Countable fan-tightness versus countable tightness

A.V. Arhangel'skii, A. Bella

Abstract. Countable tightness is compared to the stronger notion of countable fan-tightness. In particular, we prove that countable tightness is equivalent to countable fan-tightness in countably compact regular spaces, and that countable fan-tightness is preserved by pseudo-open compact mappings. We also discuss the behaviour of countable tightness and of countable fan-tightness under the product operation.

Keywords: tightness, fan-tightness, countably compact spaces, pseudo-compact space, P-point, biquotient mapping
Classification: 54A25

The notion of countable fan-tightness was introduced in a natural manner by the first author during his investigation on the topological properties of a function space in the topology of pointwise convergence (see [3]). In fact, the main result of [3] says that the space $C_p(X)$ has countable fan-tightness if and only if X^n is a Hurewicz space for any integer n.

The aim of this paper is to study various topological properties related to the concept of countable fan-tightness, in particular with respect to the much more familiar concept of countable tightness. Among other things, for instance, we show that these two notions coincide in any countably compact regular space.

Henceforth all spaces are assumed at least T_1 and all mappings continuous; ω denotes the set (or the discrete space) of all the intergers.

A space X has countable tightness if whenever $A \subset X$ and $x \in \overline{A}$, there exists a countable set $B \subset A$ such that $x \in \overline{B}$.

A space X has *countable fan-tightness* if for any countable family $\{A_n : n \in \omega\}$ of subsets of X satisfying $x \in \bigcap_{n\in\omega}\overline{A_n}$ it is possible to select finite sets $K_n \subset A_n$ in such a way that $x \in \overline{\bigcup_{n\in\omega}K_n}$.

It is evident that a space having countable fan-tightness has also countable tightness. The converse is in general false.

Example 1. Let us take the product of the usual convergent sequence $\omega + 1$ with the discrete space ω. The quotient space of it obtained by identifying all non isolated points is called *Fréchet-Urysohn fan*; it is usually denoted by S_ω. This fan is a typical example of a countable space with only one non-isolated point the fan-tightness of which is not countable.

Example 2. Put $O = (0,0)$, $A = \{(0,n) : n \in \omega\setminus\{0\}\}$ and $B = \{(m,n) : m,n \in \omega\setminus \{0\}\}$. On the set $X = \{O\}\cup A\cup B$ we introduce a topology in the following way. All points of B are isolated, for each $n \in \omega \setminus \{0\}$ the subspace $P_n = \{(m,n) : m \in \omega\}$ is compact and the subspace $\{O\} \cup A$ is also compact. Now take on X the finest (strongest) topology satisfying the previous conditions. This space, introduced by R. Arens [1], is usually denoted by S_2. Let us show that the fan-tightness of this space at the point O is not countable. Take $A_k = \{(m,n) : n > k, m, n \in \omega \setminus \{0\}\}$. Then one can not choose finite subsets of A_k in such a way that their union would have O as a limit point.

Clearly, if the fan-tightness of a space X is countable, and Y is a subspace of X, then the fan-tightness of Y is also countable.

Remark 1. In the definition of fan-tightness, the family $\{A_n : n \in \omega\}$ can be assumed decreasing, without any loss of generality. Indeed, suppose that the definition of the countable fan-tightness holds for any decreasing family of sets in a space X, and let $\{A_n : n \in \omega\}$ be a (not necessarily decreasing) family of subsets of X such that $x \in \cap_{n\in\omega}\overline{A_n}$ for some $x \in X$. Put $A'_n = \cup_{k\geq n} A_k$ and select finite sets $K'_n \subset A'_n$ in such a way that $x \in \overline{\cup_{n\in\omega} K'_n}$. Let $K_n = \cup_{m\leq n} K'_m \cap A_n$. Clearly $\cup_{n\in\omega} K_n = \cup_{n\in\omega} K'_n$ and therefore $x \in \overline{\cup_{n\in\omega} K_n}$.

Lemma 1. *Let B and $\{A_n : n \in \omega\}$ be subsets of a space X satisfying the following conditions:*

(i) $\overline{A_n} \cap B \neq \emptyset$ *for every $n \in \omega$;*
(ii) *every sequence $\{x_n : n \in \omega\}$ such that $x_n \in A_n$ has a limit point in X;*
(iii) *for every neighbourhood U of B in X there is a neighbourhood V of B in X such that $\overline{V} \subset U$;*
(iv) *if $C \subset X$ and $\overline{C} \cap B \neq \emptyset$ then there is a countable subset $M \subset C$ such that $\overline{M} \cap B \neq \emptyset$.*

Then there exist finite sets $K_n \subset A_n$ such that $\overline{\cup_{n\in\omega} K_n} \cap B \neq \emptyset$.

