A Tauberian theorem for distributions *

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Abstract. The well-known general Tauberian theorem of N. Wiener is formulated and proved for distributions in the place of functions and its Ganelius' formulation is corrected. Some changes of assumptions of this theorem are discussed, too.

Keywords: Tauberian theorem, distribution, convolution, Fourier transform

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Introduction and notation

We consider spaces of functions and distributions defined on $\mathbb{R}$ only. However, the assertions of the parts 1–5 of this paper are valid generally on $\mathbb{R}^N$.

In 1932 N. Wiener [13] proved this general Tauberian theorem:

Theorem A. *Let* $\mathbf{1}^\circ$ $k \in \mathcal{L}(\mathbb{R})$, *its Fourier transform* $\widehat{k}(x) \neq 0$ *for all* $x \in \mathbb{R}$,
$\mathbf{2}^\circ$ *a function* f *is measurable and bounded in* $\mathbb{R}$ *and*
$\mathbf{3}^\circ$ $\lim\limits_{x\to\infty} k * f\,(x) = 0.$
Then $\lim\limits_{x\to\infty} h * f\,(x) = 0$ *for every* $h \in \mathcal{L}(\mathbb{R})$.

The analogy of Theorem A for distributions is mentioned in 1971 by T. Ganelius [2, p. 13]. Ganelius writes $F = o(1)$ for a distribution $F \in \mathcal{S}'$ if and only if

$$\lim_{x\to\infty} F * \varphi\,(x) = 0 \quad \text{for every} \quad \varphi \in \mathcal{S}. \tag{1}$$

To avoid the confusion with the classical meaning of the symbol $o(1)$, we shall write $F = o'(1)$ instead of $F = o(1)$ in the case (1).

Theorem B. *Let* $\mathbf{1}^\circ$ $K \in \mathcal{O}'_C$, $\widehat{K}(x) \neq 0$ *for all* $x \in \mathbb{R}$,
$\mathbf{2}^\circ$ $F \in \mathcal{S}'$ *and*
$\mathbf{3}^\circ$ $K * F = o'(1)$.
Then $H * F = o'(1)$ *for every* $H \in \mathcal{O}'_C$.

Particularly, with $H = \delta$ we get $F = o'(1)$.

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Reading the proof of this theorem ([2, p. 14]) a reader may observe one unconvincing step in the end of the proof: the relation $H * F = o'(1)$ is deduced without any explanation from the two right facts, namely $(H * F) * \chi\,(x) = o(1)$, $x \to \infty$, for any $\chi \in \widehat{\mathcal{D}} := \{\widehat{\varphi}\,;\, \varphi \in \mathcal{D}\}$ and $\widehat{\mathcal{D}}$ is dense in $\mathcal{S}$.

In the present paper we show the non-validity of Theorem B by constructing a counterexample in the part 6 and, moreover, we prove two other Tauberian theorems for distributions in the parts 12 and 13.

The functions are usually denoted by the small letters, the distributions by the capital ones. We use the notation of spaces of distributions by L. Schwartz [12]: $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ is the space of all tempered distributions, $\mathcal{O}'_C$ stands for the rapidly decreasing distributions, $\mathcal{E}'$ for the distributions with compact support, $\mathcal{D}'_{\mathcal{L}_1}$ or shortly $\mathcal{D}'_{\mathcal{L}}$ is the space of all integrable distributions, $\mathcal{B}'$ is the space of all bounded distributions, finally $\dot{\mathcal{B}}'$ denotes the closure of $\mathcal{E}'$ in $\mathcal{B}'$ as the dual space of $\mathcal{D}_{\mathcal{L}}$. For a distribution F, we often use the symbol $F(x)$ instead of F. This does not mean that F is a function of x. For example, $\langle S(x), \varphi(x)\rangle$ means $\langle S, \varphi\rangle$.

