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Some remarks on a class of weight functions

Loredana Caso, Maria Transirico

Abstract. In this paper we obtain some results about a class of functions $\varrho : \Omega \to R_+$, where Ω is an open set of R^n, which are related to the distance function from a fixed subset $S_\varrho \subset \partial\Omega$. We deduce some imbedding theorems in weighted Sobolev spaces, where the weight function is a power of a function ϱ.

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Introduction

Let Ω be an open subset of R^n.

In [T_4] M. Troisi has studied the class $\mathcal{A}(\Omega)$ of functions $\varrho : \Omega \to R_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<\varrho(y)}} \left|\log \frac{\varrho(x)}{\varrho(y)}\right| < +\infty \,. \tag{1}$$

Typical examples of functions $\varrho \in \mathcal{A}(\Omega)$ are the function

$$x \in R^n \to 1 + a|x| \,, \qquad a \in]0,1[\,,$$

and, if $\Omega \neq R^n$ and S is a nonempty subset of $\partial\Omega$, the function

$$x \in \Omega \to a\,\mathrm{dist}(x,S) \,, \qquad a \in]0,1[\,.$$

For any $\varrho \in \mathcal{A}(\Omega)$ we put

$$S_\varrho = \{z \in \partial\Omega \,:\, \lim_{x\to z} \varrho(x) = 0\} \,. \tag{2}$$

We remark (see, e.g., [T_4], [CCD_1]) that if $\varrho \in \mathcal{A}(\Omega)$ and $S_\varrho \neq \emptyset$, then ϱ is related to the distance function from S_ϱ.

For examples and properties of functions $\varrho \in \mathcal{A}(\Omega)$ we refer to [T_4] and also to [CCD_1], [TT], [DT].

For a treatment of weight functions as the distance function from a nonempty subset of the boundary of a bounded open set of R^n or weight functions related to such distance function, and for related problems see, e.g., [K], [KJF].

In some papers (see, e.g., [F_1], [S_1], [MT_2], [T_1], [CCD_1]) some classes of weighted Sobolev spaces have been studied, where the weight function is a power of a function $\varrho \in \mathcal{A}(\Omega)$.

In various papers (see, e.g., [MT_1], [IMT], [IT], [T_2], [S_2], [T_3], [F_2], [Sg], [ST], [GTT], [DT], [CCD_2]) many applications of such spaces to the study of boundary value problems for elliptic and quasielliptic differential equations have been studied, also in unbounded open sets.

In particular in [CCD_1] the authors, for fixed $\varrho \in \mathcal{A}(\Omega)$, have studied the operator

$$u \to gu \,, \tag{3}$$

where g is singular near S_ϱ, as an operator defined in a weighted Sobolev space, denoted by $W^{r,p}_q(\Omega)$ (see n. 1 for such definition), and which takes values in $L^p(\Omega)$, where the weight function is a power of ϱ. They have given conditions on ϱ (e.g. S_ϱ closed), g and Ω, so that the operator defined by (3) is bounded and other conditions in order that it is compact.

As an application (see [CCD_2]) the authors have studied the Dirichlet problem in an open set, not necessarily bounded, for variational second order elliptic equations with coefficients singular near S_ϱ. They have obtained an existence and uniqueness theorem for the solution in the closure of $C^\infty_o(\Omega)$ in $W^{1,2}_q(\Omega)$.

In this paper our purpose is to give a contribution to the study of functions of $\mathcal{A}(\Omega)$.

We state some suitable characterizations of S_ϱ, from which, in particular, we deduce that S_ϱ is a closed subset of $\partial\Omega$ (see n. 1).

Because of these results, we can give (see n. 2) a contribute to the study of some functions which are singular near S_ϱ, as the function g in (3). Furthermore we obtain (see n. 3) a remarkable improvement of the imbedding results of [CCD_1].

1. Some properties of functions of $\mathcal{A}(\Omega)$

For all $x \in R^n$ and for all $r \in R_+$ we set

$$B(x,r) = \{y \in R^n : |y - x| < r\} \,.$$

If A is a Lebesgue measurable subset of R^n, $1 \le p \le +\infty$, and $f \in L^p(A)$ we put

$$\|f\|_{L^p(A)} = |f|_{p,A} \,.$$

Let Ω be an open subset of R^n, $n \ge 2$. We put

$$\Omega(x,r) = \Omega \cap B(x,r) \qquad \forall\, x \in R^n, \quad \forall\, r \in R_+ \,.$$

We denote by $\mathcal{A}(\Omega)$ the class of functions $\varrho : \Omega \to R_+$ verifying (1).