PROOF: We shall call a point $x \in X$ *special*, if there is a sequence $\xi = \{x_n : n \in \omega\}$ such that $x_n \in A_n$ and x is a limit point of ξ. Let S be the set of all special points of X. Let us show that $\overline{S} \cap B \neq \emptyset$. Assume the contrary. Then the set $U = X \setminus \overline{S}$ is a neighbourhood of B. By (iii), there is a neighbourhood V of B in X such that $\overline{V} \subset U$. By the condition (i), $A_n \cap V \neq \emptyset$. Therefore we can choose $x_n \in A_n \cap V$ for each $n \in \omega$. By (ii), there is a limit point y of the sequence $\xi = \{x_n : n \in \omega\}$ in X. Clearly, y is a special point and hence $y \in S$. But $y \in \overline{V} \subset U = X \setminus \overline{S}$ and thus we reach a contradiction. Since $\overline{S} \cap B \neq \emptyset$, using (iv) there exists a countable set $M \subset S$ such that $\overline{M} \cap B \neq \emptyset$. Put $M = \{y_n : n \in \omega\}$ and for every n select a sequence $\xi^n = \{z_i^n : i \in \omega\}$ in accordance with the definition of special point. To finish, define $K_n = \{z_n^1, z_n^2, \dots, z_n^n\}$. □

After this lemma the next assertions are easily proved.

Proposition 1. *Let X be a space regular at all points of a subspace Y, which is countably compact in X. Further, let B be a compact subset of Y and $\{A_n :$*

$n \in \omega\}$ a countable family of subsets of Y such that the following conditions are satisfied:

(1) *$\overline{A_n} \cap B \neq \emptyset$ for each $n \in \omega$;*

(2) *If $C \subset X$ and $\overline{C} \cap B \neq \emptyset$, then there is a countable set $M \subset C$ such that $\overline{M} \cap B \neq \emptyset$.*

Then there are finite subsets $K_n \subset A_n$ such that $\overline{\cup_{n\in\omega} K_n} \cap B \neq \emptyset$.

PROOF: In the notations of Lemma 1, condition (1) is condition (i) and condition (2) is condition (iv). Condition (ii) is satisfied automatically, since Y is countably compact in X and condition (iii) is also satisfied, since B is compact and X is regular at each point of Y. Therefore, it is enough to apply the lemma. □

Theorem 1. *Let X be a space and $Y \subset X$. If Y is countably compact in X, X is regular at each point of Y and $t(y, X) \leq \omega$ for each $y \in Y$, then Y has countable fan-tightness.*

PROOF: To prove this we only need to apply Proposition 1, taking B to be the one point set $\{y\}$, where $y \in Y$ is the point at which we are verifying that the fan-tightness is countable. □

Corollary 1. *Let X be a Tychonoff space. If $at(X) \leq \omega$ (that is, $t(x, \beta X) \leq \omega$ for each $x \in X$ [4]), then X has countable fan-tightness.*

Corollary 2. *A countably compact regular space has countable tightness if and only if it has countable fan-tightness.*

The particular case of Corollary 2 concerning compact Hausdorff spaces was announced without proof in [3].

Remark 2. S_ω cannot be embedded into a countably compact regular space of countable tightness since such a space has countable fan-tightness, by Corollary 2.

It is natural to ask whether every regular space of countable fan-tightness can be embedded into a countably compact regular space of countable tightness. We will show that this in general cannot be done even for a countable space with only one non-isolated point.

Proposition 2. *If $p \in \omega^* = \beta(\omega) \setminus \omega$ then the single ultrafilter space $\omega \cup \{p\}$ (with the topology induced from $\beta(\omega)$) has countable fan-tightness if and only if p is a P-point in ω^*.*

PROOF: Let $\{A_n : n \in \omega\}$ be a family of subsets of ω such that $p \in \cap_{n\in\omega} \overline{A_n}$. As in Remark 1, we may assume that this family is decreasing. If p is a P-point in ω^*, then there exists a set $B \in p$ such that $B \setminus A_n$ is finite for each n. Put $K_n = (A_n \setminus A_{n+1}) \cap B$. Each K_n is finite and $\cup_{n\in\omega} K_n = A_0 \cap B \in p$. Conversely, suppose that $\omega \cup \{p\}$ has countable fan-tightness and let $\{A_n : n \in \omega\}$ be a set of members of p. We may assume that the family $\{A_n : n \in \omega\}$ is decreasing and select finite sets $K_n \subset A_n$ witnessing that $\omega \cup \{p\}$ has countable fan-tightness at p. Letting $B = \cup_{n\in\omega} K_n$, we have $B \in p$ and $B \setminus A_n \subset K_0 \cup \cdots \cup K_{n-1}$. Thus p is a P-point in ω^*. □