If $f \in \mathcal{L}\,(\mathbb{R})$, we define the Fourier transform $\widehat{f}$ of the function f by

$$\widehat{f}(t) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{itx}\,\mathrm{d}x; \tag{2}$$

and we use the following well-known properties of this transform. If $g(x) = f(-x)$, then

$$\widehat{g}(t) = \widehat{f}(-t), \tag{3}$$

so $\widehat{f}$ is even if f is even; if $g(x) = f(x)\, e^{aix}$, then

$$\widehat{g}(t) = \widehat{f}(t+a); \tag{4}$$

if $g(x) = f(x-a)$, then

$$\widehat{g}(t) = \widehat{f}(t) \cdot e^{ait}; \tag{5}$$

if $g(x) = x f(x)$, then

$$\widehat{g} = -i\,\widehat{f}\,'; \tag{6}$$

if $g(x) = \frac{1}{a}\,\widehat{f}(\frac{x}{a})$, then

$$\widehat{g}(t) = f(at). \tag{7}$$

If $\varphi \in \mathcal{S}$, so we have $\widehat{\varphi} \in \mathcal{S}$ as well and

$$\varphi(x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\varphi}(t)\, e^{-itx}\,\mathrm{d}t. \tag{8}$$

If $\varphi, \psi \in \mathcal{S}$, then the convolution $\varphi * \psi \in \mathcal{S}$, $\varphi \cdot \psi \in \mathcal{S}$ and we have

$$\tag{9} \begin{aligned} \widehat{\varphi * \psi} &= \sqrt{2\pi}\, \widehat{\varphi}\, \widehat{\psi}, \\ \widehat{\varphi\psi} &= \frac{1}{\sqrt{2\pi}}\, \widehat{\varphi} * \widehat{\psi}. \end{aligned}$$

For $K \in \mathcal{S}'$ the Fourier transform $\widehat{K} \in \mathcal{S}'$ is defined by

$$\langle \widehat{K}, \varphi \rangle = \langle K, \widehat{\varphi} \rangle, \qquad \varphi \in \mathcal{S}.$$

If $K \in \mathcal{L}(\mathbb{R})$, then this definition of $\widehat{K}$ coincides with (2). If $K \in \mathcal{D}'_{\mathcal{L}}$, then $\widehat{K}$ is a continuous function $\widehat{K}(x) = \frac{1}{\sqrt{2\pi}} \langle K(t), e^{itx} \rangle_t$ (see [12, VII, § 7, Example 4]); if $K \in \mathcal{O}'_C$, then $\widehat{K} \in \mathcal{E}$. Let $\mathbb{1}$ denote the unit function: $\mathbb{1}(x) = 1$, $x \in \mathbb{R}$ and δ the Dirac distribution $\langle \delta, \varphi \rangle = \varphi(0)$, $\varphi \in \mathcal{D}$. Then we have

$$\tag{10} \widehat{\delta} = \frac{1}{\sqrt{2\pi}} \mathbb{1}, \quad \widehat{\mathbb{1}} = \sqrt{2\pi}\, \delta.$$

The convolution and the product of distributions

1. Properties of the convolution mentioned in 1–4 can be found in [1] or [11]. For any $F \in \mathcal{D}'_{\mathcal{L}}$, we define the integral of the distribution F to be equal to $\langle F, \mathbb{1} \rangle$. We extend the definition of $\langle F, \varphi \rangle$ for $F \in \mathcal{D}'$ and $\varphi \in \mathcal{E}$ such that $\varphi F \in \mathcal{D}'_{\mathcal{L}}$ writing $\langle F, \varphi \rangle = \langle \varphi F, \mathbb{1} \rangle$.

Definition. Two distributions $F, G \in \mathcal{D}'(\mathbb{R}^N)$ are called convolvable if for any test function $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we have

$$\tag{11} (F(x) \times G(y))\, \varphi(x+y) \in \big(\mathcal{D}'_{\mathcal{L}}\big)_{x,y}.$$

The convolution $F * G \in \mathcal{D}'(\mathbb{R}^N)$ of two convolvable distributions F and G is defined by

$$\tag{12} \langle F * G, \varphi \rangle = \langle (F(x) \times G(y))\, , \varphi(x+y) \rangle, \qquad \varphi \in \mathcal{D}(\mathbb{R}^N).$$

