Oblique derivative problem for elliptic equations in non-divergence form with VMO coefficients

G. Di Fazio, D.K. Palagachev

Abstract. A priori estimates and strong solvability results in Sobolev space $W^{2,p}(\Omega)$, $1 < p < \infty$ are proved for the regular oblique derivative problem

$$\begin{cases} \sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \text{ a.e. } \Omega \\ \frac{\partial u}{\partial \ell} + \sigma(x)u = \varphi(x) \text{ on } \partial\Omega \end{cases}$$

when the principal coefficients a^{ij} are $VMO \cap L^\infty$ functions.

Keywords: oblique derivative, elliptic equation, non divergence form, VMO coefficients, strong solution

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Introduction

The present article is devoted to the development of L^p theory for the oblique derivative problem for linear uniformly elliptic equations with discontinuous coefficients.

More precisely, our aim is to derive a priori estimates and to prove existence result for the solutions of

$$\begin{cases} \mathcal{L}u \equiv \displaystyle\sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \quad \text{a.e. } \Omega \\ \mathcal{B}u \equiv \displaystyle\sum_{i=1}^n \ell_i(x)\frac{\partial u}{\partial x_i} + \sigma(x)u = \varphi(x) \quad \text{on } \partial\Omega \end{cases} \tag{0.1}$$

in the case of merely $VMO \cap L^\infty$ principal coefficients. The boundary condition above is prescribed in terms of directional derivative with respect to unit vector field $\ell(x) = (\ell_1(x), \dots, \ell_n(x))$, and it is assumed that $\ell(x)$ is nowhere tangential to the boundary $\partial\Omega$, i.e. (0.1) is a *regular* oblique derivative problem.

The solutions of (0.1) we are dealing with here, are to be referred to as *strong solutions* belonging to the Sobolev space $W^{2,p}(\Omega)$, $1 < p < \infty$. In other words, they are twice weakly differentiable functions with $L^p(\Omega)$ summable derivatives up to order 2 that satisfy $\mathcal{L}u = f$ almost everywhere in Ω and $\mathcal{B}u = \varphi$ holds true in the trace sense on $\partial\Omega$.

The classical oblique derivative problem with sufficiently smooth Hölder continuous data has been very well studied. We refer to [GT, Chapter 6] and reference therein for the known well-posedness results. Using explicit representations for solutions to derive suitable L^p estimates, it was proved by Agmon, Douglis and Nirenberg [ADN] that $a^{ij} \in C^0(\overline{\Omega})$ is a *sufficient* condition ensuring $W^{2,p}$-regularity (i.e., in order $f \in L^p(\Omega)$, $\varphi \in W^{1-1/p,p}(\partial\Omega)$ to imply $u \in W^{2,p}(\Omega)$) of (0.1) for *all* values of p in the range $(1, \infty)$. Using another approach, the same result was proved by M. Chicco [C1], whence existence theorems were obtained through Riesz-Fredholm theory.

If the principal coefficients a^{ij} are not uniformly continuous the oblique derivative problem is less studied. As far as we know there are a few results devoted to (0.1) in that case and all of them are concerned with the case $p = 2$. For two-dimensional domains Ω, G. Talenti [T] established $W^{2,2}$ solvability of (0.1) assuming $a^{ij}(x)$ to be only measurable functions. In the multi-dimensional case ($n \geq 3$) the $W^{2,2}$-regularity and invertibility properties of the operator $(\mathcal{L}, \mathcal{B})$ are proved if $a^{ij} \in W^{1,n}(\Omega)$ (cf. C. Miranda [M], M. Chicco [C2], G. Viola [V]), or if a^{ij}-s are measurable functions and satisfy the "Cordes condition" (cf. M. Chicco [C2], F. Nicolosi [NF]). The techniques used in the most of the cited results allow to extend them for p belonging to a suitable neighbourhood of $p = 2$.

Our main purpose here is to investigate (0.1) weakening the assumptions on the coefficients $a^{ij}(x)$ to the class of functions with vanishing mean oscillation (VMO). It is worth to note that both the cases $a^{ij} \in C^0(\overline{\Omega})$ and $a^{ij} \in W^{1,n}(\Omega)$ imply $a^{ij} \in VMO$ (see [CFL1]).

