On a problem of Gulevich on nonexpansive maps in uniformly convex Banach spaces

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Abstract. Let X be a uniformly convex Banach space, $D \subset X$, $f : D \to X$ a nonexpansive map, and K a closed bounded subset such that $\overline{\mathrm{co}}\, K \subset D$. If (1) $f|_K$ is weakly inward and K is star-shaped or (2) $f|_K$ satisfies the Leray-Schauder boundary condition, then f has a fixed point in $\overline{\mathrm{co}}\, K$. This is closely related to a problem of Gulevich [Gu]. Some of our main results are generalizations of theorems due to Kirk and Ray [KR] and others.

Keywords: uniformly convex, Banach space, Hilbert space, contraction, nonexpansive map, weakly inward map, demi-closed, Rothe condition, Leray-Schauder condition, (KR)-bounded, Opial's condition

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The well-known theorem of Browder-Göhde-Kirk assures existence of a fixed point for nonexpansive maps $f : K \to K$ where K is a *bounded* closed convex subset of a uniformly convex Banach space X. In [KR], Kirk and Ray showed that f can be replaced by a weakly inward nonexpansive map $f : K \to X$ while the boundedness of K can be replaced by that of the geometric estimator

$$G(x, fx) = \{z \in K : \|z - x\| \geq \|z - fx\|\}$$

for some $x \in K$ or more general sets. Note that any fixed point of f is contained in $G(x, fx)$.

On the other hand, Gulevich [Gu] considered the situation as follows: H is a Hilbert space, K is a nonempty bounded closed (not necessarily convex) subset of H, and $f : D \subset H \to H$ is a nonexpansive map, where $\overline{\mathrm{co}}\, K \subset D$. Gulevich's basic theorem [Gu, Theorem 1] states that f has a fixed point in $\overline{\mathrm{co}}\, K$ if f satisfies the Rothe condition $f(\mathrm{Bd}\, K) \subset K$. He also raised as a problem whether H can be replaced by a uniformly convex Banach space.

In the present paper, we obtain some fixed point theorems on nonexpansive maps defined on closed (not necessarily bounded or convex) subsets of a Banach space. Our results are closely related to Gulevich's theorem and extend some known results of Kirk and Ray [KR], Goebel and Kuczumow [Go], and Browder [B1], [B2]. Moreover, we adopt more general boundary conditions on those nonexpansive maps. In fact, the weakly inwardness or the so-called Leray-Schauder condition is used in our results instead of the Rothe condition used in [Gu].

Recall that $f : K \to X$ is a *contraction* if there exists a $k \in [0, 1)$ such that

$$\|fx - fy\| \leq k\|x - y\| \quad \text{for all} \quad x, y \in K;$$

and a *nonexpansive map* if

$$\|fx - fy\| \leq \|x - y\| \quad \text{for all} \quad x, y \in K.$$

We say that f is *weakly inward* if $fx \in \overline{I}_K(x)$ for any $x \in \mathrm{Bd}\, K$ (equivalently, for any $x \in K$), where $\overline{}$, Bd, and Int denote the closure, boundary, and interior, respectively, and

$$I_K(x) = \{x + c(y - x) : y \in K,\ c \geq 1\}.$$

Note that any map satisfying the Rothe condition is weakly inward.

We begin with the following:

Theorem 0. *Let K be a closed subset of a Banach space X and $f : K \to X$ a contraction satisfying one of the following:*

(i) $f(\mathrm{Bd}\, K) \subset K$.

(ii) f *is weakly inward.*

(iii) $0 \in \mathrm{Int}\, K$ *and* $fx \neq mx$ *for all* $x \in \mathrm{Bd}\, K$ *and* $m > 1$.

Then f has a unique fixed point.

Note that Theorem 0 (i) is a particular case of Assad and Kirk [AK, Theorem 1], Theorem 0 (ii) is due to Martinez-Yanez [M, Theorem] or, in a more general form, to Zhang [Z, Theorem 3.3], and Theorem 0 (iii) to Gatica and Kirk [GK, Theorem 2.1]. There are more general results than Theorem 0. However, Theorem 0 is sufficient for our purpose. Note also that (iii) can be replaced by the following:

(iii)$'$ there exists a $w \in \mathrm{Int}\, K$ such that

$$fx - w \neq m(x - w) \quad \text{for all} \quad x \in \mathrm{Bd}\, K \quad \text{and} \quad m > 1.$$

Moreover, (i) $\Rightarrow$ (ii) and, whenever K is convex and $0 \in \mathrm{Int}\, K$, we have (ii) $\Rightarrow$ (iii).

A subset K of a vector space is said to be *star-shaped* if there exists a given point $x_0 \in K$ such that $tx_0 + (1 - t)x \in K$ for any $t \in (0, 1)$ and $x \in K$, where x_0 is called a *center* of K.

