\mathcal{P}}$ be defined as in Section 2. Put further $\tilde{v} = \eta v$, $\tilde{\varrho} = \eta^{1/\gamma}\varrho$. Then $\tilde{\varrho}$, $\tilde{v}$ satisfy, in virtue of equation $(1.1)_1$:

$$\begin{aligned} -\mu_1 \Delta\tilde{v} - (\mu_1+\mu_2)\nabla \mathrm{div}\tilde{v} + \nabla(\tilde{\varrho})^{\gamma} &= \underline{\mathcal{F}} + \underline{\mathcal{G}}, \\ \tilde{v}|_{\partial\Omega} &= 0, \end{aligned} \tag{7.54}$$

where

$$\begin{aligned} \underline{\mathcal{G}} = -\mu_1(2\nabla\eta\cdot\nabla v + \Delta\eta v) - (\mu_1+\mu_2)(\nabla v\cdot\nabla\eta + \nabla\eta \mathrm{div} v + v\cdot\nabla\nabla\eta) \\ +\nabla(\eta^{1/\gamma})\varrho + \varrho v\cdot\nabla\eta v, \\ \underline{\mathcal{F}} = -\varrho v\cdot\nabla\tilde{v} + \eta\varrho f. \end{aligned}$$

One easily verifies that

$$\|\underline{\mathcal{G}}\|_{0,2} \leq K(\|f\|_{0,\infty}, m, \Omega') \tag{7.55}$$

Therefore, the procedure (7.47)–(7.52) applied to the equation (7.54) furnishes

$$\|\tilde{v}\|_{0,3s} \leq c(\|\tilde{\varrho}\|^{\gamma}_{0,\frac{3\gamma s}{s+1}} + K(\|f\|_{0,\infty}, m, \Omega')). \tag{7.56}$$

Next consider the equation (2.4). It yields, when multiplied by $\eta\tilde{\varrho}^P$ $(P > 0, P \neq 1)$ and integrated over Ω

$$\int_{\Omega} \tilde{\varrho}^{\gamma+P}\, dx = \int_{\Omega} \tilde{\mathcal{P}}\tilde{\varrho}^P\, dx - \frac{\gamma+P}{\gamma}\frac{2\mu_1+\mu_2}{P-1}\int_{\Omega} \nabla\eta\cdot v\tilde{\varrho}^P\, dx. \tag{7.57}$$

Now, we are in the position to start with the bootstrapping. Let $\{\Omega_i'\}_{i\in\mathbb{N}}$ be a sequence of open subsets of Ω such that $\overline{\Omega'}_{i+1} \subset \Omega_i'$ and $\{\eta_i\}_{i\in\mathbb{N}}$, $\eta_i \in \mathcal{C}_0^{\infty}(\Omega)$ be a sequence of cut-off functions such that $\eta_i(x) = 1$ in Ω_i', $0 \leq \eta_i(x) \leq 1$, $\eta_{i+1} = 0$ in $\Omega - \Omega_i$, $\eta_{i+1}(x) \leq \eta_i(x)$. Define $\tilde{a}_i = a\eta_i$, where a stands for $\mathcal{P}$, v, u and $\tilde{\varrho}_i = \eta_i^{1/\gamma}\varrho$. Denote $\tilde{\mathcal{G}}_i$, $\tilde{g}_i$, $\tilde{\mathcal{F}}_i$ the functions (2.9) with η replaced by η_i in their definition. Take $q_0 = q$, $t_0 = \frac{3q_0}{3+q_0}$. Then

$$\|\tilde{\mathcal{F}}_1\|_{-1,t_0} + \|\tilde{\mathcal{G}}_1\|_{-1,t_0} + \|\tilde{g}_1\|_{0,t_0} \leq K(\|f\|_{0,\infty}, m, \Omega_1'). \tag{7.58}$$

Lemma 4.1 applied to the equation (2.8) then gives, in particular

$$\|\tilde{\mathcal{P}}_1\|_{0,t_0} \leq K(\|f\|_{0,\infty}, m, \Omega_1'). \tag{7.59}$$

The equation (7.57) then yields

$$\|\tilde{\varrho}_1\|_{\gamma+P_0}^{\gamma+P_0} \le c(\|\tilde{\mathcal{P}}_1\|_{0,t_0} + \|v\|_{0,t_0})\|\tilde{\varrho}_1\|_{0,\frac{P_0 t_0}{t_0-1}}.$$