If p is a P-point in ω^*, then $\omega \cup \{p\}$ provides an example of a countable non-first countable space having countable fan-tightness. Coming back to the question formulated before, observe that if p is a P-point of ω^*, then the space $X = \omega \cup \{p\}$ cannot be embedded into a countably compact regular space Z of countable tightness. Indeed, if X is a subspace of Z then p is not isolated in $\overline{X} \setminus \omega$, otherwise there would be a closed neighbourhood U of p in $\overline{X}$ such that $U \subset X$, contradicting the fact that X is not locally compact at p. Now, let $\{V_n : n \in \omega\}$ be a family of neighbourhoods of p in the space $\overline{X} \setminus \omega$. As Z is regular, for each n we can select a closed neighbourhood W_n of p in the space $\overline{X}$ such that $W_n \setminus \omega = \overline{W_n \cap \omega} \setminus \omega \subset V_n$. But p is a P-point in ω^* and therefore there exists $B \in p$ in such a way that $B \setminus W_n$ is finite for each n. The set $\overline{B} \setminus \omega$ is a neighbourhood of p in $\overline{X} \setminus \omega$ which is contained in every V_n, in evident contradiction with the fact that Z has countable tightness.

Furthermore, if p is not a P-point then, by Proposition 2, the fan-tightness of X is not countable. Therefore, by Corollary 2, X cannot be embedded into a regular countably compact space of countable tightness. So we have obtained a new and more instructive proof of the following result of V.I. Malykhin (see [8]):

Theorem 2. *A single ultrafilter space $\omega \cup \{p\}$ can never be embedded into a countably compact regular space of countable tightness.*

We do not know whether some regular space having not countable fan-tightness may be embedded into a pseudo-compact regular space of countable tightness. However, in certain cases this cannot happen.

Theorem 3. *Let X be a regular space with finitely many non isolated points. If X can be embedded as a dense subspace into a pseudo-compact regular space of countable tightness Y, then the fan-tightness of X is countable.*

Proof: Fix a family $\{A_n : n \in \omega\}$ of subsets of X and a point $x \in X$ such that $x \in \bigcap_{n\in\omega} \overline{A_n}$, and let $B = \{x\}$. Then all the conditions of Lemma 1 are satisfied. In particular, (ii) holds because every infinite subset of A_n, having almost all its points isolated in X and hence also in Y, must possess a limit point in Y. Therefore, applying the lemma, we are done. □

As a consequence of the above theorem, we get that S_ω cannot be embedded as a dense subspace into a pseudo-compact regular space of countable tightness. Furthermore, Malykhin's Theorem can be strengthened as follows:

Theorem 4. *No single ultrafilter space $X = \omega \cup \{p\}$ can be embedded as a dense subspace into a pseudo-compact regular space of countable tightness.*

Proof: If p is not a P-point then the result follows from Proposition 2 and Theorem 3. If p is a P-point then it is enough to apply the argument used before stating Theorem 2 (taking clearly into account that pseudo-compactness is preserved by passing to closures of open sets). □

The above theorem is no longer true for Hausdorff pseudo-compact spaces. Indeed every single ultrafilter space $X = \omega \cup \{p\}$ can be embedded as a dense subspace into a Hausdorff pseudo-compact space of countable tightness. To see this, apply the Isbell-Mrowka construction taking a maximal family of almost disjoint subsets of ω which are closed in X.

Theorem 5. *If X is a Tychonoff space such that for each Tychonoff space Y of countable tightness the tightness of $X \times Y$ is countable then the fan-tightness of X is countable.*

Proof: Let $b(X)$ be a Hausdorff compactification of X. Take Z to be the ω-closure of X in $b(X)$, that is $Z = \cup\{\overline{A} : A \subset X \text{ and } |A| \leq \omega\}$. Then the tightness of Z at all points of X is countable, by the results in [4, Theorem 3.6]. It is also clear that for each sequence of points in X there is a limit point in Z. Now let $\{A_n : n \in \omega\}$ be a countable family of subsets of X and x a point in X such that $x \in \overline{A_n}$ for each $n \in \omega$. Letting $B = \{x\}$, we can apply Lemma 1 to the space Z in the role of X in its statement. This concludes the argument. □

Remark 3. Theorem 5 cannot be reversed, even if we restrict ourselves to the class of countably compact or σ-compact spaces or topological groups. Indeed, under CH there is a countably compact regular space of countable tightness, and, therefore, of countable fan-tightness by Corollary 2, the square of which has uncountable tightness (see [9]). A much better result is obtained in [20, Theorem 5], where there is a construction in ZFC of two Fréchet-Urysohn, and hence, strongly Fréchet-Urysohn by the results in [12], σ-compact topological groups whose product has uncountable tightness. Besides, V.V. Uspenskii in [21] has constructed a countable topological group G of the countable fan-tightness such that the tightness of the product G with the Frechét-Urysohn space $S_{\mathfrak{c}}$ (defined below) is uncountable.

