On $\mathcal{L}^{2,n}_{loc}$-regularity for the gradient of a weak solution to nonlinear elliptic systems

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Abstract. Interior $\mathcal{L}^{2,n}_{loc}$-regularity for the gradient of a weak solution to nonlinear second order elliptic systems is investigated.

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1. Introduction

In this paper we consider the problem of the regularity of the first derivatives of weak solutions to a nonlinear elliptic system

$$-D_\alpha \left(A_i^\alpha \left(Du\right)\right) = 0, \qquad (i = 1, \dots, N) \tag{1}$$

in a bounded open set $\Omega \subset R^n$. Throughout the whole text we use the summation convention over repeated indexes.

If $n \geq 3$, it is known that Du may not be continuous. Examples are provided by nonregular solutions of elliptic systems presented by Nečas in [8]. Campanato in [2] proved that $Du \in \mathcal{L}^{2,\lambda}_{loc}\left(\Omega, R^N\right)$ with $\lambda(n) < n$, and $u \in C^{0,\alpha}_{loc}\left(\Omega, R^N\right)$ for some $\alpha < 1$ if $n = 3, 4$. In this paper we give sufficient condition on $\mathcal{L}^{2,n}_{loc}$-regularity for the gradient of a weak solution to (1). Recall that if $Du \in \mathcal{L}^{2,n}_{loc}$, then u is locally Zygmund continuous.

2. Preliminaries

Let Ω be a bounded open set in R^n with points $x = (x_1, \dots x_n)$, $n \geq 3$. The notation $\Omega_0 \Subset \Omega$ means that the closure of Ω_0 is contained in Ω, i.e. $\overline{\Omega}_0 \subset \Omega$. For the sake of simplicity we denote by $|\cdot|$ and $(.,.)$ the norm and scalar product in R^n, R^N and R^{nN}. If $x \in R^n$ and r is a positive real number, we set $B\left(x, r\right) = \{y \in R^n \colon |y - x| < r\}$, i.e. the open ball in $R^n, \Omega\left(x, r\right) = B\left(x, r\right) \cap \Omega$. By $\mu\left(\Omega\left(x, r\right)\right)$ we denote the n-dimensional Lebesgue measure of $\Omega\left(x, r\right)$. A bounded domain $\Omega \subset R^n$ is said to be of type $\mathcal{A}$ if there exists a constant $\mathcal{A} > 0$ such that for every $x \in \overline{\Omega}$ and all $0 < r < diam\, \Omega$ it holds $\mu\left(\Omega\left(x, r\right)\right) \geq \mathcal{A} r^n$.

Let $u \colon \Omega \to R^N$, $N \geq 1$, $u\left(x\right) = \left(u^1\left(x\right), \dots, u^N\left(x\right)\right)$ be a vector-valued function and $Du = \left(D_1 u, \dots, D_n u\right)$, $D_\alpha = \partial/\partial x_\alpha$.

By $u_{x,r} = \mu^{-1}(\Omega(x,r)) \int_{\Omega(x,r)} u(y)\, dy = \fint_{\Omega(x,r)} u(y)\, dy$ we denote mean value of u over the set $\Omega(x,r)$ provided that $u \in L^1(\Omega, R^N)$. Besides usually used spaces as $C_0^\infty(\Omega, R^N)$, the Hölder spaces $C^{0,\alpha}(\overline{\Omega}, R^N)$ and the Sobolev spaces $H^{k,p}(\Omega, R^N)$, $H^{k,p}_{loc}(\Omega, R^N)$, $H^{k,p}_0(\Omega, R^N)$ (see e.g. [1], [6], [7] for definitions and basic properties) we use the following Campanato and Morrey spaces.

Definition 1 (Campanato and Morrey spaces)**.** Let $\lambda \in [0,n]$, $q \in [1,\infty)$. The Morrey space $L^{q,\lambda}(\Omega, R^N)$ is the subspace of such functions $u \in L^q(\Omega, R^N)$ for which $||u||^q_{L^{q,\lambda}(\Omega,R^N)} = \sup\{r^{-\lambda}\int_{\Omega(x,r)} |u(y)|^q\, dy \colon r > 0, x \in \Omega\}$ is finite.

Let $\lambda \in [0, n+q]$, $q \in [1,\infty)$. The Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega, R^N)$ and $\mathcal{L}_1^{q,\lambda}(\Omega, R^N)$ are subspaces of such functions $u \in L^q(\Omega, R^N)$ for which $[u]^q_{\mathcal{L}^{q,\lambda}(\Omega,R^N)} = \sup\{r^{-\lambda}\int_{\Omega(x,r)} |u(y) - u_{x,r}|^q\, dy \colon r > 0, x \in \Omega\}$ is finite and $[u]^q_{\mathcal{L}_1^{q,\lambda}(\Omega,R^N)} = \sup\{\inf\{r^{-\lambda}\int_{\Omega(x,r)} |u(y) - P(y)|^q\, dy \colon P \in \mathcal{P}_1\} \colon r > 0, x \in \Omega\}$ is finite. Here $\mathcal{P}_1$ is the set of all polynomials in n variables and of degree ≤ 1. Let us denote $||u||_{L^{q,\lambda}}$, $||u||_{\mathcal{L}^{q,\lambda}} = ||u||_{L^q} + [u]_{\mathcal{L}^{q,\lambda}}$ and $||u||_{\mathcal{L}_1^{q,\lambda}} = ||u||_{L^q(\Omega,R^N)} + [u]_{\mathcal{L}_1^{q,\lambda}}$.