We are going to prove two kinds of results for the problem (0.1). First of them (Theorem 1.1) is the $W^{2,p}$-regularizing property of $(\mathcal{L}, \mathcal{B})$ for all $p \in (1, \infty)$, that is obtained by proving a global a priori estimate for the strong solutions of (0.1). The approach is very similar to that used by F. Chiarenza, M. Frasca and P. Longo in [CFL1], [CFL2] in the treatment of Dirichlet's problem. Since the interior estimate was proved in [CFL1], the crucial step in proving our Theorem 1.1 is the establishment of the boundary $W^{2,p}$ estimate. For, an explicit representation of solution's second order derivatives is used in terms of singular integrals of two types. The first one is a sum of singular integral operators and commutators with Calderón-Zygmund kernels, and their L^p-boundedness was carried out in [CFL2]. The second type of integrals are less singular and due to the specific boundary condition in (0.1). Roughly speaking, we are concerned with commutators of integral operators having positive kernels depending on the difference $|a^{ij}(x) - a^{ij}(y)|$. Unfortunately, the approach of [CFL2] that is based on the VMO-character of a^{ij}-s cannot be applied here to prove boundedness of the commutators because of the absolute value inside the integral. We utilize a very recent result of M. Bramanti [B] to complete the proof of Theorem 1.1.

As a by-product we prove (Theorem 1.2) the strong $W^{2,p}$-solvability of (0.1) for all the values p in $(1, \infty)$. This is reached in a standard way having in mind Theorem 1.1. The VMO-character of the coefficients again plays a crucial role through the $W^{2,p}$-regularizing properties of $(\mathcal{L}, \mathcal{B})$ and the uniqueness result (Theorem 5.1). A variant of Aleksandrov-Pucci-Bakelman maximum principle proved by Y. Luo and N. Trudinger (cf. [LT], [L]) is used to prove uniqueness for the strong solutions of (0.1).

Finally, let us note that the nonlinear as well as degenerate oblique derivative problems will be studied in forthcoming papers.

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1. Statement of the problem and main results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$. As usual, by $W^{k,p}(\Omega)$ we denote the Sobolev space of k-times weakly differentiable functions whose derivatives up to order k are L^p-summable over Ω and $\|\cdot\|_{W^{k,p}(\Omega)}$ is the norm in this space. To describe regularity properties of functions defined on the boundary $\partial\Omega$ we use the standard notation $W^{s,p}(\partial\Omega)$ ($s > 0$ — non integer) for fractional order Sobolev spaces (see [AR] for the details). In what follows we adopt the standard convention that repeated indices indicate summation from 1 to n, and $D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$.

Our goal is to develop L^p theory for the regular oblique derivative problem for second order linear elliptic equations:

$$\left\{\begin{array}{ll} \mathcal{L}u \equiv a^{ij}(x)D_{ij}u = f(x) & \text{a.e. } \Omega \\ \mathcal{B}u \equiv \dfrac{\partial u}{\partial \ell} + \sigma(x)u = \varphi(x) & \text{on } \partial\Omega. \end{array}\right. \tag{1.1}$$

To be more precise, we are interested in proving $W^{2,p}(\Omega)$ a priori estimate for strong solutions of (1.1) and deriving strong solvability result for that problem in the case when the coefficients $a^{ij}(x)$ of the elliptic operator are merely $VMO \cap L^\infty$ functions.

Throughout the paper we shall assume that $\mathcal{L}$ is *uniformly elliptic operator with VMO coefficients:*

$$\left\{\begin{array}{l} \lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \ \text{ a.a. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n,\ \lambda = \text{const} > 0, \\ a^{ij}(x) \in VMO \cap L^\infty(\Omega), \quad a^{ij}(x) = a^{ji}(x). \end{array}\right. \tag{1.2}$$

As it concerns the boundary operator $\mathcal{B}$ it is prescribed by a directional derivative with respect to the unit vector field $\ell(x) = (\ell_1(x), \ldots, \ell_n(x))$ $(|\ell(x)| = 1)$

defined on $\partial\Omega$. We are dealing with the case of *regular boundary operator,* i.e. the field $\ell(x)$ is never tangential to the boundary:

$$\left\{\begin{array}{l} \ell(x)\cdot\nu(x)=\ell_i(x)\nu_i(x)>0 \ \text{ on } \partial\Omega, \quad \sigma(x)<0, \\ \ell_i(x),\ \sigma(x)\in C^{0,1}(\partial\Omega) \text{ (Lipschitz)}, \ \partial\Omega\in C^{1,1}, \end{array}\right. \tag{1.3}$$

where $\nu(x)=(\nu_1(x),\dots,\nu_n(x))$ is the unit inner normal to $\partial\Omega$.

The main results we want to prove are the following ones.

Theorem 1.1. *Suppose conditions* (1.2) *and* (1.3) *to be fulfilled and* $u\in W^{2,q}(\Omega)$, $1<q\le p<\infty$. *Let* $\mathcal{L}u\in L^p(\Omega)$ *and* $\mathcal{B}u\in W^{1-1/p,p}(\partial\Omega)$.

