Pseudocompactness and the cozero part of a frame

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Abstract. A characterization of the cozero elements of a frame, without reference to the reals, is given and is used to obtain a characterization of pseudocompactness also independent of the reals. Applications are made to the congruence frame of a σ-frame and to Alexandroff spaces.

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A *frame* (respectively, a σ*-frame*) is a bounded lattice L with top e and bottom 0, which is complete (respectively, countably complete) and which satisfies $x \wedge \bigvee S = \bigvee\{x \wedge t \mid t \in S\}$ for $x \in L$ and any $S \subseteq L$ (respectively, any countable $S \subseteq L$). The categories **Frm** and σ **Frm** have as maps the homomorphisms which preserve the respective operations $\wedge$ and $\bigvee$. The lattice $\mathfrak{O}X$ of open sets of a topological space X is a frame and the lattice $\mathrm{Coz}\, X$ of cozero sets of X is a σ-frame. A cozero element of a frame L is defined in the natural way and Reynolds [17] shows that the set of all cozero elements of a frame L, denoted by $\mathrm{Coz}\, L$, is an Alexandroff algebra, and, in the case that L is completely regular, $\mathrm{Coz}\, L$ generates L by join. $\mathrm{Coz}\, L$ has been shown to play a role similar to that of its archetype $\mathrm{Coz}\, X$ for topological spaces, and this role has been quite extensively investigated in relation to compactness [4], [19], realcompactness [14], [9], [15], uniformity [19] and dimension theory [4]. We may regard Coz as a functor from **Frm** to σ **Frm** which has, as left adjoint, the functor $\mathfrak{H} : \sigma\,\mathbf{Frm} \to \mathbf{Frm}$, where $\mathfrak{H}L$ is the frame of all σ-ideals of L, that is, those ideals of L closed under countable joins. This adjoint pair restricts to an equivalence of the category **Reg** σ **Frm** of all regular σ-frames with the category of all regular Lindelöf frames.

The purpose of this paper is to characterize firstly the cozero elements of a frame and subsequently pseudocompactness of a frame, without reference to the reals. Although these characterizations have been known and used for some time, the proofs appear here for the first time.

We recall some basic notions and facts about frames and σ-frames. For further information see Johnstone [11] on frames and Madden [13] on σ-frames.

We write $a \prec b$ (a is *rather below* b or a is *well inside* b) if there is a *separating element* $s \in L$ such that $a \wedge s = 0$, $s \vee b = e$. If L has arbitrary joins then $a \prec b$ if and only if $b \vee a^* = e$, where a^* is the *pseudocomplement* of a. A frame (σ-frame) L is *regular* if each $a \in L$ is a join (countable join) of elements $c \prec a$. Recall, for any a and b in L, $a \prec\!\prec b$ (a is *completely below*, or *really inside*, b) means there exists an *interpolating sequence* $(c_{nk})_{n=0,1,2,\dots;k=0,1,\dots,2^n}$ between a and b, where $c_{00} = a$, $c_{01} = b$, $c_{nk} = c_{n+1\ 2k}$, $c_{nk} \prec c_{n\ k+1}$.

Note that any interpolating sequence (c_{nk}) between a and b determines a *scale* between a and b, that is, a family $(c_q \mid q \in \mathbb{Q} \cap [0,1])$ such that $c_0 = a$, $c_1 = b$ and $c_r \prec c_s$ whenever $r < s$: put

$$c_q = \bigvee\{c_{nk} \mid \tfrac{k}{2^n} \leq q\}.$$

Analogously, $a \prec\!\prec b$ implies the existence of a *decreasing scale* between a and b, that is, a family $(c_q \mid q \in \mathbb{Q} \cap [0,1])$ for which $c_1 = a$, $c_0 = b$ and $c_s \prec c_t$ whenever $t < s$.

A frame is *completely regular* if each $a \in L$ is a join of elements $z \prec\!\prec a$. In the case that L is *normal*, that is whenever $a \vee b = e$ in L there exist $c \prec a$, $d \prec b$ with $c \vee d = e$, $\prec$ interpolates. Regular σ-frames are normal [4] and are thus precisely the Alexandroff algebras of Reynolds; in this case $\prec\ =\ \prec\!\prec$. A frame L is *compact* if the top e is a *compact element*, that is whenever $e = \vee A$ for $A \subseteq L$ then $e = \vee B$ for some finite $B \subseteq A$. The *compact completely regular coreflection* $\mathfrak{K}L$ of any frame L is given as the subframe of the frame of all ideals of L consisting of those ideals which are completely regular, that is, those ideals J such that for each $x \in J$ there exists $a \in J$ with $x \prec\!\prec a$. The coreflection map $\kappa_L : \mathfrak{K}L \to L$ is given by join [5].