Recall that a space X is Fréchet-Urysohn provided that for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence in A converging to x. X is said to be strongly Fréchet-Urysohn (or countably bisequential) if for every decreasing family $\{A_n : n \in \omega\}$ of subsets of X such that there is a point $x \in \cap_{n\in\omega}\overline{A_n}$, then it is possible to select $x_n \in A_n$ in such a way that the sequence $\{x_n : n \in \omega\}$ converges to x.

A typical example of a Fréchet-Urysohn, non-strongly Fréchet-Urysohn space is S_ω.

It was shown in [14], that each regular Fréchet-Urysohn space which is not strongly Fréchet-Urysohn, contains a topological copy of S_ω. Therefore, the next assertion holds:

Proposition 3 ([14, 16b]). *A regular Fréchet-Urysohn space is strongly Fréchet-Urysohn if and only if it has countable fan-tightness.*

Let $S_{\mathfrak{c}}$ be the space obtained by identifying the limit points of continuum many convergent sequences. In [18, Corollary 1.3] it is shown that a regular Fréchet-Urysohn space X is strongly Fréchet-Urysohn, provided that $X \times S_{\mathfrak{c}}$ has countable tightness. On the other hand, we have:

Proposition 4. *If X is a regular countably compact space of countable tightness, then the tightness of $X \times S_{\mathfrak{c}}$ is countable.*

Proof: The natural projection of $X \times S_{\mathfrak{c}}$ onto $S_{\mathfrak{c}}$ is a closed continuous mapping, since X is countably compact and $S_{\mathfrak{c}}$ is Fréchet-Urysohn. It remains to refer to a theorem in [4], which tells us that if all fibers have countable tightness, the mapping is closed and continuous, the domain is a regular space, and the range space is countably tight, then the tightness of the domain space is countable. □

In view of the result discussed above, it is reasonable to ask:

Question 1. *Let X be a regular space. Is it true that X has countable fan-tightness whenever $X \times S_{\mathfrak{c}}$ has countable tightness?*

Clearly a positive answer to this question would be a strengthening of Theorem 5.

A particular case of the above question is:

Question 2. *Let X be a Tychonoff space such that $C_p(X) \times S_{\mathfrak{c}}$ is a space of countable tightness. Is then true that X is a Hurewicz space?*

In [18, Proposition 1.1] it is proved that if X is a regular space for which $X \times S_{\mathfrak{c}}$ is a k-space, then the closure of every countable subset of X is locally countably compact. This, together with Corollary 2, leads to the next result:

Theorem 6. *Let X be a regular space of countable tightness. If $X \times S_{\mathfrak{c}}$ is a k-space, then X has countable fan-tightness.*

Theorem 7. *A hereditarily normal sequential space is strongly Fréchet-Urysohn if and only if it has countable fan-tightness.*

Proof: In [7] Kannan proved that a hereditarily normal sequential space is Fréchet-Urysohn if and only if it does not contain any copy of S_2. It remains to refer to Proposition 3. □

Observe that in Kannan's Theorem the space under consideration need not be strongly Fréchet-Urysohn. For instance, S_ω is a hereditarily normal Fréchet-Urysohn space which is not strongly Fréchet-Urysohn.

Corollary 3. *A regular sequential space with a countable network* (*in particular a countable regular sequential space*) *is strongly Fréchet-Urysohn if and only if it has countable fan-tightness.*

Recall, that a space X is *symmetrizable*, if there exists a non-negative real-valued function d on $X \times X$ satisfying the following conditions:

$d(x,y) = 0$ if and only if $x = y$;
$d(x,y) = d(y,x)$ for every $x, y \in X$;
A set $A \subset X$ is closed if and only if $d(x,A) = \inf\{d(x,y) : y \in A\} > 0$ for every $x \in X \setminus A$.