Remark 1. It is worth to recall a trivial however important property saying that $\int_\Omega |u - u_\Omega|^2\, dx = \min\{\int_\Omega |u - c|^2\, dx \colon c \in R^N\}$ for every $u \in L^2(\Omega, R^N)$.

Definition 2. The Zygmund class $\Lambda^1(\overline{\Omega}, R^N)$ is the subspace of such functions $u \in C^0(\overline{\Omega}, R^N)$ for which $[u]_{\Lambda^1(\overline{\Omega},R^N)} = \sup\{|u(x) + u(y) - 2u((x+y)/2)| / |x-y| : x, y, (x+y)/2 \in \overline{\Omega}\}$ is finite.

For more details see [1], [4], [6], [7]. In particular, we will use the following result.

Proposition 1. *Let Ω be of type $\mathcal{A}$ and $1 \le q < \infty$. Then it holds*

(a) $L^{q,\lambda}(\Omega, R^N)$, $\mathcal{L}^{q,\lambda}(\Omega, R^N)$ *and* $\mathcal{L}_1^{q,\lambda}(\Omega, R^N)$ *equipped with norms* $||u||_{L^{q,\lambda}}$, $||u||_{\mathcal{L}^{q,\lambda}}$ *and* $||u||_{\mathcal{L}_1^{q,\lambda}}$ *are Banach spaces.*
(b) $\mathcal{L}^{q,\lambda}(\Omega, R^N)$ *is isomorphic to the* $C^{0,(\lambda-n)/q}(\overline{\Omega}, R^N)$ *if* $n < \lambda \le n+q$,
(c) $L^{q,n}(\Omega, R^N)$ *is isomorphic to the* $L^\infty(\Omega, R^N) \subsetneq \mathcal{L}^{q,n}(\Omega, R^N)$,
(d) $\mathcal{L}_1^{2,n+2}(\Omega, R^N)$ *is isomorphic to the* $\Lambda^1(\overline{\Omega}, R^N)$,
(e) $C^{0,1}(\overline{\Omega}, R^N) \subsetneq \Lambda^1(\overline{\Omega}, R^N) \subsetneq \bigcap_{0<\alpha<1} C^{0,\alpha}(\overline{\Omega}, R^N)$.

Further, we suppose

(i) there is an $M > 0$ such that for every $p \in R^{nN}$

$$|A_i^\alpha(p)| \le M(1 + |p|), \tag{2}$$

(ii) $A_i^\alpha(p)$ are differentiable functions on R^{nN} with the bounded and continuous derivatives, i.e.

$$\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}(p)\right| \le M \quad \text{for every} \;\; p \in R^{nN}, \tag{3}$$

(iii) the strong ellipticity condition, i.e. there exists $\nu > 0$ such that for every $p, \xi \in R^{nN}$

$$\frac{\partial A_i^\alpha}{\partial p_\beta^j}(p)\, \xi_\alpha^i \xi_\beta^j \ge \nu |\xi|^2. \tag{4}$$

From (ii) it follows (see [3, p. 169]) the existence of a real function $\omega(s)$ defined on $[0,\infty)$, which is nonnegative, bounded, nondecreasing, concave, $\omega(0) = 0$ (moreover, ω is right continuous at 0 for uniformly continuous $\partial A_i^\alpha / \partial p_\beta^j$) and such that for all $p, q \in R^{nN}$

$$\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}(p) - \frac{\partial A_i^\alpha}{\partial p_\beta^j}(q)\right| \le \omega\left(|p-q|^2\right). \tag{5}$$

By a weak solution of (1) we mean a function $u \in H^{1,2}\left(\Omega, R^N\right)$ satisfying

$$\int_\Omega A_i^\alpha (Du)\, D_\alpha \varphi^i \, dx = 0 \tag{6}$$

for every $\varphi \in H_0^{1,2}\left(\Omega, R^N\right)$.

We will also consider the pair of complementary Young functions

$$\Phi(t) = t \ln_+ at \;\; \text{for} \;\; t \ge 0, \qquad \Psi(t) = \begin{cases} t/a & \text{for} \quad 0 \le t < 1, \\ e^{t-1}/a & \text{for} \quad t \ge 1, \end{cases} \tag{7}$$

where $a > 0$ is a constant, $\ln_+ at = 0$ for $0 \le t < 1/a$ and $\ln_+ at = \ln at$ for $t \ge 1/a$. Recall Young's inequality

$$ts \le \Phi(t) + \Psi(s), \quad t, s \ge 0.$$

For our consideration we also need to introduce quasiconvex functions.