1. The cozero part of a frame

We first recall a few basic facts, following the presentation of Banaschewski-Mulvey [6]; for a slightly different, but equivalent treatment see Johnstone [11].

The *frame of reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by the ordered pairs (p,q) of rational numbers $p, q \in \mathbb{Q}$ subject to the relations

(i) $(p,q) \wedge (s,t) = (p \vee s, q \wedge t)$.
(ii) $(p,q) \vee (s,t) = (p,t)$ whenever $p \leq s < q \leq t$.
(iii) $(p,q) = \bigvee\{(s,t) \mid p < s < t < q\}$.
(iv) $e = \bigvee\{(p,q) \mid p, q \in \mathbb{Q}\}$.

It should be remarked that (iii) implies $(p,q) = 0$ whenever $p \geq q$, a condition which is often added redundantly to the above list.

We note that $\mathcal{L}(\mathbb{R})$ is completely regular: if $p < s < t < q$ then $(s,t) \prec (p,q)$ — consider $\bigvee\{(u,s) \vee (t,v) \mid u < s, t < v\}$ — and therefore also $(s,t) \prec\!\prec (p,q)$. The definition of $\mathcal{L}(\mathbb{R})$ immediately implies that, for any frame L, a map into L from the set of all pairs (p,q) determines a homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$ iff it transforms the above relations into identities in L.

Now, a *continuous real function* on a frame L is a homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$, and its *cozero element* $\mathrm{coz}(\varphi)$ is defined as

$$\mathrm{coz}(\varphi) = \bigvee\{\varphi(p,0) \vee \varphi(0,q) \mid p, q \in \mathbb{Q}\}.$$

Note that the justification for this terminology lies in the fact that, for any topological space X, one has a one-one onto map

$$\mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathfrak{O}X) \to \mathbf{Top}(X, \mathbb{R})$$

taking each $\varphi : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ to $\tilde{\varphi} : X \to \mathbb{R}$ where

$$p < \tilde{\varphi}(x) < q \qquad \text{iff} \qquad x \in \varphi(p,q).$$

Obtaining $\tilde{\varphi}$ from φ is an easy consequence of the result that any homomorphism $\xi : \mathcal{L}(\mathbb{R}) \to \mathbf{2}$ determines $\lambda \in \mathbb{R}$ such that $p < \lambda < q$ iff $\xi(p,q) = 1$: λ is given by the Dedekind cut (V, W) where

$$V = \{r \in \mathbb{Q} \mid \xi(r,q) = 1 \text{ for some } q \in \mathbb{Q}\},$$
$$W = \{s \in \mathbb{Q} \mid \xi(p,s) = 1 \text{ for some } p \in \mathbb{Q}\}$$

(Johnstone [11], p. 125). On the other hand, any continuous $f : X \to \mathbb{R}$ obviously defines $\varphi : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ by $\varphi(p,q) = \{x \in X \mid p < f(x) < q\}$ such that $\tilde{\varphi} = f$.

Proposition 1. *For any frame L, the following is equivalent for $a \in L$:*

(1) $a \in \operatorname{Coz} L$.
(2) $a = \bigvee x_n$ *where* $x_n \prec\!\prec a$*, for all* $n = 1, 2, \dots$.
(3) $a = \bigvee a_n$ *where* $a_n \prec\!\prec a_{n+1}$*, for all* $n = 1, 2, \dots$.

Proof: (1) ⇒ (2). Any element of $\mathcal{L}(\mathbb{R})$ is a join of countably many elements completely below it, and any $\varphi : \mathcal{L}(\mathbb{R}) \to L$ preserves that fact.

(2) ⇒ (3). Given $a = \bigvee x_n$ as stated, define a_n inductively by

$$a_0 = x_0, a_n \vee x_{n+1} \prec\!\prec a_{n+1} \prec\!\prec a,$$

using the fact that $\prec\!\prec$ interpolates and is stable under binary joins.