Obviously ϱ verifies (1) if and only if there exists $\gamma \in R_+$ such that

$$(1.1) \qquad \gamma^{-1}\varrho(y) \leq \varrho(x) \leq \gamma\varrho(y) \qquad \forall\, y \in \Omega \quad \text{and} \quad \forall\, x \in \Omega \cap B(y, \varrho(y))\,.$$

We remark that for any $\varrho \in \mathcal{A}(\Omega)$ there exist $a \in R_+$ and $b \in]0,1]$ such that

$$(1.2) \qquad \varrho(x) \leq a + b|x| \qquad \forall\, x \in \Omega$$

(see, e.g., (19) and (20) of [TT]).

We denote by $\mathcal{A}_o(\Omega)$ the class of measurable functions $\varrho \in \mathcal{A}(\Omega)$.

From (1.1) and (1.2) follows that for all $\varrho \in \mathcal{A}_o(\Omega)$ we have

$$(1.3) \qquad \varrho \in L^\infty_{loc}(\bar{\Omega})\,, \qquad \varrho^{-1} \in L^\infty_{loc}(\Omega)\,.$$

As we will see in (1.5) the second relation of (1.3) can be improved.

For all $\varrho \in \mathcal{A}(\Omega)$ we denote by S_ϱ the set defined by (2).

It is well-known (see, e.g., [T4]) that, if $\varrho \in \mathcal{A}(\Omega)$ and $S_\varrho \neq \emptyset$, then

$$(1.4) \qquad \varrho(x) \leq \operatorname{dist}\,(x, S_\varrho) \qquad \forall\, x \in \Omega\,.$$

We prove the following

Lemma 1.1. *For all $\varrho \in \mathcal{A}(\Omega)$ and for all $z \in \partial\Omega$, the following statements are equivalent:*

(1) $z \in S_\varrho$,
(2) $\varrho(x) \leq |x - z| \quad \forall\, x \in \Omega$,
(3) $\inf_{\Omega(z,r)} \varrho = 0 \quad \forall\, r \in R_+$.

Proof: (1) $\Rightarrow$ (2) is a consequence of (1.4). (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are evident. To prove (3) $\Rightarrow$ (2), we observe that if there exists $x_1 \in \Omega$ such that $\varrho(x_1) > |x_1 - z|$ and if we put $\tau = \varrho(x_1) - |x_1 - z|$, we have

$$|x - x_1| < \varrho(x_1) \qquad \forall\, x \in \Omega(z, \tau)\,,$$

from which, by (1.1), follows

$$\gamma^{-1}\varrho(x_1) \leq \varrho(x) \leq \gamma\varrho(x_1) \qquad \forall\, x \in \Omega(z, \tau)\,,$$

and so we have $\inf_{\Omega(z,\tau)} \varrho > 0$. □

Theorem 1.1. *If $\varrho \in \mathcal{A}(\Omega)$, then S_ϱ is a closed subset in $\partial\Omega$.*

Proof: Let $z \in \partial\Omega \setminus S_\varrho$. As a consequence of (3) of Lemma 1.1 there exists $\tau \in R_+$ such that $\inf_{\Omega(z,\tau)} \varrho > 0$. From this we have

$$\inf_{\Omega(y, \tau - |y - z|)} \varrho > 0 \qquad \forall\, y \in B(z, \tau) \cap \partial\Omega\,,$$

and then, again from (3) of Lemma 1.1, $B(z, \tau) \cap \partial\Omega \subset \partial\Omega \setminus S_\varrho$. Thus we obtain our statement. □

Remark 1.1. If $\varrho \in \mathcal{A}(\Omega)$, for any compact set $\Omega_o \subset \overline{\Omega} \setminus S_\varrho$, from (1.1) and (3) of Lemma 1.1 we deduce easily that $\inf_{\Omega_o} \varrho > 0$. It follows that if $\varrho \in \mathcal{A}_o(\Omega)$ then

$$\varrho^{-1} \in L^\infty_{loc}(\overline{\Omega} \setminus S_\varrho)\,. \tag{1.5}$$