Question 3. *Is every regular symmetrizable space with countable fan-tightness first countable?*

We have the following partial answer:

Theorem 8. *If X is a regular symmetrizable space with countable fan-tightness, then X is first countable provided that one of the condition below holds:*

(a) *CH;*
(b) *X is metalindelöf;*
(c) *X is $\aleph_0$-monolithic;*
(d) *All points in X are G_δ's.*

Proof: We first prove that condition (d) is sufficient, and then we reduce all other conditions to condition (d). In [6, Lemma 6.11] it is shown that a regular symmetrizable space whose points are G_δ's, is either first countable or contains a closed copy of S_2. To complete the proof in case of (d), it is enough to refer to Example 2.

Since every regular symmetrizable Fréchet-Urysohn space is first countable [2], and X has countable tightness, it is enough to check that under any of the given conditions (a), (b), (c), every closed separable subspace Y of X is first countable. Indeed, we are reduced to verify that every point of Y is of type G_δ in Y. In case (a), taking into account the regularity of the space, we see that every point of Y has a local base of cardinality at most $\aleph_1$ and consequently, by [16, Proposition 3], every point of Y is of type G_δ. In case (b), we have that Y is Lindelöf and hence, hereditarily Lindelöf, by [11, Theorem 2]. In case (c), by definition Y has a countable network and, therefore, it is hereditarily Lindelöf. In both cases we see that every point of Y is of type G_δ. □

Observe, that it remains unknown whether all points of a regular symmetrizable space must necessarily be of type G_δ.

The fan-tightness plays also a role in the structure of closed mappings and of the images under such mappings.

Recall that a Lashnev space is a closed image of a metrizable space. A Lashnev space is always Fréchet-Urysohn, and it is well known (see [10]) that a Lashnev space is metrizable if and only if it is strongly Fréchet-Urysohn. This assertion can be reformulated as follows:

Proposition 5. *A Lashnev space is metrizable if and only if it has countable fan-tightness.*

In [17, Theorem 1.4] it is shown that a subspace of a countable product of Lashnev spaces is metrizable, provided it is strongly Fréchet-Urysohn. This result naturally suggests the following:

Question 4. *Is a subspace of a product of countably many Lashnev spaces metrizable, provided that it has countable fan-tightness?*

The next theorem should be compared with an analogous one due to E. Michael.

Given a space X and a set $A \subset X$, we say that A is pseudo-compact in X if every locally finite family of open subsets of X intersecting A is finite.

Theorem 9. *Let $f : X \to Y$ be a closed mapping. If Y has countable fan-tightness, then the boundary of every fiber of f is pseudo-compact in X.*

Proof: Let us proceed by contradiction. Select $y \in Y$ and assume that $f^{-1}(y) \setminus int(f^{-1}(y))$ is not pseudo-compact in X. Thus there exists an infinite locally finite family $\{U_n : n \in \omega\}$ of open subsets of X such that $U_n \cap (f^{-1}(y) \setminus int(f^{-1}(y))) \neq \emptyset$. For every $n \in \omega$ we have $y \in \overline{f(U_n) \setminus \{y\}}$. Every finite set $K_n \subset f(U_n) \setminus \{y\}$ is of the form $f(H_n)$ for some finite set $H_n \subset U_n$. Now the local finiteness of $\{U_n : n \in \omega\}$ implies that $\cup_{n\in\omega} H_n$ is closed in X and the closedness of f implies that $\cup_{n\in\omega} K_n$ is a closed set missing y. This contradicts the countable fan-tightness of Y, and the proof is complete. □

It is easy to realize that in the above theorem countable fan-tightness cannot be replaced by countable tightness. For instance, consider the canonical closed mapping used in the definition of S_ω.

It is also interesting to compare Theorem 9 with Theorem 1.4b in [5], this result says that if the tightness of Y^2 is countable, then every discrete family of open subsets of X intersecting the boundary of a fiber of f must have cardinality smaller than the continuum.

Since every pseudo-compact closed subspace of a normal space is countably compact, we get:

Corollary 4. *If X is a normal space, Y a space with countable fan-tightness and $f : X \to Y$ is a closed mapping, then the boundary of every fiber of f is countably compact.*

Corollary 5. *If $f : X \to Y$ is a closed mapping of a Diedonne complete* (*in particular, of a paracompact*) *space onto a space with countable fan-tightness, then f is compact covering, and even inductively perfect* [2].

Let us have a look at the behaviour of fan-tightness under mappings. This will allow us to obtain results on the fan-tightness of products.

Theorem 10. *Let f be a closed mapping of a regular space X onto a space Y of countable fan-tightness, and let us also assume that the fan-tightness of each fiber of f is countable. Then the fan-tightness of X is also countable.*

Proof: Let $x \in X$ and let $\{A_n : n \in \omega\}$ be a countable family of subsets of X such that $x \in \cap_{n\in\omega} \overline{A_n}$. Put $y = f(x)$ and $P = f^{-1}(y)$.