Then $u\in W^{2,p}(\Omega)$ *and the following estimate*

$$\|u\|_{W^{2,p}(\Omega)}\le C\left(\|u\|_{L^p(\Omega)}+\|\mathcal{L}u\|_{L^p(\Omega)}+\|\mathcal{B}u\|_{W^{1-1/p,p}(\partial\Omega)}\right) \tag{1.4}$$

holds true where the constant C *depends on* n, p, λ, $\partial\Omega$, ℓ, σ *and the VMO-moduli of the coefficients* $a^{ij}(x)$.

Theorem 1.2. *Assume the conditions* (1.2) *and* (1.3) *to be satisfied.*

Then the problem (1.1) *admits a unique strong solution* $u\in W^{2,p}(\Omega)$ *for each* $f\in L^p(\Omega)$ *and* $\varphi\in W^{1-1/p,p}(\partial\Omega)$, $1<p<\infty$.

Moreover, there is a constant C *(independent of* u*) such that*

$$\|u\|_{W^{2,p}(\Omega)}\le C\left(\|f\|_{L^p(\Omega)}+\|\varphi\|_{W^{1-1/p,p}(\partial\Omega)}\right). \tag{1.5}$$

Remark 1.1. 1. By virtue of Sobolev imbedding theorem and Morrey's lemma the solution of (1.1) belongs to the Hölder class $C^{1,1-n/p}(\overline{\Omega})$ if $p>n$, and the boundary condition in (1.1) is satisfied in classical sense.

2. In the case $p\ge n$ the estimate (1.5) follows immediately from (1.4). In fact, the L^p-norm of u can be estimated in terms of f and φ through the maximum principle of Aleksandrov type (see [LT, Theorem 3.1], [L, Theorem 1.5]).

3. The main results of this paper still hold true for the elliptic operator with lower order terms

$$a^{ij}(x)D_{ij}+b^i(x)D_i+c(x),$$

supposing in addition to $a^{ij}\in VMO\cap L^\infty$ that $b^i\in L^q$, $c\in L^r$ where $q>n$ if $p\le n$, $q=p$ if $p>n$ and $r>n/2$ if $p\le n/2$, $r=p$ if $p>n/2$, and requiring $c(x)\le 0$ a.e. Ω in the assumptions of Theorem 1.2.

4. The requirement $\sigma(x)<0$ on $\partial\Omega$ in (1.3) is not necessary for the proof of Theorem 1.1. Indeed, we use it in Section 3 in order to derive solution's representation formula, but the problem (1.1) with an arbitrary $\sigma(x)$ always may be reduced to the case considered here by setting $u(x)=v(x)e^{F(x)}$ where $F(x)\in C^{1,1}(\overline{\Omega})$ and $\partial F/\partial\ell=-1-\sigma(x)$ on $\partial\Omega$. As it concerns the existence result (Theorem 1.2) $\sigma(x)<0$ is an essential assumption ensuring uniqueness for the oblique derivative problem.

2. Real analysis auxiliary results

For the sake of completeness we recall here some definitions and known results. We will use the John-Nirenberg space BMO of the functions of bounded mean oscillation and its subspace VMO introduced in [JN] and [S], respectively. A locally integrable function f defined on $\mathbb{R}^n$ lies in the space BMO if

$$\sup_B \frac{1}{|B|}\int_B |f(x)-f_B|\,dx = \|f\|_* < \infty, \tag{2.1}$$

where B ranges in the class of the balls in $\mathbb{R}^n$ and $f_B = |B|^{-1}\int_B f(x)\,dx$. If $f\in BMO$, set

$$\eta(r) = \sup_{\varrho\le r} \frac{1}{|B|}\int_B |f(x)-f_B|\,dx, \tag{2.2}$$

where now B is a ball of radius ϱ. We say that $f\in VMO$ if $\lim_{r\to 0}\eta(r)=0$ and we call $\eta(r)$ *the VMO-modulus* of the function f. In a similar way we define the spaces BMO and VMO of functions defined on Ω, replacing B in (2.1) and (2.2) by the intersection of balls with a bounded open set $\Omega\subset\mathbb{R}^n$. Having a function f defined on Ω that belongs to VMO it is possible to extend it to the all $\mathbb{R}^n$ preserving the VMO-modulus by virtue of [A, Proposition 1.3] if in addition the boundary $\partial\Omega$ is $C^{1,1}$-smooth. In the following we shall use this result without explicit reference.

Finally, it is a well known fact that for given $f\in VMO$ with modulus $\eta(r)$, we can find a sequence $\{f_h\}$ of $C^\infty(\mathbb{R}^n)$ functions with moduli $\eta_h(r)$, converging to f in VMO and such that $\eta_h(r)\le\eta(r)$ for all $h\in\mathbb{N}$.