The convolution $F * \varphi$ of a distribution with a test function can be defined equivalently to be the function

$$\tag{13} F * \varphi\, (x) = \langle F(t), \varphi(x-t) \rangle_t.$$

2. Definition. Two distributions $F, G \in \mathcal{S}'(\mathbb{R}^N)$ are called $\mathcal{S}'$-convolvable if the relation (11) holds even for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Then the convolution $F * G$ is called $\mathcal{S}'$-convolution, $F * G \in \mathcal{S}'(\mathbb{R}^N)$ and (12) holds for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

3. If $G \in \mathcal{D}'(\mathbb{R}^N)$, denote $\breve{G}(x) = G(-x)$.

Equivalent definition. *Two distributions* $F, G \in \mathcal{S}'(\mathbb{R}^N)$ *are* $\mathcal{S}'$*-convolvable if and only if* $F \cdot (\check{G} * \varphi) \in \mathcal{D}'_{\mathcal{L}}$ *for any* $\varphi \in \mathcal{S}(\mathbb{R}^N)$*. Then*

$$\langle F * G, \varphi \rangle = \langle F, \check{G} * \varphi \rangle.$$

Notation. The convolution of distributions as an operation on $\mathcal{D}'(\mathbb{R}^N)$ is commutative but not associative. If $F, G, H \in \mathcal{D}'(\mathbb{R}^N)$ are such that

$$F * (G * H) = G * (H * F) = H * (F * G), \tag{14}$$

we denote $F * (G * H)$ simply by $F * G * H$. Let us note that the symbol $F * G * H$ is not used in this sense commonly.

4. Let us recall

$$\begin{aligned}
\mathcal{O}'_C &= \left\{ F \in \mathcal{S}' \,;\, F * \varphi \in \mathcal{S} \text{ for all } \varphi \in \mathcal{S} \right\}, \\
\mathcal{D}'_{\mathcal{L}} &= \left\{ F \in \mathcal{S}' \,;\, F * \varphi \in \mathcal{L} \text{ for all } \varphi \in \mathcal{S} \right\} \quad \text{and} \\
\mathcal{B}' &= \left\{ F \in \mathcal{S}' \,;\, F * \varphi \text{ is a bounded function for all } \varphi \in \mathcal{S} \right\}.
\end{aligned}$$

It is well known that a distribution $F \in \mathcal{O}'_C$ is $\mathcal{S}'$-convolvable with any $G \in \mathcal{S}'$. Distributions $F \in \mathcal{B}'$, $G \in \mathcal{D}'_{\mathcal{L}}$ are $\mathcal{S}'$-convolvable and $F * G \in \mathcal{B}'$. If $F, G \in \mathcal{D}'_{\mathcal{L}}$, then $F * G \in \mathcal{D}'_{\mathcal{L}}$. The formula (13) is valid in addition for $F \in \mathcal{S}'$, $\varphi \in \mathcal{S}$ or for $F \in \mathcal{B}'$, $\varphi \in \mathcal{D}_{\mathcal{L}}$ or for $F \in \mathcal{D}'_L$, $\varphi \in \mathcal{B}$.

Proposition. 1° *Let distributions* F, G *be* $\mathcal{S}'$*-convolvable and* $H \in \mathcal{O}'_C$*. Then the relation* (14) *is satisfied.*

2° *Let* $G, H \in \mathcal{D}'_{\mathcal{L}}$ *and* $F \in \mathcal{B}'$*. Then the relation* (14) *is satisfied.*

Proof: The part 1° of the Proposition will be proved in several steps. It is well known that functions $f, g, h \in \mathcal{S}$ are $\mathcal{S}'$-convolvable and the convolution $f * g * h$ is associative. So, using the equivalent Definition 3, we have for any $\varphi \in \mathcal{S}$

$$\begin{gathered}
\langle (F * g) * h, \, \varphi \rangle \; = \; \langle F * g, \, \check{h} * \varphi \rangle \; = \\
\langle F, \, \check{g} * \check{h} * \varphi \rangle \; = \; \langle F, \, (g * h)\check{} * \varphi \rangle \; = \; \langle F * (g * h), \, \varphi \rangle.
\end{gathered}$$

Thus we have proved $(F * g) * h = F * (g * h)$. Using this fact, we can prove in the same way $(F * G) * h = F * (G * h)$ and finally $(F * G) * H = F * (G * H)$.