Definition 3 ([5, p. 4]). A function $G\colon [0,\infty) \to R$ is said to be quasiconvex (quasiconcave) on $[0,\infty)$ if there exist a convex (concave) function g ($\tilde g$) and a constant $c > 0$ ($\tilde c > 0$) such that

$$g(t) \le G(t) \le cg(ct), \quad (\tilde g(t) \le G(t) \le \tilde c \tilde g(\tilde c t)) \quad \text{for} \;\; t \ge 0.$$

Next, we will need the following properties of quasiconvex functions:

Lemma 1 ([5, p. 4]). *Let us consider three statements:*

(a) *$G(t)$ is quasiconvex (quasiconcave) on $[0,\infty)$;*
(b) *for all $t_1,t_2\in[0,\infty)$ and all $\lambda\in(0,1)$*

$$G\big(\lambda t_1+(1-\lambda)t_2\big)\le k_1\big(\lambda G\left(k_1t_1\right)+\left(1-\lambda\right)G\left(k_1t_2\right)\big)$$

$$\Big(\lambda G\left(t_1\right)+\left(1-\lambda\right)G\left(t_2\right)\le l_1G\big(l_1\left(\lambda t_1+\left(1-\lambda\right)t_2\right)\big)\Big);$$

(c) *there exists a constant k_2 (l_2) such that for all $u\in L^2_{loc}\left(\Omega,R^N\right)$ and all balls $B\left(x,r\right)\subset\Omega$*

$$G\Big(\quad \fint_{B(x,r)} |u|^2\,dy\Big)\le k_2 \fint_{B(x,r)} G\left(k_2|u|^2\right)dy,$$

$$\Big(\fint_{B(x,r)} G\left(|u|^2\right)dy\le l_2G\Big(l_2 \fint_{B(x,r)} |u|^2\,dy\Big)\Big).$$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proposition 2. *For all $u,v\in L^2_{loc}\left(\Omega,R^N\right)$, all balls $B\left(x,r\right)\subset\Omega$ and an arbitrary nondecreasing quasiconvex function G on $[0,\infty)$ we have*
(a)

$$\int_{B(x,r)} G(|u+v|^2)\,dy\le\frac{k_1}{2}\Big(\int_{B(x,r)} G\big(4k_1\,|u|^2\big)\,dy+\int_{B(x,r)} G\big(4k_1\,|v|^2\big)\,dy\Big),$$

(b)

$$\int_{B(x,r)} G\big(\big|u-u_{x,r}\big|^2\big)\,dy\le c_1\int_{B(x,r)} G\big(c_2\,|u-c|^2\big)\,dy,$$

where $c_1=\max\{k_1/2,k_2\}$, $c_2=\max\{4k_1,4k_1k_2\}$ and $c\in R$ is arbitrary.

PROOF: (a) It follows from Lemma 1 (b).

(b) From (a) we get

$$\int_{B_r} G\big(\left|u-u_{x,r}\right|^2\big)\,dy \le \frac{k_1}{2}\Big(\int_{B_r} G\big(4k_1\left|u-c\right|^2\big)\,dy + \int_{B_r} G\big(4k_1\left|c-u_{x,r}\right|^2\big)\,dy\Big).$$

Now, by means of Hölder's inequality and Lemma 1 (c)

$$\begin{aligned}\int_{B_r} G\big(4k_1\left|c-u_{x,r}\right|^2\big)\,dy &= \mu\,(B_r)\,G\big(4k_1\left|c-u_{x,r}\right|^2\big)\\ &= \mu\,(B_r)\,G\Big(4k_1\Big|c-\fint_{B_r} u\,(y)\;dy\Big|^2\Big) = \mu\,(B_r)\,G\Big(\frac{4k_1}{\mu^2\,(B_r)}\Big|\int_{B_r}(u(y)-c)\,dy\Big|^2\Big)\\ &\le \mu\,(B_r)\,G\Big(\fint_{B_r} 4k_1\left|u\,(y)-c\right|^2\;dy\Big) \le k_2\int_{B_r} G\big(4k_1k_2\left|u(y)-c\right|^2\big)\,dy\end{aligned}$$

and the result follows easily. □

Lemma 2 ([9, p. 37])**.** *Let $\varphi\colon[0,\infty]\to[0,\infty]$ be a monotone function which is absolutely continuous on every closed interval of finite length. If $v\ge 0$ is measurable and $E(t)=\{x\in R^n\colon v(x)>t\}$, then*

$$\int_{R^n}\varphi\circ v\,dx = \int_0^\infty \mu(E(t))\varphi'(t)\,dt.$$

Proposition 3. *Let $v\in L^2_{loc}(\Omega,R^m)$, $B(x,\sigma)\subset\Omega$, $a>0$ and $s\in[1,\infty)$ be arbitrary. If the inequality*

$$\fint_{B(x,\tau\sigma)}\left|v-v_{x,\tau\sigma}\right|^2 dy \le \fint_{B(x,\sigma)}\left|v-v_{x,\sigma}\right|^2 dy$$

holds for some $\tau\in(0,1)$, then there exists a constant b such that

$$\fint_{B(x,\tau\sigma)}\ln_+^s\Big(a|v-v_{x,\tau\sigma}|^2\Big)\;dy \le b\fint_{B(x,\sigma)}\ln_+^s\Big(a|v-v_{x,\sigma}|^2\Big)\;dy.$$

For the constant b we have the following estimate

$$b\le h\left(\fint_{B(x,\sigma)}\left|v-v_{x,\sigma}\right|^2\;dy\right)\left(\fint_{B(x,\sigma)}\ln_+^s\left(a\left|v-v_{x,\sigma}\right|^2\right)\;dy\right)^{-1},$$

where $h(t) = (s/e(s-1))^{s/(s-1)}\, at$, $t \in [0, e^{s/(s-1)}/a]$ *and* $\ln^{s/(s-1)}(at)$, $t \in (e^{s/(s-1)}/a, \infty)$.