In the forthcoming sections we shall derive various L^p estimates for the solutions of the oblique derivative problem. For this reason we recall here some real analysis results about singular integrals and commutators. We refer to [CFL2] and references therein for the proof of these results.

Let us start with the following definition.

Definition. *Let $k:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a measurable function. Then k is said to be a Calderón-Zygmund kernel if*

(i) $k\in C^\infty(\mathbb{R}^n\setminus\{0\})$,
(ii) $k(x)$ *is homogeneous of degree* $-n$,
(iii) $\int_\Sigma k(x)\,d\sigma_x = 0$ *where* $\Sigma=\{x\in\mathbb{R}^n: |x|=1\}$.

The useful properties of the integral operators with Calderón-Zygmund kernels are summarized in the following result.

Theorem 2.1 ([CFL1], [CFL2])**.** *Let Ω be an open subset of $\mathbb{R}^n$ and let $k:\Omega\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{R}$, be such that*

(i) $k(x,\cdot)$ *is a Calderón-Zygmund kernel for almost all* $x\in\Omega$,
(ii) $\max_{|j|\le 2n}\left\|\frac{\partial^j}{\partial y_j}k(x,y)\right\|_{L^\infty(\Omega\times\Sigma)} = M<\infty$.

If $f \in L^p(\Omega)$, $1 < p < +\infty$, $a \in L^\infty(\mathbb{R}^n)$, $x \in \Omega$ *and* $\varepsilon > 0$, *set*

$$K_\varepsilon f(x) = \int_{\substack{|x-y|>\varepsilon \\ y\in\Omega}} k(x, x-y) f(y)\, dy$$

and

$$C_\varepsilon[a, f](x) = \int_{\substack{|x-y|>\varepsilon \\ y\in\Omega}} k(x, x-y)[a(x) - a(y)] f(y)\, dy.$$

Then, for any $f \in L^p(\Omega)$, *there exist* Kf *and* $C[a, f] \in L^p(\Omega)$ *such that*

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_{L^p(\Omega)} = \lim_{\varepsilon\to 0} \|C_\varepsilon[a, f] - C[a, f]\|_{L^p(\Omega)} = 0.$$

Furthermore, there exists a constant $c = c(n, p, M)$ *such that*

$$\|Kf\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)} \quad \text{and} \quad \|C[a, f]\|_{L^p(\Omega)} \le c\|a\|_*\|f\|_{L^p(\Omega)}.$$

As a consequence of Theorem 2.1 and the VMO assumption on $a(x)$ we have the following theorem, which plays an essential role in establishing the L^p-boundedness for the singular commutators appearing in the representation formula for the second derivatives of solutions to (1.1).

Theorem 2.2 ([CFL1], [CFL2]). *Let* k *be as in Theorem* 2.1 *and suppose* $a \in L^\infty(\Omega) \cap VMO$.

Then for each $\varepsilon > 0$ *there exists a positive constant* $\varrho_0 = \varrho_0(\varepsilon, \eta)$ *such that*

$$\|C[a, f]\|_{L^p(\Omega\cap B_r)} \le c\, \varepsilon \|f\|_{L^p(\Omega\cap B_r)} \qquad \forall\, f \in L^p(\Omega \cap B_r), \quad 1 < p < \infty$$

for any ball B_r, $r < \varrho_0$, *where* $c = c(n, p, M, \eta)$ *and* η *is the VMO modulus of* a.

In what follows we denote by $\boldsymbol{a}_n(x)$ the last row (column) of the coefficients' matrix of the elliptic operator, i.e. $\boldsymbol{a}_n(x) = (a^{1n}(x), \dots, a^{nn}(x))$, and define the "generalized reflection":

$$T(x, y) = x - \frac{2x_n}{a^{nn}(y)}\, \boldsymbol{a}_n(y), \qquad T(x) = T(x, x). \tag{2.3}$$

Finally, $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ as usual.

The boundary representation formula we shall derive in Section 3 involves some *less singular integral operators and commutators.* The next results will give us the bounds for such operators.