Proof of the part 2°: Due to the part 1° we have

$$(F * G) * H = F * (G * H) \tag{15}$$

for $F \in \mathcal{B}'$, $G \in \mathcal{D}'_{\mathcal{L}}$ and $H \in \mathcal{S}$. By [12, VI, § 8, Theorem XXVI 2°] the convolution considered as a map from $\mathcal{D}'_{\mathcal{L}} \times \mathcal{D}'_{\mathcal{L}}$ into $\mathcal{D}'_{\mathcal{L}}$ or as a map from $\mathcal{D}'_{\mathcal{L}} \times \mathcal{B}'$ into $\mathcal{B}'$ is continuous. Since $\mathcal{S}$ is dense in $\mathcal{D}'_{\mathcal{L}}$, (15) can be extended for $H \in \mathcal{D}'_{\mathcal{L}}$ and the proposition is proved. □

5. A sequence of test functions $\varphi_n \in \mathcal{D}$ is called a regular delta-sequence if it satisfies $\varphi_n \geq 0$, $\int \varphi_n = 1$, $\operatorname{supp} \varphi_n \to \{0\}$. The (multiplicative) product of two distributions F, G is defined to be the distribution $\lim\limits_{n\to\infty} F \cdot (G * \varphi_n)$ if this limit exists in $\mathcal{D}'$ for any regular delta-sequence (then it does not depend on the choice of the regular delta-sequence). The multiplicative product is commutative, but not associative.

Proposition (The exchange formula, Hirata, Ogata [6])**.** *Let $F, G \in \mathcal{S}'$. If the distributions F, G are $\mathcal{S}'$-convolvable, then the multiplicative product $\widehat{F}\widehat{G}$ exists and $\widehat{F * G} = \sqrt{2\pi}\, \widehat{F}\widehat{G}$.*

6. Counterexample to Theorem B.

We shall construct a distribution $F \in \mathcal{S}'$ and functions $k, h \in \mathcal{S}$ such that $\widehat{k}(t) \neq 0$ for all $t \in \mathbb{R}$, $k * F(x) = o(1)$, $x \to \infty$, however the relation $h * F(x) = o(1)$, $x \to \infty$, does not hold. It means $F = o'(1)$ is not valid, which contradicts Theorem B.

Take a function $\gamma \in \mathcal{S}$ such that

$$\gamma \text{ is even, } \gamma \geq 0,\ \widehat{\gamma} \in \mathcal{D}([-1,1]),\ \widehat{\gamma} \geq 0,\ \textstyle\int \widehat{\gamma} = \sqrt{2\pi}\gamma(0) = 1. \tag{16}$$

For instance, denoting $\widehat{u}(t) = \exp(4t^2 - 1)^{-1}$ for $|t| < \frac{1}{2}$, $\widehat{u}(t) = 0$ for $|t| \geq \frac{1}{2}$, we can take $\widehat{\gamma} = c\widehat{u} * \widehat{u}$ for a suitable $c > 0$.

Put $\widehat{\beta}(t) = \chi_{[-\frac{3}{2},\frac{3}{2}]}(t) * 2\,\widehat{\gamma}(2t)$, i.e. $\widehat{\beta}$ is a regularization of the characteristic function of the interval $[-\frac{3}{2}, \frac{3}{2}]$. Then

$$\widehat{\beta} \in \mathcal{D}([-2,2]),\ 0 \leq \widehat{\beta} \leq 1,\ \widehat{\beta}(t) = 1 \quad \text{for} \quad t \in [-1,1] \tag{17}$$

and the function $\widehat{\beta}$ provides us a locally finite decomposition of the unit

$$\mathbb{1}(t) = 1 = \sum_{m=-\infty}^{\infty} \widehat{\beta}(t - 3m), \quad t \in \mathbb{R}.$$