Proof: We set $E_{\tau\sigma}(t) = \{y \in B(x,\tau\sigma)\colon |v - v_{x,\tau\sigma}|^2 > t\}$ for $t \geq 0$ and $0 < \tau \leq 1$. From Lemma 2 and by means of integration by parts we get

$$\fint_{B_{\tau\sigma}} \ln_+^s \left(a|v - v_{\tau\sigma}|^2\right) dy = \frac{s}{\mu(B_{\tau\sigma})} \int_{1/a}^{\infty} \mu\left(E_{\tau\sigma}(t)\right) \frac{\ln^{s-1}(at)}{t}\, dt$$

$$= \frac{s}{\mu(B_{\tau\sigma})} \left[\frac{\ln^{s-1}(at)}{t} \int_0^t \mu\left(E_{\tau\sigma}(\lambda)\right) d\lambda \right]_{1/a}^{\infty}$$

$$+ \frac{s}{\mu(B_{\tau\sigma})} \int_{1/a}^{\infty} \left(\int_0^t \mu\left(E_{\tau\sigma}(\lambda)\right) d\lambda \right) \frac{\ln^{s-1}(at) - (s-1)\ln^{s-2}(at)}{t^2}\, dt.$$

For the sake of simplicity we put $V_r = \fint_{B(x,r)} \left|v - v_{x,r}\right|^2 dy$. The first integral is zero and on the second integral we can use the mean value theorem for the integrals and we have for some $1/a < \xi_{\tau\sigma}, \xi_\sigma < \infty$,

$$\fint_{B_{\tau\sigma}} \ln_+^s \left(a|v - v_{\tau\sigma}|^2\right) dy = sV_{\tau\sigma} \int_{\xi_{\tau\sigma}}^{\infty} \frac{\ln^{s-1}(at) - (s-1)\ln^{s-2}(at)}{t^2}\, dt$$

$$= \frac{s\ln^{s-1}(a\xi_{\tau\sigma})}{\xi_{\tau\sigma}} V_{\tau\sigma} = \frac{\xi_\sigma \ln^{s-1}(a\xi_{\tau\sigma})}{\xi_{\tau\sigma}\ln^{s-1}(a\xi_\sigma)} \frac{V_{\tau\sigma}}{V_\sigma} \fint_{B_\sigma} \ln_+^s \left(a|v - v_{x,\sigma}|^2\right) dy$$

$$= b(\tau) \fint_{B_\sigma} \ln_+^s \left(a|v - v_{x,\sigma}|^2\right) dy.$$

Now the result follows from Lemma 1 (c). □

3. The result

For $x \in \Omega$, $r > 0$ we set $U_r = U(x,r) = \fint_{\Omega(x,r)} \left|Du - (Du)_{x,r}\right|^2 dy$, $d_x = dist(x, \partial\Omega)$ and $\alpha_n = \mu\left(B(0,1)\right)$. We define $\mathcal{S}_0 = \{x \in \Omega\colon \overline{\lim}_{r\to 0+} U(x,r) > 0\}$.

Remark 2. Let u be a solution of (1). It is well known (see [9, pp. 75, 122]) that $\lim_{r\to 0+} U(x,r) = 0$ for all $x \in \Omega \setminus E$ where $n-2+\beta$ dimensional Hausdorf measure $H^{n-2+\beta}(E) = 0$ for every $\beta > 0$.

Now we can formulate the main theorem.

Theorem. *Let $u \in H^{1,2}\left(\Omega, R^N\right)$ be a weak solution to the nonlinear system* (1) *under the hypotheses* (i), (ii), (iii). *Let $x \in \mathcal{S}_0$ be arbitrary and suppose that there exists $d \in (0, d_x/2)$ such that*

$$(8) \qquad \frac{Kl_2\omega^2}{\nu^2}\Bigg(b \fint_{B(x,2d)} \ln_+^{q/(q-1)}\Big(\frac{4l_2\omega^2\left|Du-(Du)_{x,2d}\right|^2}{CU_{2d}}\Big)\,dy\Bigg)^{1-1/q} < \frac{1}{4}\tau^n,$$

where $K = c\,(n,N,q)\,(M/\nu)^8$, $\tau = (2^{n+5}A)^{-1/2}$, l_2, A are the constants from Lemma 1 (c), *Lemma* 3, $\omega = \omega(2^n l_2 U_{2d})$, *$\omega$ is from* (5), $C = 2^{n-8}\nu^2\tau^n/\alpha_n A$ *and b is the constant from Proposition* 3 *for the case $a = 1/CU_{2d}$, $\sigma = 2d$, $v = 2\sqrt{l_2}\omega Du$, $s = q/(q-1)$ where $q \in (1, n/(n-2)]$. Then there exists a ball $B(x, r_x) \subset \Omega$ such that $Du \in \mathcal{L}^{2,n}\left(B(x,r_x), R^{nN}\right)$ and*

$$(9) \quad [Du]^2_{\mathcal{L}^{2,n}(B(x,r_x),R^{nN})} \le \max\{2^n(4A\tau^{-n}+1)U_{2d}, \mu^{-1}(B_{2d}) \int_\Omega |Du-(Du)_\Omega|^2\,dx\}.$$