Theorem 2.3 ([CFL2]). *Let $f \in L^p(\mathbb{R}^n_+)$, $1 < p < \infty$, $a \in VMO \cap L^\infty(\mathbb{R}^n)$ and set for $x \in \mathbb{R}^n_+$*

$$\widetilde{K} f(x) = \int_{\mathbb{R}^n_+} \frac{f(y)}{|T(x) - y|^n}\, dy, \qquad \widetilde{C}[a, f](x) = \int_{\mathbb{R}^n_+} \frac{a(x) - a(y)}{|T(x) - y|^n} f(y)\, dy.$$

Then there exists a constant $c = c(n, p)$ such that

$$\|\widetilde{K} f\|_{L^p(\mathbb{R}^n_+)} \le c \|f\|_{L^p(\mathbb{R}^n_+)} \quad \text{and} \quad \|\widetilde{C}[a, f]\|_{L^p(\mathbb{R}^n_+)} \le c\, \|a\|_*\, \|f\|_{L^p(\mathbb{R}^n_+)}.$$

Our representation formula contains a special term due to the presence of first order differential operator on the boundary. To estimate it we need L^p-bounds of commutators with absolute value inside the integral. The following result, that we report here in the euclidean case, is proved in the general setting of homogeneous spaces by M. Bramanti [B, Theorem 0.1 and Example 0.2].

Theorem 2.4 ([B]). *Let $a \in BMO(\mathbb{R}^n_+)$. Define for $f \in L^p(\mathbb{R}^n_+)$, $1 < p < \infty$,*

$$\widetilde{C}_a f(x) = \int_{\mathbb{R}^n_+} \frac{|a(x) - a(y)|}{|T(x) - y|^n} f(y)\, dy.$$

Then we have the bound

$$\|\widetilde{C}_a f\|_{L^p(\mathbb{R}^n_+)} \le c\, \|a\|_*\, \|f\|_{L^p(\mathbb{R}^n_+)}$$

with $c = c(n, p)$.

Remark 2.1. It is evident from the proofs of [CFL2, Theorem 2.5] and [B, Theorem 0.1] that the last two results still hold true if the kernel $|T(x) - y|^{-n}$ is replaced by an equivalent function.

3. Boundary representation formula

Now we will derive a representation formula for the solution of the oblique derivative problem with constant coefficients operators.

Let $B_\varrho = \{x \in \mathbb{R}^n : |x| < \varrho\}$, $B^+_\varrho = B_\varrho \cap \{x_n > 0\}$, $C_\varrho = B_\varrho \cap \{x_n = 0\}$, $n \ge 3$, and consider the uniformly elliptic operator

$$\mathcal{L}_0 \equiv a^{ij} D_{ij}$$

with constant coefficients a^{ij}. Let

$$\mathcal{B}_0 \equiv \frac{\partial}{\partial \ell} + \sigma = \ell_i D_i + \sigma$$

be the boundary operator prescribed by directional derivative with respect to a constant vector field $\ell = (\ell_1, \dots, \ell_n)$, $|\ell| = 1$, and assume that $\sigma = \text{const} < 0$ and $\mathcal{B}_0$ is a regular operator on $\{x_n = 0\}$, i.e. $\ell_n > 0$.

Before stating our result we introduce some more notations. The matrix $\{a^{ij}\}_{i,j=1}^n$ of the coefficients of $\mathcal{L}_0$ will be denoted by $\boldsymbol{a}$ and $\boldsymbol{A} = \{A^{ij}\}$ will be the inverse matrix $\boldsymbol{a}^{-1}$. The symbol $\Gamma(\xi)$, $\xi = (\xi_1, \dots, \xi_n)$, stands for the normalized fundamental solution of the operator $\mathcal{L}_0$, i.e.,

$$\Gamma(\xi) = \frac{1}{n(2-n)\omega_n(\det \boldsymbol{a})^{1/2}} \left(A^{ij}\xi_i\xi_j\right)^{\frac{2-n}{2}}, \tag{3.1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Finally, for an arbitrary $\xi \in \mathbb{R}^n$ we set

$$T(\xi) = \xi - \frac{2\xi_n}{a^{nn}} \boldsymbol{a}_n$$

as in (2.3), where the vector $\boldsymbol{a}_n$ is the last row (column) of the matrix $\boldsymbol{a}$.

Lemma 3.1. *Suppose that $u(x) \in C^\infty(B_{2\varrho})$ satisfies*

$$\begin{cases} \mathcal{L}_0 u = f & \text{in } B^+_{2\varrho} \\ \mathcal{B}_0 u = 0 & \text{on } C_{2\varrho} \end{cases} \tag{3.2}$$

and let supp $u \subset B_\varrho$. *Then*

$$u(x) = \int_{B^+_{2\varrho}} G(x,y) f(y)\, dy, \tag{3.3}$$

where

$$G(x,y) = \Gamma(x-y) - \Gamma(T(x)-y) + \theta(T(x)-y) \tag{3.4}$$

and

$$\theta(\xi) = \frac{2}{n\omega_n(\det \boldsymbol{a})^{1/2}} \frac{\ell_n}{a^{nn}} \int_0^\infty \frac{e^{\sigma s}\,(\xi + sT(\ell))_n}{\left(A^{ij}(\xi+sT(\ell))_i(\xi+sT(\ell))_j\right)^{n/2}}\, ds. \tag{3.5}$$