Let the functions k, h be defined by

$$\widehat{k}(t) := \sum_{m=-\infty}^{\infty} 2^{-2|m|}\, \widehat{\beta}(t - 3m), \tag{18}$$

and

$$\widehat{h}(t) := \sum_{m=-\infty}^{\infty} 2^{-|m|}\, \widehat{\beta}(t - 3m). \tag{19}$$

Evidently $\widehat{k}, \widehat{h} \in \mathcal{S}$, so $k, h \in \mathcal{S}$. Further, let us define F to be such a distribution that its Fourier transform is the smooth function

$$\widehat{F}(t) := -i\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\widehat{\gamma}(t-3n)\,e^{2^n\pi it}, \quad t \in \mathbb{R}. \tag{20}$$

We know by (16) that the supports of the functions $t \mapsto \widehat{\gamma}(t-3n)\,e^{2^n\pi it}$ are pairwise disjoint for $n = 0, 1, 2\ldots$, so $\widehat{F}$ is the derivative of a bounded function, thus $\widehat{F} \in \mathcal{S}'$ and $F \in \mathcal{S}'$ are well-defined. Using the exchange formula (Proposition 5) we obtain by (20)

$$(k * F)\widehat{\ } = \sqrt{2\pi}\,\widehat{k}\,\widehat{F} = -\sqrt{2\pi}\,i\,\widehat{k}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\widehat{\gamma}(t-3n)\,e^{2^n\pi it}, \quad t \in \mathbb{R}.$$

Since the sum (18) is locally finite and by (16), (17) we have

$$\widehat{\beta}(t-3n) = 1,\ \widehat{\beta}(t-3m) = 0 \ \text{ for } \ t \in \operatorname{supp}(t \mapsto \gamma(t-3n)),\ m \neq n,$$

it follows

$$\widehat{k * F} = -\sqrt{2\pi}\,i\sum_{n=0}^{\infty}2^{-2n}\frac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{\gamma}(t-3n)\,e^{2^n\pi it}\right), \quad t \in \mathbb{R}. \tag{21}$$

The series (21) is a sum of functions from $\mathcal{D}$ and is convergent in $\mathcal{L}(\mathbb{R})$, as

$$\int\left|2^{-2n}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{\gamma}(t-3n)\,e^{2^n\pi it}\right)\right|\,\mathrm{d}t \le 2^{-2n}\|\widehat{\gamma}'\|_{\mathcal{L}} + 2^{-n}\|\widehat{\gamma}\|_{\mathcal{L}}.$$

It follows that $k * F$ is the uniform limit of a sequence of functions from $\mathcal{S}$ and so $\lim\limits_{|x|\to\infty} k * F(x) = 0$, i.e. $k * F = o'(1)$ and the assumptions of Theorem B are satisfied. On the other hand,

$$\widehat{h * F}\,(t) = -\sqrt{2\pi}\,i\sum_{n=0}^{\infty}2^{-n}\frac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{\gamma}(t-3n)\,e^{2^n\pi it}\right). \tag{22}$$

This can be proved by (19) and (20) in the same way as (21). It is clear that the series (22) is convergent in $\mathcal{E}$. Since its partial sums are uniformly bounded, the series also converges in $\mathcal{S}'$ and we obtain by (4), (5) and (6)

$$\begin{aligned} h * F\,(x) &= x\sqrt{2\pi}\sum_{n=0}^{\infty}2^{-n}\gamma(x-2^n\pi)\,e^{-3in(x-2^n\pi)} \\ &= x\sqrt{2\pi}\sum_{n=0}^{\infty}2^{-n}\gamma(x-2^n\pi)\,e^{-3inx}, \quad x \in \mathbb{R}. \end{aligned} \tag{23}$$

If $x = 2^m\pi$, $m \in \mathbb{N}$, then $e^{-3inx} = 1$ and $\gamma(x-2^n\pi) \ge 0$ for $n = 0, 1, 2, \ldots$. It follows

$$h * F\,(2^m\pi) \ge 2^m\pi\sqrt{2\pi}\,2^{-m}\gamma(0) = \pi$$

(see (16)), so the relations $h * F\,(x) = o(1)$, $x \to \infty$, and $F = o'(1)$ cannot take place. $\square$

7. The next propositions show the connection of the concept $o'(1)$ with classical concepts of the theory of distributions.