□

If $r \in N$, $1 \le p \le +\infty$, $q \in R$ and $\varrho \in \mathcal{A}_o(\Omega)$, we denote by $W^{r,p}_q(\Omega)$ the space of distributions u on Ω such that $\varrho^{q+|\alpha|-r}\, \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le r$ with the norm

$$\|u\|_{W^{r,p}_q(\Omega)} = \sum_{|\alpha|\le r} |\varrho^{q+|\alpha|-r}\, \partial^\alpha u|_{p,\Omega}\,. \tag{1.6}$$

We put

$$W^{0,p}_q(\Omega) = L^p_q(\Omega)\,.$$

2. The spaces $K^p_q(\Omega)$

Let us fix $\varrho \in \mathcal{A}_o(\Omega)$.

We consider the spaces $K^p_q(\Omega)$, $\tilde{K}^p_q(\Omega)$, $\overset{o}{K}{}^p_q(\Omega)$, $1 \le p < +\infty$, $q \in R$, defined in [CCD$_1$] in correspondence with the family of open sets $\Omega(x, \varrho(x))$, $x \in \Omega$.

Let us recall that:

$K^p_q(\Omega)$ is the space of functions $g \in L^p_{loc}(\bar{\Omega} \setminus S_\varrho)$ such that

$$\|g\|_{K^p_q(\Omega)} = \sup_{x\in\Omega} \left(\varrho^{q-n/p}(x)\, |g|_{p,\Omega(x,\varrho(x))} \right) < +\infty\,, \tag{2.1}$$

with the norm defined by (2.1),

$\tilde{K}^p_q(\Omega)$ is the closure of $L^\infty_q(\Omega)$ in $K^p_q(\Omega)$,

$\overset{o}{K}{}^p_q(\Omega)$ is the closure of $C^\infty_o(\Omega)$ in $K^p_q(\Omega)$.

For some properties of the spaces $K^p_q(\Omega)$, $\tilde{K}^p_q(\Omega)$ and $\overset{o}{K}{}^p_q(\Omega)$ we refer to [CCD$_1$], [CCD$_2$].

In order to recall a result of [CCD$_1$] which we will use later, we introduce the following notations (see, e.g., n. 1 of [CCD$_1$]).

We denote by α a function of $C^{(0,1)}(\bar{\Omega}) \cap C^\infty(\Omega)$ such that $\alpha(x) \sim \operatorname{dist}(x, \partial\Omega)$ and we put

$$\Omega_k = \{x \in \Omega :\ |x| < k,\ \alpha(x) > 1/k\}\,, \qquad \forall\, k \in N\,.$$

We denote, furthermore, by $(\psi_k)_{k\in N}$ a sequence of functions in $C^\infty_o(\Omega)$ such that

$$0 \le \psi_k \le 1\,, \qquad \psi_k|_{\Omega_k} = 1\,, \qquad \operatorname{supp} \psi_k \subset \Omega_{2k}\,.$$

The following result holds (see Lemma 2 of [CCD$_1$]): a function $g \in \overset{o}{K}{}^p_q(\Omega)$ if and only if $g \in K^p_q(\Omega)$ and

$$\lim_{k\to\infty} \|(1-\psi_k)\, g\|_{K^p_q(\Omega)} = 0\,. \tag{2.2}$$

Because of this result, of Theorem 1.1 and of Remark 1.1, we can prove the following condition so that a function in $K^p_q(\Omega)$ is in $\overset{o}{K}{}^p_q(\Omega)$.

Lemma 2.1. *If $g \in K^p_q(\Omega)$, $1 \le p < +\infty$, $q \in R$, and if moreover*

$$\lim_{|x|\to+\infty} \varrho^q(x)\, g(x) = 0\,, \tag{2.3}$$

$$\lim_{x\to x_o} \varrho^q(x)\, g(x) = 0 \qquad\qquad \forall\, x_o \in S_\varrho\,, \tag{2.4}$$

then $g \in \overset{o}{K}{}^p_q(\Omega)$.

Proof: Let us fix $\epsilon > 0$.