(3) ⇒ (1). For each n, let $(c_{nr} \mid r \in \mathbb{Q} \cap [0,1])$ be a descending scale between a_n and a_{n+1}, and define

$$c_r = \begin{cases} 0 & \text{for } \ r > 1 \\ c_{n(n+1)(r - \frac{1}{n+1})} & \text{for } \ \frac{1}{n} \geq r \geq \frac{1}{n+1} \\ e & \text{for } \ r \leq 0. \end{cases}$$

Note that, for all $r, s \in \mathbb{Q}$, $r < s$ implies $c_s \prec\!\prec c_r$, and

$$c_{\frac{1}{n}} = c_{n1} = a_n.$$

Now define

$$\varphi(p,q) = \bigvee \{c_{p'} \wedge c^*_{q'} \mid p < p' < q' < q\}.$$

To see that this indeed determines a homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$, it has to be checked that the relations (i)–(iv) are transformed into identities in L.

Re (i):

$$\varphi(p,q) \wedge \varphi(r,s) = \bigvee\{c_{p'} \wedge c^*_{q'} \wedge c_{r'} \wedge c^*_{s'} \,|\, p < p' < q' < q,\ r < r' < s' < s\},$$

and since $c_{p'} \wedge c_{r'} = c_{p' \vee r'}$ and $c^*_{q'} \wedge c^*_{s'} = c^*_{q' \wedge s'}$, this is $\varphi(p \vee r, q \wedge s)$, as desired.

Re (ii): Given $p \leq s < q \leq t$, it is obvious that

$$\varphi(p,q) \vee \varphi(s,t) \leq \varphi(p,t).$$

For the reverse inequality, note first that, for p' and t' such that $p < p' < t' < t$,

$$(*) \qquad c_{p'} \wedge c^*_{t'} \leq \varphi(p,q) \vee \varphi(s,t)$$

whenever $s < p'$ or $t' < q$. Hence it remains to consider the case $p' \leq s$, $q \leq t'$. For this, pick r and r' such that $s < r < r' < q$ and compute

$$\begin{aligned}
(c_{p'} \wedge c^*_{t'}) &\wedge ((c_{p'} \wedge c^*_{r'}) \vee (c_r \wedge c^*_{t'})) \\
&= (c_{p'} \wedge c^*_{t'} \wedge c^*_{r'}) \vee (c_{p'} \wedge c_r \wedge c^*_{t'}) \\
&= (c_{p'} \wedge c^*_{r'}) \vee (c_r \wedge c^*_{t'}) \\
&= (c_{p'} \vee c_r) \wedge (c_{p'} \vee c^*_{t'}) \wedge (c^*_{r'} \vee c_r) \wedge (c^*_{r'} \vee c^*_{t'}) \\
&= c_{p'} \wedge c^*_{t'}
\end{aligned}$$

the final step since $c_r \leq c_{p'}$ and $c^*_{r'} \leq c^*_{t'}$ because $p' \leq r$ and $r' \leq t'$, while $c_{t'} \prec c_{p'}$ and $c_{r'} \prec c_r$ because $p' < t'$ and $r < r'$. It follows that, in the present case, we again have the inequality $(*)$, and this proves the desired result.

Re (iii):

$$\bigvee\{\varphi(r,s) \,|\, p < r < s < q\} = \bigvee\{c_{r'} \wedge c^*_{s'} \,|\, p < r < r' < s' < s < q\} = \varphi(p,q).$$

Re (iv):

$$\bigvee\{\varphi(p,q) \,|\, p, q \in \mathbb{Q}\} = \bigvee\{c_p \wedge c^*_q \,|\, p < q\} = e$$

since $c_{-1} \wedge c^*_2 = e$. Now we have

$$\begin{aligned}
\mathrm{Coz}(\varphi) &= \bigvee\{(c_p \wedge c^*_r) \vee (c_s \wedge c^*_q) \,|\, p < r < 0,\ 0 < s < q\} \\
&= \bigvee\{c_s \wedge c^*_q \,|\, 0 < s < q\} \\
&= \bigvee\{c_s \,|\, 0 < s \leq 1\} \leq a,
\end{aligned}$$

the last step since each c_s, $0 < s \leq 1$, is below some a_n and hence below a; on the other hand, each a_n is a c_s, and therefore $\mathrm{Coz}(\varphi) = a$. □

Corollary 1. *Whenever $x \prec\prec y$ in L, there exists $z \in \operatorname{Coz} L$ such that $x \prec\prec z \prec\prec y$.*

PROOF: If (z_{nk}) is any interpolating sequence between x and y then $z = \bigvee\{z_{nk} \mid \frac{k}{2^n} < \frac{1}{2}\}$ is a cozero element by the proposition, and clearly $x \prec\prec z \prec\prec y$. □