Proposition 4. *Set $\omega_\infty = \lim_{t\to\infty}\omega(t)$, $V_1 = c_1\,(M/\nu)^{3n+8}\,(\omega_\infty/\nu)^2$ and $V_2 = c_2\,(M/\nu)^{3n+6}\,(\omega_\infty/\nu)^2$. If*

$$(10) \qquad V_2 \le e^q \;\&\; q^{q-1}V_1V_2^{1-1/q} < 1 \;\text{ or }\; V_2 > e^q \;\&\; V_1\ln^{q-1}V_2 < 1,$$

then condition (8) *holds for every $x \in \mathcal{S}_0$. Here $q \in (1, n/(n-2)]$, $c_1 = c_1\,(n,N,q)$ and $c_2 = c_2\,(n,N)$.*

PROOF: Let $x \in \mathcal{S}_0$ and $d \in (0, d_x/2)$ be arbitrary such that $U\,(x,2d) > 0$. From Proposition 3 it follows that the left hand side of (8) is equal or less than $Kl_2\omega_\infty^2 h^{1-1/q}\left(4\omega_\infty^2 U_{2d}\right)/\nu^2$. From the definition of the function $h\,(t)$ and assumption (10) it follows that (8) is satisfied.

□

Example. We can consider the system (1) for $n = 3$, $N = 2$ where $A_i^\alpha(p) = \left(a\,\delta_{ij}\delta_{\alpha\beta} + b\,\delta_{i\alpha}\delta_{j\beta}\arctan|p|^2/2\pi\right)p_\beta^j$, a, b are constants, $0 < b/6 < a$. We have

$$\frac{\partial A_i^\alpha}{\partial p_\beta^j}(p)\,\xi_\alpha^i\xi_\beta^j \ge (a-b/6)\,|\xi|^2, \quad \forall\,\xi, p \in R^6,$$

$\omega_\infty \le b$ and $\left|\partial A_i^\alpha/\partial p_\beta^j(p)\right| \le M = a+b$. Setting $P = b/a$ we get that $V_1 < 4c_1P^2\,(1+P)^{3n+8}/(1-P/6)^{3n+10}$, $V_2 < 4c_2P^2\,(1+P)^{3n+6}/(1-P/6)^{3n+8}$ and it is not difficult to see that (10) is satisfied for P sufficiently small.

Corollary 1. *Let $\Omega_0 \Subset \Omega$ be arbitrary and of type $\mathcal{A}$ and the assumptions of Theorem be satisfied for every $x \in \overline{\Omega}_0 \cap \mathcal{S}_0$. Then there are constants $\mathcal{U}, d_0, r_0 > 0$ such that $Du \in \mathcal{L}^{2,n}\left(\Omega_0, R^{nN}\right)$ and the following estimate*

$$[Du]^2_{\mathcal{L}^{2,n}(\Omega_0,R^{nN})} \le \max\Big\{2^n(4A\tau^{-n}+1)\,\mathcal{U},\quad \mu^{-1}(B_{2d_0})\int_\Omega |Du-(Du)_\Omega|^2\,dx,\quad (\mathcal{A}r_0^n)^{-1}\int_{\Omega_0}\left|Du-(Du)_{\Omega_0}\right|^2\,dx\Big\} \tag{11}$$

holds.

Proof: From Remark 2, Theorem and the definition of the set $\mathcal{S}_0$ it follows that for every $x \in \overline{\Omega}_0$ there exists $B(x,r_x) \subset \Omega$ such that $Du \in \mathcal{L}^{2,n}\left(B(x,r_x), R^{nN}\right)$. As $\overline{\Omega}_0$ is the compact set and the system balls $\{B(x,r_x)\}$ covers of $\overline{\Omega}_0$ we can choose a finite subcover $\{B(x_j,r_{x_j})\}_{j=1}^m$. If we set $\mathcal{U} = \max\{U(x_j,2d_{x_j})\colon 1 \le j \le m\}$, $r_0 = \min\{r_{x_j}\colon 1 \le j \le m\}$ and $d_0 = \min\{d_{x_j}\colon 1 \le j \le m\}$, then the estimate follows from Remark 1. □

Corollary 2. *Let the assumptions of Corollary* 1 *be satisfied. Then $u \in \Lambda^1\left(\overline{\Omega}_0, R^N\right)$.*

Proof: It follows from Proposition 1 (d), Poincaré's inequality and Corollary 1. □

4. Lemmas

The statement of the following lemma is well known (see e.g. [1], [3], [7], [8]).

Lemma 3. *Let $v \in H^{1,2}\left(\Omega, R^N\right)$ be a weak solution to the system* (1) *satisfying* (i), (ii) *and* (iii), *where $\partial A_i^\alpha/\partial p_\beta^j$ are the constants. Then there exists a constant $A = c\,(n,N)\,(M/\nu)^6$ such that for every $x \in \Omega$ and $0 < \sigma \le R \le dist\,(x,\partial\Omega)$ the following estimate holds*

$$\int\limits_{B(x,\sigma)} |Dv(y)-(Dv)_{x,\sigma}|^2\,dy \le A\left(\frac{\sigma}{R}\right)^{n+2}\int\limits_{B(x,R)} |Dv\,(y)-(Dv)_{x,R}|^2\,dy.$$

The following lemma is possible to derive by the difference quotient method (see e.g. [1], [3], [7], [8]).