Case 1. There is an infinite subset M of ω such that $x \in \cap_{n\in M} \overline{A_n \cap P}$. Since the fan-tightness of the space P is countable, it is clear how to conclude the argument.

Case 2. There is an infinite subset $L \subset \omega$ such that $x \notin \overline{A_n \cap P}$ for each $n \in L$. In this case we replace A_n by $A_n \setminus P$ for each $n \in L$. We can also treat L as ω. Thus, for the remaining part of the proof we assume that $A_n \cap P = \emptyset$ for each $n \in \omega$.

Put $B_n = f(A_n)$. Then, by continuity of f, $y = f(x) \in \cap\{\overline{B_n} : n \in \omega\}$. A point z of the set P will be called nice if we can choose finite subsets $K_n \subset A_n$ in such a way that $z \in \overline{\cup_{n\in\omega} K_n}$. Our first step will be to show that x is in the closure of the set N of all nice points of P. Take any neighbourhood U of x in X. Then $x \in \overline{U \cap A_n}$. Put $C_n = f(U \cap A_n)$. Clearly, $y \in \overline{C_n}$. It follows that there are finite sets $H_n \subset C_n$ such that $y \in \overline{\cup_{n\in\omega} H_n}$. We can fix a finite set $K_n \subset U \cap A_n$ such that $H_n = f(K_n)$. Let $S = \cup_{n\in\omega} K_n$. Obviously $f(S)$ contains y in its closure. Therefore, $\overline{S} \cap P \neq \emptyset$, by closedness of f. It is clear that all points of the set $\overline{S} \cap P$ are nice and belong to the closure of U. Since X is regular it follows that x is in the closure of N. Now we shall need the fact that the set N is ω-closed. To see this, let $E = \{y_i : i \in \omega\}$ be a countable subset of N. For each $i \in \omega$ and each $n \in \omega$ we fix finite subsets $K_{n,i} \subset A_n$ such that $y_i \in \overline{\cup_{n\in\omega} K_{n,i}}$. Put $K_1 = K_{1,1}$, $K_2 = K_{2,1} \cup K_{2,2}, \dots, K_n = \cup_{i\le n} K_{n,i}$, and so on. Then $K_n \subset A_n$, K_n is finite, and $y_i \in \overline{\cup_{n\in\omega} K_n}$ for each $i \in \omega$. It follows that the closure of the set E consists only of nice points and consequently N is ω-closed.

Now we are ready to complete the argument. Since $x \in \overline{N} \subset B$ and the tightness of the space P is countable, it follows that x is actually an element of N. The fact that x is a nice point concludes the proof. □

Corollary 6. *If X is a compact space of countable tightness and Y is a regular space of countable fan-tightness then the fan-tightness of $X \times Y$ is countable.*

The above corollary is similar to the well known result of Malykhin concerning the tightness.

Notice that in Corollary 6 one cannot replace compactness by countable compactness (at least consistently). However we have:

Corollary 7. *If X is a countably compact regular space of countable tightness, and Y is a sequential space of countable fan-tightness, then the product $X \times Y$ has countable fan-tightness.*

PROOF: This follows from Corollary 2 and the well known fact that the natural projection of $X \times Y$ onto Y is a closed mapping. □

Remark 4. In view of Corollary 7, it would be very interesting to know whether it is consistent with ZFC that every countably compact regular space of countable tightness is sequential. If this were the case, then consistently the square of every countably compact regular space of countable tightness would have been a space of countable tightness.

A mapping is said to be *nowhere constant* provided that all its fibers have an empty interior. This notion leads to another consequence of Theorem 10, applied in a combination with Corollaries 2 and 4.

Corollary 8. *Let X be a normal space of countable tightness. If there exists a nowhere constant closed mapping from X into some space of countable fan-tightness then X has countable fan-tightness.*

Theorem 11. *Let X be a space, and let X^ω_b be the set X^ω, equipped with the box topology. If X^ω_b is countably tight, then X is of countable fan-tightness.*

Proof: Let $\{A_n : n \in \omega\}$ be a family of subsets of X and $x \in \cap_{n\in\omega}\overline{A_n}$. If $A \subset X^\omega$ is the set defined by $A = \prod_{n\in\omega} A_n$ and $p \in X^\omega$ is the point whose coordinates are all equal to x then obviously $p \in \overline{A}$ in the space X^ω_b. By the hypothesis, there exists a set $M = \{z^n : n \in \omega\} \subset A$ such that $p \in \overline{M}$. Now let us denote by z^n_i the i-th coordinate of the point z^n. Clearly $z^n_i \in A_i$ for each $i \in \omega$. Putting $K_i = \{z^1_i, z^2_i, \dots, z^i_i\}$, it is not difficult to check that $x \in \overline{\cup_{i\in\omega} K_i}$. □

A similar result cannot be proved for the usual product topology as next example shows.