For the K and f in Theorem 0, we say that K is (KR)-*bounded* or bounded in the sense of Kirk-Ray [KR] if, for some bounded set $A \subset K$, the set

$$G(A) = \bigcap_{u \in A} G(u, fu)$$

is either empty or bounded.

The following is a generalization of the almost fixed point property of bounded closed subsets of a Banach space for nonexpansive maps.

Theorem 1. *Let X be a Banach space, K a closed subset of X, and $f : K \to X$ a nonexpansive map such that K is (KR)-bounded and one of the following holds:*

(i) *K is star-shaped and $f(\operatorname{Bd} K) \subset K$.*
(ii) *K is star-shaped and f is weakly inward.*
(iii) *$0 \in \operatorname{Int} K$ and $fx \neq mx$ for all $x \in \operatorname{Bd} K$ and $m > 1$.*

Then there exists a bounded sequence $\{x_n\}$ in K such that $\|x_n - fx_n\| \to 0$ as $n \to \infty$.

Proof: For cases (i) and (ii) we may without loss of generality assume that 0 is the center. For $\alpha \in (0,1)$, define $f_\alpha : K \to X$ by $f_\alpha x = \alpha f x$ for $x \in K$. Then clearly f_α is a contraction. We show that one of (i)–(iii) in Theorem 0 holds for f_α:

(i) Since K is star-shaped at center 0, we have $\alpha K \subset K$. Since $f(\operatorname{Bd} K) \subset K$, for $x \in \operatorname{Bd} K$, we have $f_\alpha x = \alpha f x \in \alpha K \subset K$. Therefore, $f_\alpha(\operatorname{Bd} K) \subset K$.

(ii) From $fx \in \overline{I}_K(x)$, we have $f_\alpha x = \alpha f x \in \alpha \overline{I}_K(x) \subset \overline{I}_K(x)$ since $I_K(x)$ is a star-shaped set with center 0. See Zhang [Z, Theorem 1.2]. Note that (i) $\Rightarrow$ (ii).

(iii) Suppose that $f_\alpha x = mx$ for some $x \in \operatorname{Bd} K$ and $m > 1$, then $fx = \alpha^{-1} f_\alpha x = (\alpha^{-1} m)x$ and $\alpha^{-1} m > 1$, which contradicts our assumption.

Therefore, by Theorem 0, f_α has a fixed point $x_\alpha \in K$. Suppose that the set $\{x_\alpha : \alpha \in (0,1)\}$ is not bounded. Then it is possible to choose $\alpha \in (0,1)$ so that

$$\sup_{u \in A} \|fu\| \leq \inf_{u \in A} \|x_\alpha - u\|$$

and in addition, if $G(A) \neq \emptyset$, then α may also be chosen so that

$$\|x_\alpha\| > \sup\{\|x\| : x \in G(A)\}.$$

Therefore, for each $u \in A$,

$$\begin{aligned} \|x_\alpha - fu\| &= \|\alpha f x_\alpha - fu\| \leq \alpha \|f x_\alpha - fu\| + (1-\alpha)\|fu\| \\ &\leq \alpha \|x_\alpha - u\| + (1-\alpha)\|x_\alpha - u\| = \|x_\alpha - u\|. \end{aligned}$$

This implies $x_\alpha \in G(A)$, which is a contradiction. Thus $M = \sup\{\|x_\alpha\| : \alpha \in (0,1)\} < \infty$ and we have

$$\|x_\alpha - f x_\alpha\| = (\alpha^{-1} - 1)\|x_\alpha\| \leq (\alpha^{-1} - 1)M,$$

yielding $\|x_\alpha - f x_\alpha\| \to 0$ as $\alpha \to 1$. This completes our proof. □

Note that Kirk and Ray [KR, Theorem 2.3] obtained Theorem 1 (ii) for the case K is convex. In the second half of the proof of Theorem 1, we followed that of [KR, Theorem 2.3]. Note that Theorem 1 (i) generalizes Dotson [D, Theorem 1].

A Banach space X is said to satisfy *Opial's condition* if, whenever a sequence $\{x_n\}$ converges weakly to $x_0 \in X$, then

$$\liminf_{n\to\infty}\|x_n - x\| > \liminf_{n\to\infty}\|x_n - x_0\|$$

for all $x \in X$, $x \neq x_0$. Opial [O] showed that if C is a weakly compact subset of a Banach space X satisfying this condition and $f : C \to X$ is nonexpansive, then $I - f$ is demi-closed ([B2], [Gö]); that is, if $\{x_n\} \subset C$ satisfies $x_n \to x$ weakly while $(I - f)x_n \to y$ strongly, then $(I - f)x = y$, where I is the identity map on C.

Examples of spaces satisfying Opial's condition are Hilbert spaces, l^p $(1 \leq p < \infty)$, and uniformly convex Banach spaces with weakly continuous duality maps.