Corollary 2. $\operatorname{Coz} L$ *is a regular sub-σ-frame of L.*

PROOF: $\operatorname{Coz} L$ is closed in L under (i) countable joins by the proposition, and (ii) binary meets since $\prec\prec$ is stable under binary meet. Concerning regularity, note first that, for any $x, y \in \operatorname{Coz} L$, $x \prec\prec y$ in L implies $x \prec y$ in $\operatorname{Coz} L$: take any $z \in L$ such that $x \prec\prec z \prec\prec y$ and then $t \in \operatorname{Coz} L$ such that $z^* \prec\prec t \prec\prec x^*$ by Corollary 1 and the fact that $z^* \prec\prec x^*$; it follows that $x \wedge t = 0$ and $y \vee t = e$. Now, for any $a \in \operatorname{Coz} L$, if $a = \bigvee x_n$ where $x_n \prec\prec a$ for $n = 1, 2, \ldots$, take $a_n \in \operatorname{Coz} L$ such that $x_n \prec\prec a_n \prec\prec a$. Then $a = \bigvee a_n$ and $a_n \prec a$ in $\operatorname{Coz} L$, showing regularity. □

Corollary 3. *For any $a, b \in L$, $a \prec\prec b$ iff $a \leq c$, $d \leq b$ where $c, d \in \operatorname{Coz} L$ such that $c \prec d$ in $\operatorname{Coz} L$.*

PROOF: ($\Rightarrow$) Use Corollary 1 twice to obtain $c, d \in \operatorname{Coz} L$ such that $a \prec\prec c \prec\prec d \prec\prec b$. Then, by the preceding proof, we also have $c \prec d$ in $\operatorname{Coz} L$.

($\Leftarrow$) Since $\operatorname{Coz} L$ is a regular σ-frame, its relation $\prec$ interpolates, and hence $c \prec d$ in $\operatorname{Coz} L$ implies $c \prec\prec d$ in L. □

For the following, recall that an element a in a frame L is called *Lindelöf* whenever $a = \bigvee S$ implies $a = \bigvee T$ for some countable $T \subseteq S$, and L is called Lindelöf whenever $e \in L$ is Lindelöf. Then we have:

Corollary 4. *In any Lindelöf completely regular frame L, $a \in L$ is cozero iff it is Lindelöf.*

PROOF: ($\Rightarrow$) Since $a = \bigvee x_n$ where $x_n \prec\prec a$ for all n, $a \leq \bigvee S$ implies $e = x_n^* \vee \bigvee S$ for each n, and hence $e = x_n^* \vee \bigvee T_n$ for some countable $T_n \subseteq S$ so that $x_n \leq \bigvee T_n$ and consequently $a \leq \bigvee T$ for the countable set $T = \bigcup T_n$.

($\Leftarrow$) Since $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$ by complete regularity, any Lindelöf a is a join of countably many $x \prec\prec a$, and hence cozero by the proposition. □

Our final result in this setting relates $\operatorname{Coz}(L)$ to $\operatorname{Coz}(\mathfrak{K}L)$ for the compact completely regular coreflection $\kappa_L : \mathfrak{K}L \to L$ of L.

Corollary 5. *For any $a \in L$, $a \in \operatorname{Coz} L$ iff $a = \kappa_L(J)$ for $J \in \operatorname{Coz}(\mathfrak{K}L)$.*

PROOF: Since homomorphisms clearly take a cozero element to a cozero element we only have to prove ($\Rightarrow$). Now, for $a \in \operatorname{Coz} L$, let $a = \bigvee a_n$ where $a_n \prec\prec a_{n+1}$ for all n by the proposition. Then, the ideal J generated by these a_n is completely regular so that $J \in \mathfrak{R}L$; further, being countably generated, it is Lindelöf and hence cozero by Corollary 4, while $a = \bigvee J = \kappa_L(J)$. □

Remark. Corollary 4 is contained in Madden-Vermeer [14], included here for the sake of completeness. We note, though, that our proof is different from theirs.