Proof: Without loss of generality we may suppose $\varrho = 1$ above. Let $\boldsymbol{P} = \{P^{ij}\}$ be a non-singular $n \times n$ matrix such that $\boldsymbol{P}\boldsymbol{a}\boldsymbol{P}^t = A\boldsymbol{Id}$ ($\boldsymbol{P}^t$ is the transposed of $\boldsymbol{P}$). After the linear change $\tilde{x} = x\boldsymbol{P}^t$, problem (3.2) becomes

$$\begin{cases} \Delta \tilde{u} = \tilde{f} \ \text{ in } \widetilde{B}^+_2 = \boldsymbol{P}(B^+_2) \\ \dfrac{\partial \tilde{u}}{\partial \tilde{\ell}} + \sigma \tilde{u} = 0 \ \text{ on } \widetilde{C}_2 = \boldsymbol{P}(C_2), \end{cases} \tag{3.6}$$

where $u(x) \to \tilde{u}(\tilde{x})$, $f(x) \to \tilde{f}(\tilde{x})$ under the transformation and $\tilde{\ell} = \ell \boldsymbol{P}^t$. Further, we may take $P^{nj} = 0$ for $1 \le j \le n-1$ and $P^{nn} = 1/\sqrt{a^{nn}}$.

It is clear that the change $x \to \tilde{x} = x\boldsymbol{P}^t$ maps the half-space $\{x_n > 0\}$ onto $\{\tilde{x}_n > 0\}$ ($\tilde{x}_n = x_n/\sqrt{a^{nn}}$) and therefore $\widetilde{C}_2 = \boldsymbol{P}(C_2) = \widetilde{B}_2 \cap \{\tilde{x}_n = 0\}$. Moreover, $\tilde{\ell}_n = \ell_n/\sqrt{a^{nn}} > 0$, i.e. the new oblique derivative problem (3.6) is regular too.

Let $\tilde{\Gamma}(\eta)$, $\eta \in \mathbb{R}^n$, be the fundamental solution of Laplace's equation

$$\tilde{\Gamma}(\eta) = \frac{1}{n(2-n)\omega_n} |\eta|^{2-n}.$$

It is well known [GT, Section 6.7] that the solution $\tilde{u}(\tilde{x})$ of (3.6) can be represented as

$$\tilde{u}(\tilde{x}) = \int_{\tilde{B}_2^+} \tilde{G}(\tilde{x}, \tilde{y}) \tilde{f}(\tilde{y})\, d\tilde{y}, \tag{3.7}$$

where

$$\tilde{G}(\tilde{x}, \tilde{y}) = \tilde{\Gamma}(\tilde{x} - \tilde{y}) - \tilde{\Gamma}(\tilde{x} - \tilde{y}^*) - 2\tilde{\ell}_n \int_0^\infty e^{\sigma s} D_{\tilde{x}_n} \tilde{\Gamma}(\tilde{x} - \tilde{y}^* + s\tilde{\ell})\, ds \tag{3.8}$$

and $\tilde{y}^*$ is the reflected point $(\tilde{y}_1, \dots, \tilde{y}_{n-1}, -\tilde{y}_n)$.

In order to derive the representation (3.3) for the solution of the original problem we set $\xi = x - y$, $\eta = \tilde{x} - \tilde{y} = \xi \boldsymbol{P}^t$. Then

$$|\eta|^2 = \eta \cdot \eta = (\eta (\boldsymbol{P}^{-1})^t \boldsymbol{A} \boldsymbol{P}^{-1}) \cdot \eta = \eta (\boldsymbol{P}^t)^{-1} \boldsymbol{A} \cdot \eta (\boldsymbol{P}^t)^{-1} = \xi \boldsymbol{A} \cdot \xi = A^{ij} \xi_i \xi_j.$$

Thus

$$\tilde{\Gamma}(\tilde{x} - \tilde{y}) = \tilde{\Gamma}((x-y)\boldsymbol{P}^t) = (\det \boldsymbol{a})^{1/2} \Gamma(x - y).$$