For $P_0 = \frac{q_0(t_0-1)}{t_0}$, one gets

$$\|\tilde{\varrho}_1\|_{0,q_1} \le K(\|f\|_{0,\infty}, m, \Omega_1'), \quad q_1 = \gamma + \frac{q_0(t_0-1)}{t_0} = \frac{2q_0+3\gamma-3}{3}. \tag{7.60}$$

The reader easily verifies that $q_1 > q_0$, $q_1 > 2\gamma$ but still $q_1 < \frac{7}{3}\gamma - 1 < 3\gamma$. Now, for $s \in (2, \frac{q_1}{3\gamma-q_1}]$ we have $\frac{3\gamma s}{\gamma+1} \le q_1$; therefore, (7.56) furnishes

$$\|\tilde{v}_1\|_{0,3s_1} \le K(\|f\|_{0,\infty}, m, \Omega_1'), \quad s_1 = \frac{q_1}{3\gamma - q_1}. \tag{7.61}$$

After this we find (compare with (7.58))

$$\begin{aligned} &\|\tilde{\mathcal{F}}_2\|_{-1,t_1} + \|\tilde{\mathcal{G}}_2\|_{-1,t_1} + \|\tilde{g}_2\|_{0,t_1} \le K(\|f\|_{0,\infty}, m, \Omega_2'), \\ &\qquad t_1 = \frac{3q_1}{6\gamma - 2q_1 + 3}. \end{aligned} \tag{7.62}$$

Lemma 4.1 applied to the equation (2.8) then gives (compare with (7.59))

$$\|\tilde{\mathcal{P}}_2\|_{0,t_1} \le K(\|f\|_{0,\infty}, m, \Omega_2'). \tag{7.63}$$

Finally (7.57) furnishes

$$\|\tilde{\varrho}_2\|_{\gamma+P_1}^{\gamma+P_1} \le (\|\tilde{\mathcal{P}}_2\|_{0,t_1} + \|\tilde{v}_1\|_{0,t_1})\|\tilde{\varrho}_2\|_{0,\frac{P_1 t_1}{t_1-1}}$$

which becomes, when $P_1 = \frac{q_1(t_1-1)}{t_1}$, in virtue of (7.61), (7.63)

$$\|\tilde{\varrho}_2\|_{0,q_2} \le K(\|f\|_{0,\infty}, m, \Omega_2'), \quad q_2 = \gamma + \frac{q_1(t_1-1)}{t_1} = \frac{5q_1 - 3\gamma - 3}{3}. \tag{7.64}$$

The reader finds $q_1 < q_2 < 3\gamma$, $q_2 - q_1 > \frac{\gamma-3}{3}$. We are thus forced to repeat the procedure (starting from (7.61)) several times getting on each step

$$\|\tilde{v}_i\|_{0,3s_i} \le K(\|f\|_{0,\infty}, m, \Omega_i'), \quad s_i = \frac{q_i}{3\gamma - q_i}, \tag{7.65}$$

$$\begin{aligned} &\|\tilde{\mathcal{F}}_{i+1}\|_{-1,t_i} + \|\tilde{\mathcal{G}}_{i+1}\|_{-1,t_i} + \|\tilde{g}_{i+1}\|_{0,t_i} \le K(\|f\|_{0,\infty}, m, \Omega_{i+1}'), \\ &\qquad t_i = \frac{3q_i}{6\gamma - 2q_i + 3}, \end{aligned} \tag{7.66}$$

$$\|\tilde{\mathcal{P}}_{i+1}\|_{0,t_i} \le K(\|f\|_{0,\infty}, m, \Omega'_{i+1}), \tag{7.67}$$

$$\|\tilde{\varrho}_{i+1}\|_{0,q_{i+1}} \le K(\|f\|_{0,\infty}, m, \Omega'_{i+1}), \quad q_{i+1} = \frac{5q_i - 3\gamma - 3}{3}\,. \tag{7.68}$$

Certainly, $q_{i+1} - q_i > \frac{\gamma-3}{\gamma}$, hence there exists $i_0 \in \mathbb{N}$ such that

$$q_{i_0} > 3\gamma. \tag{7.69}$$

In such a case, we have (with properly chosen Ω'_i's)

$$\|v\|_{0,3s,\Omega'} \le K(\|f\|_{0,\infty}, m, \Omega'), \quad s \in (1,\infty). \tag{7.70}$$

The estimates (7.70), (7.56) complete the proof of the estimate (7.46).