We close with some additional comments. It is obvious that $\operatorname{Coz} L$ depends functorially on L, providing a functor $\operatorname{Coz} : \mathbf{Frm} \to \mathbf{Reg}\,\sigma\,\mathbf{Frm}$. Moreover, this functor has as left adjoint the restriction of $\mathfrak{H} : \sigma\,\mathbf{Frm} \to \mathbf{Frm}$ with adjunction maps

$$\begin{array}{ccc} \mathfrak{H}\operatorname{Coz} L & \to & L \\ J & \rightsquigarrow & \bigvee J \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \to & \operatorname{Coz}\mathfrak{H}M. \\ a & \rightsquigarrow & \downarrow a \end{array}$$

In particular, the latter is an isomorphism, a result originally due to Reynolds [16] (see also Johnstone [11]). The following proof is substantially simpler than the earlier ones, due to the advantages provided by Proposition 1. Obviously, in $\mathfrak{H}M$, $I \prec J$ iff there exists $a \prec b$ in M such that $I \subseteq \downarrow a$, $\downarrow b \subseteq J$, and $a \prec b$. In particular, if $J \in \operatorname{Coz}(\mathfrak{H}M)$ and therefore $J = \bigvee J_n$ where $J_n \prec\!\prec J$ for each n, one has elements $a_n, b_n \in M$ such that

$$J_n \subseteq \downarrow a_n,\ \downarrow b_n \subseteq J,\ a_n \prec b_n$$

for each n. Now, for $a = \bigvee a_n$, $a \in J$ and $J_n \subseteq \downarrow a$ for all n, showing that $J = \downarrow a$. Conversely, the regularity of M says that, for any $a \in M$, $a = \bigvee a_n$ where $a_n \prec a$ for each n, and in $\mathfrak{H}M$ this means $\downarrow a = \bigvee \downarrow a_n$ and $\downarrow a_n \prec\!\prec \downarrow a$. Hence the adjunction map is an isomorphism.

2. Pseudocompactness

For any frame L, $\varphi : \mathcal{L}(\mathbb{R}) \to L$ is called *bounded* if $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if all $\varphi : \mathcal{L}(\mathbb{R}) \to L$ are bounded.

We observe that these notions are in accord with the corresponding classical ones for a topological space X: The one-one onto map

$$\mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathfrak{O}X) \to \mathbf{Top}(X, \mathbb{R})$$

mentioned earlier, taking φ to $\tilde{\varphi}$ such that

$$p < \tilde{\varphi}(x) < q \ \text{ iff } \ x \in \varphi(p,q),$$

shows that φ is bounded (some $\varphi(p,q) = X$) iff $\tilde{\varphi}$ is bounded (for some p, q, $p < \tilde{\varphi}(x) < q$ for all x), and hence *a space X is pseudocompact iff the frame $\mathfrak{O}X$ is pseudocompact.*

Our aim is to characterize pseudocompact frames in a variety of ways.

Proposition 2. *For any frame L, the following is equivalent:*

(1) *L is pseudocompact.*
(2) *Any sequence $a_0 \prec\!\prec a_1 \prec\!\prec a_2 \prec\!\prec \dots$ such that $\bigvee a_n = e$ in L terminates, that is, $a_k = e$ for some k.*
(3) *The σ-frame $\operatorname{Coz} L$ is compact.*
(4) *The frame $\mathfrak{H}\operatorname{Coz} L$ is compact.*

PROOF: (1) $\Rightarrow$ (2). Given $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \dots$ such that $\bigvee a_n = e$, let $(c_{nq} \mid q \in \mathbb{Q} \cap [0,1])$ be a scale between a_n and a_{n+1} for each n, put

$$c_r = \begin{cases} 0 & (r < 0) \\ c_{nr-n} & (n \le r \le n+1) \end{cases}$$

and define

$$\varphi(p,q) = \bigvee\{c_{p'}^* \wedge c_{q'} \mid p < p' < q' < q\}.$$

It is easy to verify, by essentially the same arguments as in the proof of Proposition 1, that the relations (i)–(iv) are transformed into identities in L. The resulting $\varphi : \mathcal{L}(\mathbb{R}) \to L$ being bounded by pseudocompactness, we have $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$. Now, for any $k \ge q$,

$$\begin{aligned} a_k &\ge \bigvee\{c_{q'} \mid p < q' < q\} \\ &\ge \bigvee\{c_{p'}^* \wedge c_{q'} \mid p < p' < q' < q\} = \varphi(p,q), \end{aligned}$$

and hence $a_k = e$, as desired.

(2) $\Rightarrow$ (3). If $\bigvee a_n = e$ in $\operatorname{Coz} L$, and $a_n = \vee a_{nk}$ where $a_{nk} \prec\prec a_{nk+1}$ by Proposition 1, put $c_n = a_{1n} \vee a_{2n} \vee \cdots \vee a_{nn}$. Then $c_n \prec\prec c_{n+1}$ and $\vee c_n = e$, hence $c_n = e$ for some n, and consequently also $a_1 \vee \cdots \vee a_n = e$, proving compactness.

(3) $\Rightarrow$ (4). By Proposition 3 of Banaschewski [2], $\mathfrak{H}M$ is compact for any compact σ-frame.