To evaluate $\tilde{\Gamma}(\tilde{x} - \tilde{y}^*)$ in (3.8) we denote $\tilde{y}^* = \tilde{y}\tilde{\boldsymbol{R}}$ where $\tilde{\boldsymbol{R}}$ is a symmetric $n \times n$-matrix with entries $\tilde{R}^{ij} = \delta_{ij}$ (Kronecker's delta) for $i,\ j < n$, $\tilde{R}^{in} = \tilde{R}^{ni} = 0$ for $i < n$, and $\tilde{R}^{nn} = -1$. Direct calculations based on the representation of the matrix $\boldsymbol{P}$ show that $\tilde{\boldsymbol{R}}\boldsymbol{P} = A\boldsymbol{P}\boldsymbol{R}$ where $\boldsymbol{R} = \{R^{ij}\}$ is a matrix with entries $R^{ij} = \delta_{ij}$ for $j < n$, $R^{in} = -2a^{in}/a^{nn}$, $R^{nn} = -1$. We have

$$\tilde{x} - \tilde{y}^* = x\boldsymbol{P}^t - y\boldsymbol{P}^t \tilde{\boldsymbol{R}} = (x - y\boldsymbol{R}^t)\boldsymbol{P}^t$$

and it follows as above

$$\tilde{\Gamma}(\tilde{x} - \tilde{y}^*) = \tilde{\Gamma}((x - y\boldsymbol{R}^t)\boldsymbol{P}^t) = (\det \boldsymbol{a})^{1/2} \Gamma(x - T(y))$$

(recall $T(y) = y\boldsymbol{R}^t = y - 2(y_n/a^{nn})\boldsymbol{a}_n$).

In order to calculate the third term in (3.8) we have

$$
\begin{aligned}
D_{\tilde{x}_n}\tilde{\Gamma}(\tilde{x}-\tilde{y}^*+s\tilde{\ell}) &= \frac{1}{n\omega_n}\frac{(\tilde{x}-\tilde{y}^*+s\tilde{\ell})_n}{|\tilde{x}-\tilde{y}^*+s\tilde{\ell}|^n} = \frac{1}{n\omega_n}\frac{\Big((x-T(y)+s\ell)\boldsymbol{P}^t\Big)_n}{|(x-T(y)+s\ell)\boldsymbol{P}^t|^n}\\
&= \frac{1}{n\omega_n\sqrt{a^{nn}}}\frac{(x-T(y)+s\ell)_n}{\Big(A^{ij}(x-T(y)+s\ell)_i(x-T(y)+s\ell)_j\Big)^{n/2}}
\end{aligned}
$$

as a consequence of the choice of $\boldsymbol{P}$.

So, changing the variables $\tilde{y} = y\boldsymbol{P}^t$ in (3.7) and using $\det \boldsymbol{P} = (\det \boldsymbol{a})^{-1/2}$ we obtain

$$
u(x) = \int_{B_2^+} G_1(x,y)f(y)\,dy
$$

where

$$
\begin{aligned}
G_1(x,y) &= \Gamma(x-y) - \Gamma(x-T(y))\\
&\quad - \frac{2}{n\omega_n(\det \boldsymbol{a})^{1/2}}\frac{\ell_n}{a^{nn}}\int_0^\infty \frac{e^{\sigma s}\,(x-T(y)+s\ell)_n}{\Big(A^{ij}(x-T(y)+s\ell)_i(x-T(y)+s\ell)_j\Big)^{n/2}}\,ds
\end{aligned}
$$

and Γ is given by (3.1).

In order to show (3.3), (3.4) it remains to note that $T^2(\xi) = T(T(\xi)) = \xi$ and $\boldsymbol{R}^t\boldsymbol{A}\boldsymbol{R} = \boldsymbol{A}$ whence

$$
A^{ij}(T(\xi))_i(T(\xi))_j = T(\xi)\boldsymbol{A}\cdot T(\xi) = \xi\boldsymbol{R}^t\boldsymbol{A}\cdot\xi\boldsymbol{R}^t = \xi\boldsymbol{A}\cdot\xi = A^{ij}\xi_i\xi_j
$$

and

$$
\begin{aligned}
&\Gamma(x-T(y)) = \Gamma(T^2(x)-T(y)) = \Gamma(T(T(x)-y)) = \Gamma(T(x)-y),\\
&(x-T(y)+s\ell)_n = -(T(x)-y+sT(\ell))_n,\\
&A^{ij}(x-T(y)+s\ell)_i(x-T(y)+s\ell)_j =\\
&\qquad\qquad = A^{ij}(T(x)-y+sT(\ell))_i(T(x)-y+sT(\ell))_j.
\end{aligned}
$$