Proposition. *The following statements are equivalent for any* $F \in \mathcal{D}'$ *:*

1° $F \in \dot{\mathcal{B}}'$,
2° $F * \varphi \in \dot{\mathcal{B}}$ *for all* $\varphi \in \mathcal{D}$,
3° $\lim\limits_{|x|\to\infty} F * \varphi\,(x) = 0$ *for all* $\varphi \in \mathcal{D}$.

Proof: The equivalence of 2° and 3° is clear because $F*\varphi \in \mathcal{E}$ and $(F*\varphi)^{(n)} = F*\varphi^{(n)}$. Further, by the observation in Schwartz [12] at the end of VI, § 8, $F \in \dot{\mathcal{B}}'$ if and only if $\lim\limits_{|h|\to\infty} \langle T(t),\, \varphi(t+h)\rangle_t = 0$ for all $\varphi \in \mathcal{D}$. The last statement is equivalent to 3°, thus the proposition is proved. □

8. Proposition. *The following statements are equivalent for any* $F \in \mathcal{S}'$ *:*

1° $F * \varphi \in \dot{\mathcal{B}}$ *for all* $\varphi \in \mathcal{D}$,
2° $F * \varphi \in \dot{\mathcal{B}}$ *for all* $\varphi \in \mathcal{S}$.

Proof: Let 1° be satisfied. Then $F \in \dot{\mathcal{B}}' \subset \mathcal{B}' = (\mathcal{D}_{\mathcal{L}})'$, by Proposition 7. Thus $F \in \big(\mathcal{U}_n\big)^0$ for some $n \in \mathbb{N}$, where $\mathcal{U}_n$ is a neighbourhood of zero in $\mathcal{D}_{\mathcal{L}}$ of the form

$$\mathcal{U}_n = \Big\{\, \psi \in \mathcal{D}_{\mathcal{L}}\,;\ \textstyle\int|\psi| < \frac{1}{n},\ \int|\psi'| < \frac{1}{n},\ \dots,\ \int|\psi^{(n)}| < \frac{1}{n} \Big\}.$$

It means that for all $\psi \in \mathcal{U}_n$ we have

$$|\langle F, \psi\rangle| \le 1. \tag{24}$$

Let $\varepsilon > 0$. Since $\mathcal{D}$ is dense in $\mathcal{S}$, every $\varphi \in \mathcal{S}$ can be written in the form

$$\varphi = \varphi_0 + \psi, \quad \text{where } \ \varphi_0 \in \mathcal{D},\ \psi \in \varepsilon\,\mathcal{U}_n. \tag{25}$$

It follows

$$F * \varphi = F * \varphi_0 + F * \psi$$

with $F * \varphi_0 \in \dot{\mathcal{B}}$ by 1° and $|F * \psi\,(x)| \le \varepsilon$ for all $x \in \mathbb{R}$ by (24). Since ε was arbitrarily chosen, we have $\lim\limits_{|x|\to\infty} F * \varphi\,(x) = 0$. By the same argument, $\lim\limits_{|x|\to\infty} \big(\frac{\mathrm{d}}{\mathrm{d}x}\big)^\alpha F * \varphi\,(x) = \lim\limits_{|x|\to\infty} F * \varphi^{(\alpha)}\,(x) = 0$, i.e. 2° holds. The opposite implication is obvious. □

9. Proposition. *Let* $F \in \mathcal{S}'(\mathbb{R})$ *be a distribution whose support is bounded from the right. Then* $F = o'(1)$. *Analogically, for any* $F \in \mathcal{S}'(\mathbb{R})$ *whose support is bounded from the left, we have* $\lim\limits_{x\to-\infty} F * \varphi\,(x) = 0$ *for all* $\varphi \in \mathcal{S}$.