(4) $\Rightarrow$ (1). First note that any $\varphi : \mathcal{L}(\mathbb{R}) \to L$ lifts through $\mathfrak{H} \operatorname{Coz} L$, that is, there exists $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathfrak{H} \operatorname{Coz}(L)$ such that $\bigvee \overline{\varphi} = \varphi$. Since $\varphi : \mathcal{L}(\mathbb{R}) \to L$ actually maps into $\operatorname{Coz} L$ by the first part of the proof of Proposition 1, we may define $\overline{\varphi}(p,q) \in \mathfrak{H} \operatorname{Coz} L$, for any $p, q \in \mathbb{Q}$, by

$$\overline{\varphi}(p,q) = \downarrow \varphi(p,q).$$

To see that this defines a homomorphism $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathfrak{H} \operatorname{Coz} L$ we note that the defining relations (i)–(iv) of $\mathcal{L}(\mathbb{R})$ are transformed into identities in $\mathfrak{H} \operatorname{Coz} L$ because the map

$$\mathcal{L}(\mathbb{R}) \xrightarrow{\varphi} \operatorname{Coz} L \xrightarrow{\downarrow} \mathfrak{H} \operatorname{Coz} L$$

is a σ-frame homomorphism. Further, $\vee \overline{\varphi} = \varphi$ since $\vee \overline{\varphi}(p,q) = \vee \downarrow \varphi(p,q) = \varphi(p,q)$ and the (p,q) generate $\mathcal{L}(\mathbb{R})$.

In particular, if $\mathfrak{H} \operatorname{Coz} L$ is compact then $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathfrak{H} \operatorname{Coz} L$ is obviously bounded, and this makes $\varphi = \bigvee \overline{\varphi}$ bounded, showing L is pseudocompact. □

The above result was first presented, without proof, by the second author at the 1991 Prague Topology Symposium. Before that, it had been privately communicated to a number of colleagues who then used one or the other of the conditions (2) and (3) as a definition of pseudocompactness (Baboolal and Banaschewski [1], Walters [18], and subsequently Marcus [15]). The spatial version of (1) $\Leftrightarrow$ (3) can be found in Kennison [12].

Corollary 6. *A countably generated regular frame L is pseudocompact iff it is compact.*

Proof: Since L is countably generated and regular it is also regular as a σ-frame and hence $L = \mathrm{Coz}\, L$, by normality and Proposition 1. Thus L is compact as a σ-frame, and being countably generated this makes it compact as a frame. □

Remark 1. In a somewhat similar fashion, it is an easy consequence of Proposition 2 that a completely regular frame L is compact iff it is pseudocompact and Lindelöf. A slightly different formulation of this result appears in Walters [18] whose proof makes use of the fact, due to Madden and Vermeer [14], that the composite functor $\mathfrak{H}\,\mathrm{Coz}$ delivers the coreflection of frames to regular Lindelöf frames. Since the latter, as also noted by Madden and Vermeer [14], are exactly the closed quotients of copowers of $\mathcal{L}(\mathbb{R})$, this makes $\mathfrak{H}\,\mathrm{Coz}$ an analogue of Hewitt's realcompactification. As a consequence, Walters interprets her result as a frame version of the well-known characterization of the compact Hausdorff spaces as the spaces which are both pseudocompact and realcompact. On the other hand, Marcus [15] has recently proved the compactness of pseudocompact frames that are realcompact in the sense of Schlitt, a property strictly weaker than being Lindelöf. In fact, Marcus obtains Schlitt's realcompact coreflection υL by means of a Wallman-type construction and then uses this to show L is pseudocompact iff υL is compact.

Remark 2. The implication (4) ⇒ (1) could also be obtained from the coreflection property of $\mathfrak{H}\,\mathrm{Coz}$, provided one knows that $\mathcal{L}(\mathbb{R})$ is Lindelöf. This is easy enough to prove, but the obvious argument uses the Countable Axiom of Choice while our proof is choice-free, and in fact constructively valid.