Next we prove that $\{\varrho^n, v^n\}_{n=1}^{\infty}$ satisfies the estimate

$$\|\varrho^n\|_{0,p,\Omega'} + \|v^n\|_{1,p,\Omega'} \le K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 \le p < \infty. \tag{7.71}$$

Put $\bar{q}_1 = \bar{q}$. We find

$$\|\tilde{\mathcal{F}}_2\|_{-1,\bar{q}_1} + \|\tilde{\mathcal{G}}_2\|_{-1,\bar{q}_1} + \|\tilde{g}_2\|_{0,\bar{q}_1} \le K(\|f\|_{0,\infty}, m, \Omega'_1). \tag{7.72}$$

Lemma 4.1 applied to the equation (2.8) yields, in particular

$$\|\tilde{\mathcal{P}}_2\|_{0,\bar{q}_1} \le K(\|f\|_{0,\infty}, m, \Omega'_2). \tag{7.73}$$

The equation (7.57) furnishes

$$\|\tilde{\varrho}_2\|^{\gamma+\bar{P}_1}_{\gamma+\bar{P}_1} \le (\|\tilde{\mathcal{P}}_2\|_{0,\bar{q}_1} + \|\tilde{v}_1\|_{0,\bar{q}_1})\|\tilde{\varrho}_2\|_{0,\frac{\bar{P}_1\bar{q}_1}{\bar{q}_1-1}}\,.$$

The choice $\bar{P}_1 = \bar{q}_1 - 1$ yields

$$\|\tilde{\varrho}_2\|_{0,\bar{q}_2} \le K(\|f\|_{0,\infty}, m, \Omega'_2), \quad \bar{q}_2 = \gamma + \bar{q}_1 - 1. \tag{7.74}$$

We repeat the whole procedure starting from (7.72) several times getting at the end of each step ($i \in \mathbb{N}$)

$$\|\tilde{\varrho}_i\|_{0,\bar{q}_i} \le K(\|f\|_{0,\infty}, m, \Omega'_i), \quad \bar{q}_i = \gamma + \bar{q}_{i-1} - 1 > 3\gamma + (i-1)(\gamma-1) \tag{7.75}$$

and

$$\|\tilde{\mathcal{P}}_i\|_{0,\bar{q}_i} \le K(\|f\|_{0,\infty}, m, \Omega'_i), \quad \bar{q}_i = \gamma + \bar{q}_{i-1} - 1 > 3\gamma + (i-1)(\gamma-1). \tag{7.76}$$

This yields, after the finite number of steps (with properly chosen Ω_i''s)

$$\|\varrho\|_{0,p,\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 < p < \infty \tag{7.77}$$

and

$$\|\mathcal{P}\|_{0,p,\Omega'} \leq K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 < p < \infty. \tag{7.78}$$

After this, equation (2.4) yields

$$\|\mathrm{div}\tilde{v}\|_{0,p} \leq K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 < p < \infty \tag{7.79}$$

and (7.1), (7.46), (7.77) imply

$$\|\underline{\mathcal{F}}\|_{-1,p} + \|\underline{\mathcal{G}}\|_{-1,p} \leq K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 < p < \infty. \tag{7.80}$$

Applying Lemma 4.1 to the nonhomogeneous Stokes problem (see (7.54))

$$\begin{aligned} -\mu_1\Delta\tilde{v} + \nabla\hat{\mathcal{P}} &= \underline{\mathcal{F}} + \underline{\mathcal{G}} \quad \text{in} \quad \Omega \\ \mathrm{div}\tilde{v} &= \mathrm{div}\tilde{v} \quad \text{in} \quad \Omega \\ \tilde{v}|_{\partial\Omega} &= 0, \end{aligned} \tag{7.81}$$

where

$$\hat{\mathcal{P}} = -(\mu_1 + \mu_2)\mathrm{div}\tilde{v} + \tilde{\varrho}^{\gamma},$$

we obtain, in particular

$$\|\tilde{v}\|_{1,p} \leq K(\|f\|_{0,\infty}, m, \Omega'), \quad 1 < p < \infty. \tag{7.82}$$

The estimates (7.77) and (7.82) complete the proof of estimate (7.71).

We come back to the Stokes problem (2.8). Due to the estimates (7.43), (7.71), we obtain

$$\tilde{\mathcal{F}},\ \tilde{\mathcal{G}} \in L^p(\Omega), \quad \tilde{g} \in W^{1,p}(\Omega), \quad 2\gamma \leq p < \infty. \tag{7.83}$$

This implies, by Lemma 4.2, applied to the equation (2.8), the estimate (7.45) for u and $\mathcal{P}$. The estimate of $W^{2,p}-$norm of ϕ follows from Lemma 4.5. Theorem 7.3 is thus proved. □

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