Proof (of the first part only): Choose a function $\omega \in \mathcal{E}$, such that $\operatorname{supp}\omega$ is bounded from the right and $\omega(t) = 1$ for $t \in \operatorname{supp} F$. Then for all $\varphi \in \mathcal{S}$ we have

$$F * \varphi\,(x) = \langle F(t), \varphi(x-t)\rangle = \langle F(t)\omega(t),\, \varphi(x-t)\rangle = \\ \langle F(t),\, \omega(t)\varphi(x-t)\rangle \;\longrightarrow\; 0 \quad \text{if} \quad x \to \infty,$$

since for $x \to \infty$ the functions $t \mapsto \omega(t)\varphi(x-t)$ converge to 0 in $\mathcal{S}$. □

10. Corollary. **1°** *Let $F \in \mathcal{S}'(\mathbb{R})$ and* $\operatorname{supp} F$ *is bounded from the left. Then $F = o'(1)$ if and only if $F \in \dot{\mathcal{B}}'$.*

2° *Let $F \in \mathcal{S}'(\mathbb{R})$, let $\omega \in \mathcal{E}$ be such that* $\operatorname{supp}\omega$ *is bounded from the left and $\omega(t) = 1$ for all t large enough. Then $F = o'(1)$ if and only if $\omega F \in \dot{\mathcal{B}}'$.*

11. Equivalent definition. *Let $F \in \mathcal{S}'$. Then $F = o'(1)$ if and only if $\lim\limits_{x\to\infty} F * \varphi\,(x) = 0$ for all $\varphi \in \mathcal{D}$.*

12. A Tauberian theorem for distributions can be formulated in the following way.

Theorem. *Let* **1°** *$K \in \mathcal{D}'_{\mathcal{L}}$, $\widehat{K}(t) \neq 0$ for all $t \in \mathbb{R}$,*

2° *$F \in \mathcal{B}'$ and*

3° *$K * F = o'(1)$.*

Then $H * F = o'(1)$ for any $H \in \mathcal{D}'_{\mathcal{L}}$.

Particularly, with $H = \delta$ we get $F = o'(1)$.

Remark. This formulation is one-sided, for $x \to \infty$. The theorem is valid also for $x \to -\infty$, i.e. if $F = o'(1)$ means $\lim\limits_{x\to-\infty} F * \varphi\,(x) = 0$ for all $\varphi \in \mathcal{S}$. In the two-sided formulation we can avoid the concept $o'(1)$ writing simply $K * F \in \dot{\mathcal{B}}'$, $H * F \in \dot{\mathcal{B}}'$ (see Propositions 7 and 8). This theorem is a more exact generalization for distributions of the classical Theorem A than the (non valid) Theorem B which has the harder assumption 1° and the weaker assumption 2° in comparison with this theorem.

The following proof is based on the proof of Theorem A in [4]. We divide this proof in several steps assuming in each of them that the assumptions of Theorem 12 are satisfied.

Step 1. There is a function $k \in \mathcal{D}_{\mathcal{L}}$ such that $\widehat{k}(t) \neq 0$ for all $t \in \mathbb{R}$, $k * F$ is a bounded function and $k * F\,(x) = o(1)$, $x \to \infty$. So we can replace K with k in Theorem 12.

Indeed, it is sufficient to take $k = \varphi * K$, for some $\varphi \in \mathcal{S}$ with $\widehat{\varphi}(t) \neq 0$ for all $t \in \mathbb{R}$, e.g. $\varphi(x) = e^{-x^2}$. Associativity of the convolution $\varphi * K * F$ follows by Proposition 4.1°.

Step 2. If $h \in \mathcal{L}$ and $\operatorname{supp}\widehat{h}$ is compact, then there is a (well-defined) function $g \in \mathcal{L}$, satisfying $g * k = h$.

Indeed, thanks to (9) this assertion says that $\widehat{h}/\widehat{k}$ is the Fourier transform of some function $g \in \mathcal{L}$. This is proved by Hardy [4, 12.5, Theorem 229].

STEP 3. If $h \in \mathcal{L}$ and $\operatorname{supp} \widehat{h}$ is compact, then $h * F\,(x) = o(1)$, $x \to \infty$.