From (2.3) it follows that there exists $r_\epsilon > 0$ such that

$$|\varrho^q(y)\, g(y)| < \epsilon \qquad\qquad \forall\, y \in \Omega\,, \quad |y| > r_\epsilon\,. \tag{2.5}$$

If we put

$$A_\epsilon = \{x \in \Omega : \mathrm{dist}(x, B_{r_\epsilon} \cap \Omega) < \varrho(x)\}\,,$$

from Theorem 1.3 of [T$_4$] it follows that A_ϵ is bounded.

Let $r^*_\epsilon > r_\epsilon$ such that $A_\epsilon \subset B_{r^*_\epsilon} \cap \Omega$.

We remark that if $x \in \Omega$, $|x| \ge r^*_\epsilon$ and $y \in \Omega(x, \varrho(x))$, then $|y| > r_\epsilon$.

Thus, because of (2.5), for any $k \in N$ we have

$$\begin{aligned} &\sup_{\substack{x\in\Omega\\ |x|\ge r^*_\epsilon}} \varrho^{qp-n}(x) \int_{\Omega(x,\varrho(x))} |1-\psi_k|^p\, |g|^p\, dy \\ &\le c_1 \sup_{\substack{x\in\Omega\\ |x|\ge r^*_\epsilon}} \varrho^{-n}(x) \int_{\Omega(x,\varrho(x))} |1-\psi_k|^p\, |\varrho^q g|^p\, dy \;\le c_2\, \epsilon^p\,, \end{aligned} \tag{2.6}$$

where the constants $c_1, c_2 \in R_+$ are independent of x and k.

Clearly, if $x \in \Omega$, $|x| < r^*_\epsilon$ and $y \in \Omega(x, \varrho(x))$, then $|y| < r^*_\epsilon + \sup_{\Omega\cap B_{r^*_\epsilon}} \varrho = \tilde r_\epsilon$.

From (2.4) and from Theorem 1.1 it follows that $S_\varrho \cap \overline{B}_{\tilde r_\epsilon}$ can be covering by a finite number of open balls, with center on S_ϱ, $I_{\epsilon,i}$, $i = 1, \dots, m$, such that, letting $K_\epsilon = \cup_{i=1}^m I_{\epsilon,i}$, we have

$$|\varrho^q(y)\, g(y)| < \epsilon \qquad\qquad \forall\, y \in \Omega \cap K_\epsilon\,. \tag{2.7}$$

From (2.7), for any $k \in N$ we have

$$\begin{aligned} &\sup_{\substack{x\in\Omega\\ |x|< r^*_\epsilon}} \varrho^{qp-n}(x) \int_{\Omega(x,\varrho(x))\cap K_\epsilon} |1-\psi_k|^p\, |g|^p\, dy \\ &\le c_3 \sup_{\substack{x\in\Omega\\ |x|< r^*_\epsilon}} \varrho^{-n}(x) \int_{\Omega(x,\varrho(x))\cap K_\epsilon} |1-\psi_k|^p\, |\varrho^q g|^p\, dy \;\le c_4\, \epsilon^p\,, \end{aligned} \tag{2.8}$$

where the constants $c_3, c_4 \in R_+$ are independent of x and k.

Moreover, from (1.5), we get

$$\begin{aligned}
&\sup_{\substack{x\in\Omega \\ |x|<r^*_\epsilon}} \varrho^{qp-n}(x) \int_{\Omega(x,\varrho(x))\setminus K_\epsilon} |1-\psi_k|^p \, |g|^p \, dy \\
&\le c_5 \sup_{\substack{x\in\Omega \\ |x|<r^*_\epsilon}} \int_{\Omega(x,\varrho(x))\setminus K_\epsilon} \varrho^{qp-n}(y) \, |1-\psi_k|^p \, |g|^p \, dy \qquad (2.9) \\
&\le c_6 \int_{(\Omega\cap B_{\tilde{r}_\epsilon})\setminus K_\epsilon} |1-\psi_k|^p \, |g|^p \, dy \,,
\end{aligned}$$

where the constants $c_5, c_6 \in R_+$ are independent of x and k.