Lemma 4. *Let $u \in H^{1,2}\left(\Omega, R^N\right)$ be a weak solution to the system* (1) *satisfying* (i), (ii) *and* (iii). *Then $u \in H^{2,2}_{loc}\left(\Omega, R^N\right)$ and for all $x \in \Omega$, $0 < \sigma < \varrho \le dist(x,\partial\Omega))$ we have*

$$\int\limits_{B(x,\sigma)} |D^2u|^2\,dy \le \frac{6n\,(M/\nu)^2}{(\varrho-\sigma)^2}\int\limits_{B(x,\varrho)} |Du-(Du)_{x,\varrho}|^2\,dy.$$

Lemma 5 ([6]). *Let $1 \le p,\ q < \infty$, $p^{-1} - q^{-1} \le n^{-1}$, $R > 0$, $x \in R^n$. Then for $u \in H^{1,p}\left(B(x,R), R^N\right)$ we have*

$$\left(\int_{B(x,R)} |u(y)|^q \, dy\right)^{1/q} \le cR^{1+n/q-n/p}\left(R^{-p}\int_{B(x,R)} |u(y)|^p \, dy + \int_{B(x,R)} |Du(y)|^p \, dy\right)^{1/p},$$

where $c = c(n,N,p,q)$ is a constant independent of x, R and u.

Lemma 6. *Let $u \in H^{1,2}\left(\Omega, R^N\right)$ be a weak solution to* (1) *satisfying* (i), (ii) *and* (iii). *Then for every ball $B(x,2R) \subset \Omega$ and an arbitrary constant $a > 0$ we have*

$$\int_{B(x,R)} \left|Du - (Du)_{x,R}\right|^2 \ln_+\left(a\left|Du - (Du)_{x,R}\right|^2\right) dy \le c\left(\frac{M}{\nu}\right)^2 \left(\fint_{B(x,2R)} \ln_+^{q/(q-1)}\left(4a\left|Du - (Du)_{x,2R}\right|^2\right) dy\right)^{1-1/q} \int_{B(x,2R)} \left|Du - (Du)_{x,2R}\right|^2 dy,$$

where $1 < q \le n/(n-2)$ and $c = c(n,N,q)$.

Proof: Let $x \in \Omega$ and $0 \le R \le \frac{1}{4} dist\,(x, \partial\Omega)$. We denote $B_R = B(x,R)$ for simplicity. From Lemma 4 it follows that $Du \in H^{1,2}_{loc}\left(\Omega, R^N\right)$. By means of Sobolev's imbedding theorem $H^{1,2}\left(B_R, R^N\right) \hookrightarrow L^s\left(B_R, R^N\right)$, where $B_R \subset \Omega$ be arbitrary and $1 \le s \le 2n/(n-2)$. From this we obtain by Proposition 2 (b) and Lemma 5

$$\begin{aligned}
&\int_{B_R} |Du - (Du)_R|^2 \ln_+\left(a\,|Du - (Du)_R|^2\right) dy \\
&\le 4\int_{B_R} |Du - (Du)_{2R}|^2 \ln_+\left(4a\,|Du - (Du)_{2R}|^2\right) dy \\
&\le 4\left(\int_{B_R} |Du - (Du)_{2R}|^{2q} \, dy\right)^{1/q}\left(\int_{B_R} \ln_+^{q/(q-1)}\left(4a\,|Du - (Du)_{2R}|^2\right) dy\right)^{1-1/q} \\
&\le cR^{n(1/q-1)+2}\left(R^{-2}\int_{B_R} |Du - (Du)_{2R}|^2 + \int_{B_R} \left|D^2u\right|^2 \, dy\right) \times
\end{aligned}$$

$$\times \left(\int_{B_R} \ln_+^{q/(q-1)} \left(4a \left|Du - (Du)_{2R}\right|^2\right) dy \right)^{1-1/q}$$

$$\leq c \left(\frac{M}{\nu}\right)^2 R^{-n(1-1/q)} \int_{B_{2R}} \left|Du - (Du)_{2R}\right|^2 \, dy \, \times$$

$$\times \left(\int_{B_R} \ln_+^{q/(q-1)} \left(4a \left|Du - (Du)_{2R}\right|^2\right) dy \right)^{1-1/q}$$

and we finally obtain the result. □

5. Proof of Theorem

Set $A_{ij}^{\alpha\beta}(\zeta) = \partial A_i^\alpha / \partial p_\beta^j(\zeta)$, $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}((Du)_R)$,

$$\tilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left((Du)_R + t\left(Du - (Du)_R\right)\right) dt,$$

$B_R = B(x,R)$ and $U_R = U(x,R)$ for simplicity. Then the system (1) can be rewritten as

$$-D_\alpha \left(A_{ij,0}^{\alpha\beta} D_\beta u^j\right) = -D_\alpha \left(\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right).$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$\begin{cases} - \quad D_\alpha \left(A_{ij,0}^{\alpha\beta} D_\beta v^j\right) = 0 \quad \text{in} \quad B_R \\ \qquad v - u \in H_0^{1,2}\left(B_R, R^N\right). \end{cases}$$

For every $0 < \sigma \leq R$ from Lemma 3 it follows

$$\int_{B_\sigma} \left|Dv - (Dv)_\sigma\right|^2 dy \leq A \left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} \left|Dv - (Dv)_R\right|^2 dy,$$

hence

$$\int_{B_\sigma} \left|Du - (Du)_\sigma\right|^2 dy \leq 2A \left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} \left|Dv - (Dv)_R\right|^2 dy + 2 \int_{B_R} |Dw|^2 \, dy. \tag{12}$$