PROOF: By the preceding step, pick a function $g \in \mathcal{L}$, satisfying $h = g * k$. So $h*F = g*k*F$; associativity follows by Proposition 4.2. Since by Step 1 $k*F$ is a bounded function and $k*F\,(x) = o(1)$, $x \to \infty$, we have $h*F\,(x) = g*k*F\,(x) = o(1)$, as well.

PROOF OF THEOREM 12: We have to prove that $\varphi * H * F\,(x) = o(1)$, $x \to \infty$ for any $\varphi \in \mathcal{D}$. Pick a function $\gamma \in \mathcal{S}$ satisfying (16) and designate $\alpha_1 := \gamma / \int \gamma$. Then $\alpha_1 \in \mathcal{S}$, $\alpha_1 \geq 0$, $\int \alpha_1 = 1$, $\widehat{\alpha}_1 \in \mathcal{D}$. Put further $\alpha_n(x) = n\alpha_1(nx)$, $n \in \mathbb{N}$ and write

$$\varphi * H * F = \alpha_n * \varphi * H * F + (\delta - \alpha_n) * \varphi * H * F. \tag{26}$$

Since by the assumptions $H \in \mathcal{D}'_{\mathcal{L}}$, we have $\alpha_n * H \in \mathcal{L}$ (see the notes in the paragraph 4). As the distribution $\widehat{\alpha_n * H} = \sqrt{2\pi}\,\widehat{\alpha}_n\,\widehat{H}$ has a compact support, we obtain by Step 3

$$\alpha_n * H * F\,(x) = o(1),\ x \to \infty,$$

and of course

$$\alpha_n * \varphi * H * F\,(x) = o(1),\ x \to \infty. \tag{27}$$

By the assumptions $H * F \in \mathcal{B}'$, so $\varphi * H * F \in \mathcal{B}$ (paragraph 4). It follows that the function $\varphi * H * F$ is bounded and uniformly continuous and so the functions $(\delta - \alpha_n) * \varphi * H * F$ converge uniformly to zero if $n \to \infty$. Thus by (26) the function $\varphi * H * F$ is the uniform limit of the functions $\alpha_n * \varphi * H * F$ and by (27) $\varphi * H * F\,(x) = o(1)$, $x \to \infty$, as well.

13. Corollary. *Let* **1°** $K \in \mathcal{O}'_C$, $\widehat{K}(t) \neq 0$ *for all* $t \in \mathbb{R}$,
2° $F \in \mathcal{S}'$ *be such that* $K * F \in \mathcal{B}'$ *and*
3° $K * F = o'(1)$.
Then $H * F = o'(1)$ *for any* $H \in \mathcal{D}'_{\mathcal{L}}$ *for which the* $\mathcal{S}'$*-convolution* $H * F$ *exists and is from* $\mathcal{B}'$.

PROOF: Thanks to Theorem 12 we only need to prove

$$K * H * F = o'(1), \tag{28}$$

associativity is guaranteed by Proposition 4.1°. Under the assumptions of the Corollary, the relation (28) follows from Theorem 12 used for the distributions δ and $K * F$ in the place of K and F respectively. □

Remark. Theorem 12 can be applied for $F \in \mathcal{B}'$ only while this Corollary is also applicable to $F \in \mathcal{S}' \setminus \mathcal{B}'$ under certain restrictions (namely $K \in \mathcal{O}'_C$, $K * F \in \mathcal{B}'$). Counterexample 6 shows that there are such distributions. Indeed, take F, k, h by this counterexample (formulas (18)–(20)). We have $F \in \mathcal{S}' \setminus \mathcal{B}'$ as otherwise we would have $F = o'(1)$ by Theorem 12. Further, we can easily see by (23) that $F * h$ is a bounded function, so $F * h \in \mathcal{B}'$. By the same argument $F * k \in \mathcal{B}'$. Thus, by Corollary 13 we have $F * h = o'(1)$ and yet $F * h\,(x) = o(1)$, $x \to \infty$, does not hold, as we have shown in the counterexample.

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