Remark 3. Another result concerning the compactness of certain pseudocompact frames is that every *paracompact* pseudocompact frame is compact (Banaschewski-Pultr [7]). We note, in addition, that the fact quoted at the beginning of Remark 1 may be viewed as a consequence of this, given that every regular Lindelöf frame is paracompact. For the latter, we offer the following simple proof: For any cover S of such a frame L, $T = \{x \in L \,|\, x \prec \tilde{x} \prec a, a \in S\}$ is again a cover, and we may then take $c_n \in T$, $n = 1, 2, \dots$, such that $\vee c_n = e$. Now, put $b_1 = \tilde{c}_1$ and $b_n = \tilde{c}_n \wedge c_1^* \wedge \cdots \wedge c_{n-1}^*$ for all $n > 1$. Here, also $\vee b_n = e$ since

$$(\star) \qquad b_1 \vee b_2 \vee \cdots \vee b_n = \tilde{c}_1 \vee \tilde{c}_2 \vee \cdots \vee \tilde{c}_n,$$

which is proved by induction: $n = 1$ is given by definition, and if $(\star)$ for any n then

$$\begin{aligned} b_1 \vee b_2 \vee \cdots \vee b_{n+1} &= \tilde{c}_1 \vee \cdots \vee \tilde{c}_n \vee (\tilde{c}_{n+1} \wedge c_1^* \wedge \cdots \wedge c_n^*) \\ &= (\tilde{c}_1 \vee \cdots \vee \tilde{c}_n \vee \tilde{c}_{n+1}) \wedge (\tilde{c}_1 \vee \cdots \vee \tilde{c}_n \vee c_1^*) \wedge \dots (\tilde{c}_1 \vee \cdots \vee \tilde{c}_n \vee c_n^*) \\ &= \tilde{c}_1 \vee \tilde{c}_2 \vee \cdots \vee \tilde{c}_{n+1} \end{aligned}$$

since $c_i \prec \tilde{c}_i$ for all i. The cover $\{b_n \,|\, n = 1, 2, \dots\}$ is obviously a refinement of S, and as $c_k \wedge b_n = 0$ for all $n > k$ it is locally finite.

3. The congruence frame of a σ-frame

A congruence on a σ-frame L is an equivalence relation on L which is a sub-σ-frame of the product $L \times L$. The lattice CL of all congruences on a σ-frame L, partially ordered by inclusion, is a frame and is generated by the congruences of the form $\Delta_a = \{(x,y) \,|\, x \wedge a = y \wedge a\}$ and $\nabla_b = \{(x,y) \,|\, x \vee b = y \vee b\}$ for $a, b \in L$. Δ_a and ∇_b are the principal congruences generated by $(0,a)$ and (b,e) respectively and, since $\Delta_a \cap \nabla_a = \{(x,x) \,|\, x \in L\} = \Delta$ and $\nabla_a \vee \Delta_a = L \times L = \nabla$, they are complemented elements of CL. Since CL is zero-dimensional it is, in particular, completely regular. The map $\nu_L : L \to CL$ taking a to ∇_a, is a σ-frame embedding and, if $C_\sigma L$ denotes the σ-frame generated in CL by all the congruences Δ_a, ∇_b, $a, b \in L$, then the codomain restriction of ν_L to $C_\sigma L$ is the universal σ-frame homomorphism from L with the property that each element in the image is complemented. The reader is referred to the paper of Madden [13] where these, and other, aspects of CL are presented. The following proposition may also be extracted from [13] but we include it, and a simple proof, for completeness.

Proposition 3. *The congruence frame CL of a σ-frame is a Lindelöf frame.*

PROOF: A family $\{\Theta_\alpha\}_{\alpha \in I}$ is said to be σ-directed if for every countable $J \subset I$, there is a $\beta \in I$ such that $\Theta_\alpha \subseteq \Theta_\beta$ for all $\alpha \in J$. It is a straightforward exercise to check that CL is closed under unions of σ-directed families; consequently, if $\nabla = \bigvee\limits_{\alpha \in I} \Theta_\alpha$, then

$$(0,e) \in \bigvee_{\alpha \in I} \Theta_\alpha = \cup\{\textstyle\bigvee \Theta_\beta \,|\, \beta \in J \subset I,\ J \text{ countable}\}.$$

Thus $(0,e) \in \bigvee\limits_{\beta \in J} \Theta_\beta$ for some countable $J \subset I$, and $\nabla = \bigvee\limits_{\beta \in J} \Theta_\beta$ as required. ☐

If we denote the set of Lindelöf elements of CL by $\sigma(CL)$, we have:

Corollary 7. *For every σ-frame L*

$$\operatorname{Coz} CL = \sigma(CL) = C_\sigma L.$$

PROOF: Apply Corollary 4 of Proposition 1 to obtain $\operatorname{Coz} CL = \sigma(CL)$. Since each Δ_a and ∇_b is a cozero element, $C_\sigma L$ generates $\operatorname{Coz} CL$ and, as each $\Theta \in \operatorname{Coz} CL$ is Lindelöf, $C_\sigma L = \operatorname{Coz} CL$. ☐