Now $w \in H_0^{1,2}\left(B_R, R^N\right)$ satisfies

$$\int_{B_R} A_{ij,0}^{\alpha\beta} D_\beta w^j D_\alpha \varphi^i \, dy \leq \int_{B_R} \left|A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right| \left|D_\beta u^j - \left(D_\beta u^j\right)_R\right| \left|D_\alpha \varphi^i\right| dy$$

$$\leq \left(\int_{B_R} \omega^2 \left(\left|Du - (Du)_R\right|^2\right) \left|Du - (Du)_R\right|^2 \, dy \right)^{1/2} \left(\int_{B_R} |D\varphi|^2 \, dy \right)^{1/2}$$

for any $\varphi \in H_0^{1,2}\left(B_R, R^N\right)$, where ω is from (5). Hence, choosing $\varphi = w$, we get

$$\nu^2 \int\limits_{B_R} |Dw|^2\, dy \le \int\limits_{B_R} \omega^2 \left(\left|Du - (Du)_R\right|^2\right) \left|Du - (Du)_R\right|^2 dy.$$

Now applying the Young inequality (with the complementary functions (7)) on the right-hand side, we obtain for every $\varepsilon > 0$

$$\nu^2 \int\limits_{B_R} |Dw|^2\, dy \le \varepsilon \int\limits_{B_R} |Du - (Du)_R|^2 \ln_+ \left(a\varepsilon\, |Du - (Du)_R|^2\right) dy + \frac{2}{a} \int\limits_{B_R} e^{\omega_R^2/\varepsilon - 1}\, dy, \tag{13}$$

where $\omega_R^2 = \omega^2(|Du - (Du)_R|^2)$.

From (12) and (13) it follows

$$\begin{aligned}\int\limits_{B_\sigma} |Du - (Du)_\sigma|^2\; dy \le{}& 4A\left(\frac{\sigma}{R}\right)^{n+2} \int\limits_{B_R} |Du - (Du)_R|^2\; dy \\ &+ \frac{2(2A+1)}{\nu^2}\left(\varepsilon \int\limits_{B_R} |Du - (Du)_R|^2 \ln_+ \left(a\varepsilon\, |Du - (Du)_R|^2\right) dy \right.\\ &\left. + \frac{2}{a}\int\limits_{B_R} e^{\omega_R^2/\varepsilon - 1}\, dy\right),\end{aligned} \tag{14}$$

We can estimate the right-hand side by means of Lemma 1 (c) (for the quasiconcave case), Lemma 6 and we get

$$\begin{aligned}\int\limits_{B_\sigma} |Du - (Du)_\sigma|^2\, dy \le{}& 4A\left(\frac{\sigma}{R}\right)^{n+2} \int\limits_{B_R} |Du - (Du)_R|^2\, dy \\ &+ \frac{2(2A+1)}{\nu^2}\left[\varepsilon c \left(\frac{M}{\nu}\right)^2 \left(\fint_{B_{2R}} \ln_+^{q/(q-1)} \left(4a\varepsilon|Du - (Du)_{2R}\,|^2\right) dy\right)^{1-1/q} \times\right.\\ &\left.\times \int\limits_{B_{2R}} |Du - (Du)_{2R}|^2\; dy + \frac{2\alpha_n R^n}{a} e^{l_2 \omega^2 (l_2 U_R)/\varepsilon - 1}\right].\end{aligned}$$

Setting

$$\phi\,(t) = \int\limits_{B_t} |Du - (Du)_t\,|^2\, dy,$$

$$F_\varepsilon(t) = \left(\fint_{B_t} \ln_+^{q/(q-1)} \left(4a\varepsilon|Du - (Du)_t\,|^2\right) dy\right)^{1-1/q},$$

we can rewrite the previous inequality as follows:

$$\phi(\sigma) \le 4A\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{K\varepsilon}{\nu^2}F_\varepsilon(2R)\phi(2R) + \frac{2^4\alpha_n A}{a\nu^2}e^{l_2\omega^2(2^n l_2 U_{2R})/\varepsilon-1}R^n, \tag{15}$$

where $K = c(n,N,q)(M/\nu)^8$. From the assumptions of Theorem it follows that there exists $d \in (0, d_x/2)$ such that (8) holds. Now we are going to prove that

$$\phi\left(2\tau^k d\right) \le \tau^{kn}\phi(2d) \tag{16}$$

for every natural number k and $\tau = (2^{n+5}A)^{-1/2}$. Let $k = 1$. If we put in (15) $a = 1/CU_{2d}, \varepsilon = l_2\omega^2(2^n l_2 U_{2d}), \sigma = 2\tau d$ and $R = d$ we get

$$\begin{aligned}\phi(2\tau d) &\le 2^{n+4}A\tau^{n+2}\phi(d) + \frac{Kl_2\omega^2}{\nu^2}F_\varepsilon(2d)\phi(2d) + \frac{2^4\alpha_n A}{\nu^2}CU_{2d}d^n\\ &\le 2^{n+4}A\tau^{n+2}\phi(2d) + \frac{Kl_2\omega^2}{\nu^2}b^{1-1/q}F_\varepsilon(2d)\phi(2d) + \frac{1}{4}\tau^n\phi(2d)\\ &\le \left(2^{n+4}A\tau^2 + \frac{1}{4} + \frac{1}{4}\right)\tau^n\phi(2d) = \tau^n\phi(2d).\end{aligned}$$

Thus (16) holds for $k = 1$. Consequently $U_{2\tau d} \le U_{2d}$ and by means of Proposition 3 we have $F_\varepsilon(2\tau d) \le b^{1-1/q}F_\varepsilon(2d)$.