From (2.6), (2.8) and (2.9) it follows that

$$\begin{aligned}
&\sup_{x\in\Omega} \varrho^{qp-n}(x) \int_{\Omega(x,\varrho(x))} |1-\psi_k|^p \, |g|^p \, dy \\
&\le c_7 \, (\epsilon^p + \int_{(\Omega\cap B_{\tilde{r}_\epsilon})\setminus K_\epsilon} |1-\psi_k|^p \, |g|^p \, dy) \,, \qquad (2.10)
\end{aligned}$$

where the constant $c_7 \in R_+$ is independent of x and k.

From (2.10) we obtain (2.2) and thus our statement. □

3. Imbedding results

For any $x \in R^n$ and for any $\theta \in]0, \frac{\pi}{2}[$ we denote by $C_\theta(x)$ an open indefinite cone with vertex in x and opening θ.

For a fixed $C_\theta(x)$, we put

$$C_\theta(x,r) = C_\theta(x) \cap B(x,r) \qquad \forall\, r \in R_+ \,.$$

We denote by $\Gamma(\Omega,\theta,r)$ the family of open cones C of opening θ, height r and such that $\overline{C} \subset \Omega$.

We suppose that the following condition holds:

(h_0) there exist $b \in]0,1]$ and $\theta \in]0, \frac{\pi}{2}[$ such that

$$(3.1) \qquad \forall\, x \in \Omega \qquad \exists C_\theta(x) \;:\qquad \overline{C_\theta(x, b\varrho(x))} \subset \Omega \,.$$

Remark 3.1. We remark that if, for example, $\varrho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ and Ω verifies the condition

(h_1) there exists an open subset Ω^* of R^n with the cone property such that

$$\Omega \subset \Omega^* \,, \qquad\qquad \partial\Omega \setminus S_\varrho \subset \partial\Omega^* \,,$$

then the condition (h_0) holds.

In fact, we consider $\theta \in]0, \frac{\pi}{2}[$ and $r \in R_+$ such that for all $x \in \Omega$ there exists $C_\theta(x)$ such that $\overline{C_\theta(x,r)} \subset \Omega^*$.

Let us fix $b \in]0,1]$ such that $b \operatorname{ess\,sup}_\Omega \varrho < r$. Then (see n. 2 of [CCD$_1$]) we have that for any $x \in \Omega$ it results $\overline{C_\theta(x, b\varrho(x))} \subset \Omega$. □

Let us fix $\gamma \in R_+$ such that (1.1) is verified.

For all $x \in \Omega$ and for all $\lambda \in]0,1]$ we denote by $G_{\lambda,b}(x)$ the subset of R^n union of the family of open cones $C \in \Gamma(\Omega, \theta, \lambda\gamma^{-1} b\varrho(x))$ such that $x \in C$.

We fix $\Omega_b(x)$, $x \in \Omega$, with the condition that there exists $\lambda \in]0,1]$ such that

$$G_{\lambda,b}(x) \subset \Omega_b(x) \subset \Omega(x, b\varrho(x)) \qquad \forall\, x \in \Omega\,. \tag{3.2}$$

We put, in the case $b = 1$,

$$G_\lambda(x) = G_{\lambda,1}(x)\,, \qquad \Omega(x) = \Omega_1(x) \qquad \forall\, x \in \Omega\,.$$

Remark 3.2. In n. 5 of [CCD$_1$], fixed $\varrho \in \mathcal{A}_o(\Omega)$, the authors assumed that the following hypotheses are satisfied:

(i$_1$) S_ϱ is closed, $\varrho^{-1} \in L^\infty_{loc}(\overline{\Omega} \setminus S_\varrho)$,
(i$_2$) $\exists\, \theta \in]0, \frac{\pi}{2}[$ such that

$$\forall\, x \in \Omega \qquad \exists C_\theta(x)\ :\qquad \overline{C_\theta(x, \varrho(x))} \subset \Omega\,,$$

(i$_3$) $\Omega(x)$ has the cone property with a cone $C \in \Gamma(\Omega, \theta_o, \lambda_o\, \varrho(x))$, where θ_o and λ_o are constants independent of x,
(i$_4$) r, q, p, s are numbers such that

$$r \in N, \quad q \in R, \quad 1 \le p \le s < +\infty, \quad s \ge \frac{n}{r}, \quad s > \frac{n}{r} \ \text{ if } \ \frac{n}{r} = p > 1\,. \tag{3.3}$$

Condition (i$_3$) is not really contained in n. 5 of [CCD$_1$], but in the note (1) of page 115 of [CCD$_2$] it is explained that hypothesis (i$_3$) must be added in order to prove the results of n. 5 of [CCD$_1$].