Example 3. Recall that a space X is a Hurewicz space, if for any countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X it is possible to select finitely many open sets $\mathcal{V}_n \subset \mathcal{U}_n$ in such a way that the family $\cup_{n\in\omega}\mathcal{V}_n$ covers X. Now let X be any Lindelöf space which is not a Hurewicz space and satisfies the condition: X^n is Lindelöf for each $n \in \omega$. For instance, we can take X to be the space of all irrational numbers with the usual topology (see [3]). Then the space $C_p(X)$ of all real valued functions on X in the topology of pointwise convergence as well as $C_p(X)^\omega$ have countable tightness (in case of irrational numbers these two spaces have actually countable networks). On the other hand, the fan-tightness of $C_p(X)$ is not countable, since otherwise the space X would have been a Hurewicz space (see [3]).

In [3] it was shown, that the countable fan-tightness is preserved by almost open mappings, in particular by open mappings.

Now we shall prove some other results in this direction.

Recall (see [10]) that a mapping $f : X \to Y$ is said to be *biquotient*, if whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, cover a neighbourhood of y in Y.

Similarly, we have the following concept which was studied by Siwiec and Mancuso in [15] and by Siwiec in [13]. A mapping $f : X \to Y$ is *countably biquotient* if, whenever $y \in Y$ and $\{U_n : n \in \omega\}$ is an increasing countable cover of $f^{-1}(y)$ by open subsets of X, then y is in the interior of $f(U_n)$ for some n. For example, all open mappings are biquotient, and all perfect mappings are biquotient. Moreover every pseudo-open mapping with compact fibers is also biquotient (see [13], [15]).

Theorem 12. *If X is a space of countable fan-tightness, and $f : X \to Y$ is a countably biquotient onto mapping, then Y has also countable fan-tightness.*

Proof: Let $\{B_n : n \in \omega\}$ be a family of subsets of Y and let $y \in Y$ be a point satisfying $y \in \cap_{n\in\omega}\overline{B_n}$. Following Remark 1, we may assume that the family $\{B_n : n \in \omega\}$ is decreasing. Since f is countably biquotient, there is a point $x \in f^{-1}(y)$ which is a limit point of $A_n = f^{-1}(B_n)$ for each n. Indeed, otherwise, the family $\{X \setminus \overline{f^{-1}(B_n)} : n \in \omega\}$ would be an increasing open cover of $f^{-1}(y)$ and therefore there would exist n such that $f(X \setminus \overline{f^{-1}(B_n)}) \subset Y \setminus B_n$ is a neighbourhood of y in Y—a contradiction. Since X has countable fan-tightness, there are finite subsets H_n of A_n such that $x \in \overline{\cup_{n\in\omega} H_n}$. Clearly, the finite sets $K_n = f(H_n)$ have the property we are looking for. □

Corollary 9. *Let f be a pseudo-open mapping of a space X onto a space Y such that all fibers are countably compact. If X has countable fan-tightness, then also Y has countable fan-tightness.*

Proof: It was shown in [15] that every pseudo-open mapping with countably compact fibers is countably biquotient. It remains to apply Theorem 12. □

In particular, we see that perfect mappings preserve countable fan-tightness.

It is well known that tightness is preserved by quotient mappings. The same is no longer true for countable fan-tightness, even if we assume that the mapping under consideration is closed or quotient with compact fibers. Concerning the former assertion, it is enough to remember that S_ω is a closed image of a countable metrizable space. For the latter assertion we have:

Example 4. Let R be the real line with the usual topology, and let X be the quotient space of R obtained by identifying the pairs of the form $\{\frac{1}{n}, n\}$ to a single point, for each $n \in \omega \setminus \{0\}$. The natural projection from R onto X is a quotient mapping with finite fibers, but the space X is not of countable fan-tightness. To check the last assertion, let $A_n =]n, \infty[\setminus\omega$. It is clear that $0 \in \cap_{n\in\omega}\overline{A_n}$, but $0 \notin \overline{\cup_{n\in\omega} K_n}$, for any choice of finite sets $K_n \subset A_n$.

We close the paper by calling the reader's attention to another question left open.