The following proposition may be compared with its analogue for frames [3] — where a frame is shown to be noetherian precisely when its congruence frame is compact. A σ-frame L is called noetherian when each of its elements is compact, $c \in L$ being compact whenever $c \leq \bigvee x_n$ implies $c \leq x_{n_1} \vee \cdots \vee x_{n_k}$ for some x_{n_i}. Note that this is equivalent to the Ascending Chain Condition which says that every sequence $a_1 \leq a_2 \leq a_3 \leq \ldots$ in L eventually terminates. Further, with Countable Dependent Choice, this is equivalent to the condition that every ideal in L is principal.

Proposition 4. *The following is equivalent for a σ-frame L:*

(1) *CL is a pseudocompact frame.*
(2) *$C_\sigma L$ is compact.*
(3) *L is noetherian.*
(4) *CL is the congruence lattice of L as a lattice.*
(5) *CL is a compact frame.*

Proof: (1) $\Rightarrow$ (2) By Proposition 2, $\mathrm{Coz}\, CL$ is compact, and by the corollary of Proposition 3, this is $C_\sigma L$.

(2) $\Rightarrow$ (3) Clearly the complemented elements of a compact σ-frame are compact. Hence, ∇_a is compact in $C_\sigma L$ for each $a \in L$, and since $\nu_L : L \to C_\sigma L$ is a σ-frame embedding this makes a compact in L.

(3) $\Rightarrow$ (4) If L is noetherian, then so is $L \times L$, so that countable joins in $L \times L$ are finite joins. Consequently, every lattice congruence on L is a σ-frame congruence.

(4) $\Rightarrow$ (5) Observe that CL is closed under up-directed unions and suitably modify the proof of Proposition 3.

(5) $\Rightarrow$ (1) Trivial. $\square$

We conclude with an application of Proposition 3 to obtain a generalization of a basic result on Alexandroff spaces (= cozero set spaces).

First a technical observation concerning $C_\sigma L$.

Lemma. *Let L be a sub-σ-frame of a σ-frame M such that each $a \in L$ has a complement in M and M is generated by the $a \in L$ and their complements. Then, the obvious homomorphism $h : C_\sigma L \to M$ is an isomorphism whenever M is compact.*

Proof: For any $\Theta \in C_\sigma L$, if $\Theta = \bigvee \nabla_{a_n} \cap \Delta_{b_n}$ for some $a_n, b_n \in L$, $n = 0, 1, 2, \ldots$, then $h(\Theta) = \bigvee a_n \wedge ({\sim}\, b_n)$, showing h is onto. Further, $h(\Theta) = e$ implies $(a_0 \wedge ({\sim}\, b_0)) \vee \cdots \vee (a_k \wedge ({\sim}\, b_k)) = e$ for some k by compactness, and therefore

$$(\nabla_a \cap \Delta_b) \vee \cdots \vee (\nabla_{a_k} \cap \Delta_{b_k}) = \nabla$$

by the uniqueness of the Boolean envelope of L, and finally $\Theta = \nabla$. This says h is codense and therefore one-one, by regularity. $\square$

An Alexandroff space may be described most succinctly here as a pair $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a collection of subsets of X which is regular as a sub-σ-frame of $\mathbb{P}X$, the power set of X ([9]). Gordon [10] calls the collection $B\mathcal{A}$ of subsets of X which are obtained from $\mathcal{A}$ by complementation, countable union and intersection, the Baire sets of $(X, \mathcal{A})$ and shows that $(X, B\mathcal{A})$ is never pseudocompact (as an Alexandroff space) unless X is finite. It follows from the Lemma that $B\mathcal{A}$, if compact is isomorphic as σ-frame to $C_\sigma\mathcal{A}$. In [8] it is shown that an Alexandroff space $(X, \mathcal{D})$ is pseudocompact if and only if $\mathcal{D}$ is compact as a σ-frame, thus Gordons result may be interpreted: $C_\sigma\mathcal{A}$ is never compact unless X is finite. In greater generality, and with a very simple proof, we have:

Proposition 5. *For any regular σ-frame L, $C_\sigma L$ is compact iff L is finite.*

PROOF: By Proposition 4, if $C_\sigma L$ is compact then L is noetherian. Hence by regularity $a \prec a$ for each $a \in L$, which means that L is Boolean. Thus, L is a Boolean algebra in which every ideal is principal, and it is well known that this makes L finite. The converse is obvious. □

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