In n. 5 of [CCD$_1$], under the hypotheses (i$_1$), (i$_2$), (i$_3$) and (i$_4$) the authors have proved (see Theorem 1) that for all $g \in L^s_{loc}(\overline{\Omega} \setminus S_\varrho)$ such that $\sup_{x\in\Omega}\big(\varrho^{-q+r-\frac{n}{s}}(x)|g|_{s,\Omega(x)}\big) < +\infty$ and for all $u \in W^{r,p}_q(\Omega)$, it results $gu \in L^p(\Omega)$ and

$$|gu|_{p,\Omega} \le c \sup_{x\in\Omega}\Big(\varrho^{-q+r-\frac{n}{s}}(x)|g|_{s,\Omega(x)}\Big)\|u\|_{W^{r,p}_q(\Omega)}\,, \tag{3.4}$$

where the constant $c \in R_+$ is independent of g and u.

Moreover, they have proved some consequences of the above Theorem 1.

We remark (see Theorem 1.1 and Remark 1.1) that the hypothesis (i_1) can be dropped.

Moreover we remark that the hypothesis (i_3) is not necessary in order to obtain Theorem 1 of [CCD_1]. In fact, this theorem holds with $\Omega(x) = G_\lambda(x)$, because $G_\lambda(x)$ verifies the (i_3). Then from this we deduce that the above theorem holds also without (i_3), because for any $q \in R$ and $p \in [1, +\infty[$ we have

$$\sup_{x \in \Omega} \varrho^{q-\frac{n}{p}}(x)|g|_{p,G_\lambda(x)} \le \sup_{x \in \Omega} \varrho^{q-\frac{n}{p}}(x)|g|_{p,\Omega(x)} .$$

□

From Remark 3.2 it follows that, the hypotheses (h_0) and (3.3) are enough to prove Theorem 1 of [CCD_1] in the case $\Omega_b(x)$ and thus also in the case $\Omega(x, \varrho(x))$. So we have

Theorem 3.1. *If the conditions* (h_0) *and* (3.3) *hold, then for all* $g \in K^s_{-q+r}(\Omega)$ *and for all* $u \in W^{r,p}_q(\Omega)$ *we have* $gu \in L^p(\Omega)$ *and*

$$|gu|_{p,\Omega} \le c\|g\|_{K^s_{-q+r}(\Omega)}\|u\|_{W^{r,p}_q(\Omega)} , \tag{3.5}$$

where the constant $c \in R_+$ *is independent of* g *and* u. □

Using arguments similar to those in n. 5 of [CCD_1], from Theorem 3.1 we deduce that:

(a) if the hypotheses of Theorem 3.1 hold and therefore $g \in \tilde{K}^s_{-q+r}(\Omega)$, then for any $\epsilon \in R_+$ there exists $c(\epsilon) \in R_+$ such that

$$|gu|_{p,\Omega} \le \epsilon\|u\|_{W^{r,p}_q(\Omega)} + c(\epsilon)\|u\|_{L^p_{q-r}(\Omega)} \qquad \forall\, u \in W^{r,p}_q(\Omega)\,; \tag{3.6}$$

(b) if, moreover, $g \in \overset{o}{K}{}^s_{-q+r}(\Omega)$, then for any $\epsilon \in R_+$ there exist $c(\epsilon) \in R_+$ and a bounded open set Ω_ϵ, with the cone property and with $\overline{\Omega}_\epsilon \subset \Omega$, such that

$$|gu|_{p,\Omega} \le \epsilon\|u\|_{W^{r,p}_q(\Omega)} + c(\epsilon)|u|_{p,\Omega_\epsilon} \qquad \forall\, u \in W^{r,p}_q(\Omega) \tag{3.7}$$

and we have that the operator

$$u \in W^{r,p}_q(\Omega) \to gu \in L^p(\Omega) \tag{3.8}$$

is compact.

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Dipartimento di Ingegneria dell'Informazione e Matematica Applicata, Facoltá di Scienze, Universitá di Salerno, 84081 Baronissi (SA), Italy

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