Question 5. *Is it true that the fan-tightness of every pseudo-compact Tychonoff space of countable tightness is countable?*

At the first glance, it seems that the last question must have a negative answer.

Finally, we introduce the general concept of fan-tightness. We shall define *the fan-tightness of a space X*, denoted by $vet(X)$, as the smallest cardinal number τ such that whenever $\{A_\alpha : \alpha \in \tau\}$ is a centered family of subsets of X and $x \in \cap_{\alpha\in\tau}\overline{A_\alpha}$, there exist finite sets $K_\alpha \subset A_\alpha$ such that $x \in \overline{\cup_{\alpha\in\tau} K_\alpha}$. Observe, that for any space X, if the cardinal number τ is large enough, the above condition is satisfied. Therefore, the invariant $vet(X)$ is well defined.

Notice also that, by Remark 1, this definition is equivalent in the countable case with the definition of countable fan-tightness, given at the beginning.

Furthermore, using the same argument as in the proof of Theorem 12, and taking into account a characterization of biquotient mappings, similar to the characterization of countable biquotient mappings we referred to before, we establish the following result:

Theorem 13. *If a space Y is an image of a space X under a biquotient mapping, then $vet(Y) \leq vet(X)$.*

In particular, perfect mappings, open mappings, and more generally pseudo-open compact mappings, do not increase fan-tightness.

Addendum. Question 1 has been answered positively by the second named author and Jan van Mill. The proof will appear elsewhere.

References

[1] Arens R., *Note on convergence in topology*, Math. Mag. **23** (1950), 229–234.

[2] Arhangel'skii A.V., *Mappings and spaces*, Russian Math. Surveys **21** (1966), 115–162.

[3] Arhangel'skii A.V., *Hurewicz spaces, analytic sets and fan-tightness of spaces of functions*, Soviet Math. Dokl. **33 2** (1986), 396–399.

[4] Arhangel'skii A.V., *The frequency spectrum of a topological space and the product operation*, Trans. Moscow Math. Soc. **40** (1981), 163–200.

[5] Gruenhage G., Tanaka Y., *Products of k-spaces and spaces of countable tightness*, Trans. Amer. Math. Soc. **273 1** (1982), 299–308.

[6] Harley P.W. III, Stephenson R.M., Jr., *Symmetrizable and related spaces*, Trans. Amer. Math. Soc. **219** (1976), 89–111.

[7] Kannan V., *Every compact T_5 sequential space is Fréchet*, Fund. Math. **107** (1980), 85–90.

[8] Malykhin V.I., *On countable spaces having no bicompactification of countable tightness*, Soviet Math. Dokl. **13 5** (1972), 1407–1411.

[9] Malykhin V.I., *An example of a topological group*, in Topological spaces and their mappings, Riga. Latv. Gosud. Univ. (in Russian), 1981, pp. 120–123.

[10] Michael E.A., *A quintuple quotient quest*, Gen. Top. Appl. **2** (1972), 91–138.

[11] Nedev S., *Symmetrizable spaces and final compactness*, Soviet Math. Dokl. **8** (1967), 890–892.

[12] Nyikos P.J., *Metrizability and the Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83** (1981), 793–801.

[13] Siwiec F., *Sequence-covering and countably biquotient mappings*, Gen. Top. Appl. **1** (1971), 143–154.

[14] Siwiec F., *Generalizations of the first axiom of countability*, Rocky Mount. J. Math. **5** (1975), 1–60.

[15] Siwiec F., Mancuso V.J., *Relations among certain mappings and conditions for their equivalence*, Gen. Top. Appl. **1** (1971), 34–41.

[16] Stephenson M., Jr, *Symmetrizable, $\mathcal{F}$-, and weakly first countable spaces*, Can. J. Math. **XXIX** (1977), 480–488.

[17] Tamano K., *Closed images of metric spaces and metrization*, Topology Proceedings **10** (1985), 177–186.

[18] Tanaka Y., *Product of spaces of countable tightness*, Topology Proceedings **6** (1981), 115–133.

[19] Tanaka Y., *Metrizability of certain quotient spaces*, Fund. Math. **119** (1983), 157–168.

[20] Todorcevic S., *Some applications of S and L combinatorics*, Annals New York Academy of Science, 1991, pp. 130–167.

[21] Uspenskii V.V., *Frequency spectrum of functional spaces*, Vestnik Mosk. Universiteta, Ser. Matematica **371** (1982), 31–35.

Moscow State University, Mech. Math. Faculty, Moscow, 119899, Russia

Department of Mathematics, University of Catania, Citta' universitaria, viale A. Doria 6, 95125 Catania, Italy

(*Received* September 1, 1995)