From Theorem 1, we have the following:

Theorem 2. *Let X be a Banach space, K a weakly compact subset of X, and $f : K \to X$ a nonexpansive map satisfying one of* (i)–(iii) *in Theorem* 1.

(a) *If $I - f$ is demi-closed on K, then f has a fixed point.*
(b) *If X satisfies Opial's condition, then f has a fixed point.*

Proof: Since K is closed and bounded, f satisfies all the requirements of Theorem 1. Hence, there exists a sequence $\{x_n\}$ in K such that $\|x_n - fx_n\| \to 0$ as $n \to \infty$. Since K is weakly compact, we may assume that $x_n \to x$ weakly to some $x \in K$. Since $x_n - fx_n \to 0$ strongly, $x_n - fx_n = (I - f)x_n$, and $I - f$ is demi-closed, we conclude that $(I - f)x = 0$ and hence $x = fx$. This completes our proof. □

Note that Zhang [Z, Theorem 3.8 and Corollaries 3.10, 3.11] obtained the multi-valued version of Theorem 2(ii), with different proof, and that Theorem 2(i) generalizes Dotson [D, Theorem 2]. Note also that if K is compact, then f has a fixed point in Theorem 2 without assuming the demi-closedness of $I - f$.

From Theorem 1, we also have the following:

Theorem 3. *Let X be a uniformly convex Banach space, D a subset of X, and $f : D \to X$ a nonexpansive map. Let K be a closed* (KR)*-bounded subset of X such that $\overline{\mathrm{co}}\,K \subset D$ and one of* (i)–(iii) *of Theorem* 1 *holds for $f|_K$. Then f has a fixed point in $\overline{\mathrm{co}}\,K$.*

Proof: Since $f|_K$ satisfies all the requirements of Theorem 1, there exists a bounded sequence $\{x_n\}$ in K such that $\|x_n - fx_n\| \to 0$. Since $\{x_n\}$ is contained in a bounded closed convex subset $L \subset \overline{\mathrm{co}}\,K$ and L is weakly compact, we may assume $x_n \to x_0$ weakly to some $x_0 \in L$. Since $I - f$ is demi-closed on L ([B2], [Gö]) and $(I - f)x_n \to 0$ strongly, we conclude that $(I - f)x = 0$, and hence $x = fx$. This completes our proof. □

If we can eliminate the star-shapedness in (i), then Theorem 3(i) will be the required affirmative answer to Gulevich's problem. Moreover, Gulevich [Gu] noted that, for case (i) of Theorem 3 in a Hilbert space H, f has a fixed point in K.

In case $0 \in \operatorname{Int} K$, Theorem 3 (iii) generalizes [Gu, Theorem 1].

Note that the set $A \subset K$ for the (KR)-boundedness can be chosen so that $A \subset D$ and $f(A) \subset K$. See the proof of Theorem 1. Therefore, Theorem 3 generalizes Ray [R, Lemma 1].

For $D = K = \overline{\mathrm{co}}\, K$, Theorem 3 reduces to the following:

Theorem 4. *Let X be a uniformly convex Banach space, K a closed convex subset, and $f : K \to X$ a nonexpansive map such that K is (KR)-bounded and one of* (i)–(iii) *in Theorem* 0 *holds. Then f has a fixed point.*

Note that Theorem 4 (i) and (ii) are due to Kirk and Ray [KR, Theorem 2.3], which extends Goebel and Kuczumow [Go, Theorem 6]. Also note that Theorem 4 (iii) extends Browder [B2, Theorem 1] for nonexpansive maps. For a Hilbert space X and a closed ball in X, Theorem 4 (i) is due to Browder [B1, Theorem 2], which was used to show existence of periodic solutions for nonlinear equations of evolution.

Recently Canetti, Marino, and Pietramala [CMP] obtained multi-valued versions of Theorem 4 (ii) and, under the stronger assumption of convexity, some other results similar to Theorems 1–3 for case (ii).

Finally, we note that the so-called Rothe condition (i) was first adopted by Knaster, Kuratowski, and Mazurkiewicz [KKM]. Also, the origin of the so-called Leray-Schauder condition (iii) seems to be Schaefer [S], and the following are well-known examples of that condition:

(A) $\|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2$ for $x \in \operatorname{Bd} K$.

(K) $\operatorname{Re}\langle fx, x\rangle = \|x\|^2$ for $x \in \operatorname{Bd} K$, $x \neq 0$, in a Hilbert space H.

Or more generally,

(P) $(fx, Jx) \leq (x, Jx)$ for $x \in \operatorname{Bd} K$, $0 \in \operatorname{Int} K$, where J is any duality map of X into 2^{X^*}.

Condition (A) is due to Altman [A], (K) to Krasnosel'skii [K] and Shinbrot [Sh], and (P) to Petryshyn [P].

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