Let us suppose that (16) holds for $k \ge 1$. Similarly to consideration above we have $U_{2\tau^k d} \le U_{2d}$ and $F_\varepsilon\left(2\tau^k d\right) \le b^{1-1/q}F_\varepsilon(2d)$. We will show that (16) holds for $k+1$. Setting $a = 1/CU_{2d}$, $\varepsilon = l_2\omega^2(2^n l_2 U_{2d})$, $\sigma = 2\tau^{k+1}d$ and $R = \tau^k d$ in (15) we obtain

$$\begin{aligned}\phi(2\tau^{k+1}d) &\le 2^{n+4}A\tau^{n+2}\phi\left(\tau^k d\right) + \frac{Kl_2\omega^2}{\nu^2}F_\varepsilon(2\tau^k d)\phi(2\tau^k d)\\ &\quad + \frac{2^4\alpha_n A}{\nu^2}e^{\omega^2(2^n l_2 U_{2\tau^k d})/\omega^2(2^n l_2 U_{2d})-1}\tau^{kn}CU_{2d}d^n\\ &\le 2^{n+4}A\tau^{n+2}\phi\left(2\tau^k d\right) + \frac{Kl_2\omega^2}{\nu^2}F_\varepsilon(2\tau^k d)\phi(2\tau^k d) + \frac{1}{4}\tau^{(k+1)n}\phi(2d)\\ &\le 2^{n+4}A\tau^{n+2}\tau^{kn}\phi(2d) + \frac{Kl_2\omega^2}{\nu^2}b^{1-1/q}F_\varepsilon(2d)\tau^{kn}\phi(2d) + \frac{1}{4}\tau^{(k+1)n}\phi(2d)\\ &\le \left(2^{n+4}A\tau^2 + \frac{1}{4} + \frac{1}{4}\right)\tau^{(k+1)n}\phi(2d) = \tau^{(k+1)n}\phi(2d).\end{aligned}$$

Let us consider the two possibilities:

(a) if $\tau \le t < 1$, then $t^{-n}\phi(td) \le \tau^{-n}\phi(td) \le \tau^{-n}\sup_{t\in[\tau,1)}\phi(td)$ and also

$$\phi(td) \le \left(\tau^{-n}\sup_{t\in[\tau,1)}\phi(td)\right)t^n, \tag{17}$$

(b) if $0 < t < \tau$, then there exists natural $k \ge 1$ such that $\tau^{k+1} \le t < \tau^k$. From Proposition 3, (8), (16) and (15) with $a = 1/CU_{2d}$, $\varepsilon = l_2\omega^2(2^n l_2 U_{2d})$, $\sigma = td$ and $R = \tau^k d$ we have

$$\begin{aligned}
\phi(td) &= \phi\left(\frac{t}{\tau^k}(\tau^k d)\right)\\
&\le 4A\left(\frac{t}{\tau^k}\right)^{n+2}\phi\left(\tau^k d\right) + \frac{K\varepsilon}{\nu^2}F_\varepsilon(2\tau^k d)\phi\left(2\tau^k d\right)\\
&\quad + \frac{2^4\alpha_n A}{a\nu^2}e^{l_2\omega^2(2^n l_2 U_{2\tau^k d})/\varepsilon - 1}\tau^{kn}d^n\\
&\le 4A\left(\frac{t}{\tau^k}\right)^{n+2}\tau^{kn}\phi(2d) + \frac{Kl_2\omega^2}{\nu^2}b^{1-1/q}F_\varepsilon(2d)\tau^{kn}\phi(2d) \qquad (18)\\
&\quad + \frac{2^4\alpha_n A}{\nu^2}CU_{2d}\tau^{kn}d^n\\
&\le \left(4A\left(\frac{t}{\tau^k}\right)^{n+2}\tau^{kn} + \tau^{(k+1)n}\right)\phi(2d)\\
&\le \left(4A\tau^{-n}\left(\frac{t}{\tau^k}\right)^{n+2} + 1\right)\tau^{(k+1)n}\phi(2d) < \left(4A\tau^{-n}+1\right)t^n\phi(2d).
\end{aligned}$$

In both cases (17) and (18) we obtain

$$t^{-n}\phi(td) \le c, \quad t \in (0,1],$$

where $c = \max\{\tau^{-n}\sup_{t\in[\tau,1)}\phi(td), \left(4A\tau^{-n}+1\right)\phi(2d)\} = \left(4A\tau^{-n}+1\right)\phi(2d)$. Let $0 < r < dist(B(x,r_x), \partial\Omega)$. Hence $U(y,r)$ is uniformly continuous for fixed r in $\overline{B(x,r_x)} \subset \Omega$. According to Proposition 3, the expression

$$\frac{Kl_2\omega^2}{\nu^2}\left(b\fint_{B(y,r)}\ln_+^{q/(q-1)}\left(\frac{4l_2\omega^2\left|Du-(Du)_{y,r}\right|^2}{CU(y,r)}\right)dz\right)^{1-1/q}$$

is also uniformly continuous with respect to y in $\overline{B(x,r_x)}$ and we arrive at the conclusion. □

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