$\square$

Remark 3.1. Defining $\chi = \frac{T(x)-y}{|T(x)-y|}$ we have

$$
\theta(T(x)-y) = |T(x)-y|^{2-n}\psi(\chi,|T(x)-y|),
$$

where

$$
\psi(\chi,|T(x)-y|) = \frac{2}{n\omega_n(\det \boldsymbol{a})^{1/2}}\frac{\ell_n}{a^{nn}}\int_0^\infty \frac{e^{\sigma|T(x)-y|t}\,(\chi+tT(\ell))_n}{\Big(A^{ij}(\chi+tT(\ell))_i(\chi+tT(\ell))_j\Big)^{n/2}}\,dt
$$

is a regular function. In fact, by virtue of the positiveness of the matrix $\boldsymbol{A}$ we get for some constant $\overline{\lambda} > 0$

$$A^{ij}(\chi + tT(\ell))_i(\chi + tT(\ell))_j \geq \overline{\lambda}|\chi + tT(\ell)|^2 \geq \overline{\lambda}(1 - \delta_0^2) > 0$$

since

$$|\chi + tT(\ell)|^2 = 1 + 2t\chi \cdot T(\ell) + t^2|T(\ell)|^2 \geq 1 - 2t|T(\ell)|\delta_0 + t^2|T(\ell)|^2 \geq 1 - \delta_0^2 > 0$$

as consequence of the obliqueness of the field ℓ: $\chi \cdot T(\ell) \geq -|T(\ell)|\delta_0$, $\delta_0 < 1$ (the angle between χ and $T(\ell)$ is less than π).

Moreover, since $D_\xi^\alpha \psi(\chi, |\xi|) = O(|\xi|^{-|\alpha|})$ as $|\xi| \to 0$ the following estimate holds true

$$|D_\xi^\alpha \theta(\xi)| \leq C(n, |\alpha|, \ell, \boldsymbol{a})|\xi|^{2-n-|\alpha|}.$$

4. L^p a priori estimates

In this section we prove $W^{2,p}$ a priori estimate for the solutions of (1.1) as stated in Theorem 1.1.

Hereafter we shall denote by $\Gamma(x, \xi)$ and $\theta(x, \xi)$ the functions defined by (3.1) and (3.5) respectively, in the case when the coefficients a^{ij} (and therefore A^{ij}) depend on x. Further

$$\Gamma_i(x, \xi) = \frac{\partial \Gamma(x, \xi)}{\partial \xi_i}, \ \Gamma_{ij}(x, \xi) = \frac{\partial^2 \Gamma(x, \xi)}{\partial \xi_i \partial \xi_j},$$
$$\theta_i(x, \xi) = \frac{\partial \theta(x, \xi)}{\partial \xi_i}, \ \theta_{ij}(x, \xi) = \frac{\partial^2 \theta(x, \xi)}{\partial \xi_i \partial \xi_j}$$

and let

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha}{\partial \xi^\alpha} \Gamma_{ij}(x, \xi) \right\|_{L^\infty(\Omega \times \{|\xi|=1\})}.$$

If $T(x, y)$ is the reflection defined by (2.3) we define the vector $A(y) = (A_1(y), \dots, A_n(y))$ by

$$A(y) = \frac{\partial}{\partial x_n} T(x, y), \quad \text{i.e.} \quad A(y) = \left(-2\frac{a^{1n}(y)}{a^{nn}(y)}, \dots, -2\frac{a^{n-1,n}(y)}{a^{nn}(y)}, -1 \right).$$

Finally, the VMO-moduli of the coefficients $a^{ij}(x)$ of the operator $\mathcal{L}$ will be denoted by $\eta_{ij}(r)$ and $\eta(r) = (\eta_{ij}^2(r))^{1/2}$.

We start by recalling the following interior estimate proved by Chiarenza, Frasca and Longo in [CFL1].

Theorem 4.1. *Suppose that the operator $\mathcal{L}$ satisfies assumption* (1.2). *Then for all p, q: $1 < q \le p < \infty$ and $u \in W^{2,q}_{\rm loc}(\Omega)$ such that $\mathcal{L}u \in L^p_{\rm loc}(\Omega)$ we have $u \in W^{2,p}_{\rm loc}(\Omega)$. Moreover, given $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a constant $c = c(n, p, \lambda, M, \mathrm{dist}(\Omega', \partial\Omega''), \eta)$ such that*

$$\|u\|_{W^{2,p}(\Omega')} \le c\left(\|u\|_{L^p(\Omega'')} + \|\mathcal{L}u\|_{L^p(\Omega'')}\right).$$

The boundary estimate we need will be derived in several steps. First of all, we consider the simplest case of homogeneous boundary condition with constant coefficients boundary operator. The strategy is to use Lemma 3.1 in order to represent the second derivatives of the solution. As in Section 3 the symbols $\mathcal{L}_0$ and $\mathcal{B}_0$ stand for the elliptic and boundary operator, respectively, with constant coefficients. Moreover, without loss of generality we may assume that the solutions u of (1.1) we are dealing with are supported in the ball B_ϱ.

Lemma 4.2. *Let $u \in W^{2,p}(B_\varrho^+)$ be a solution of the equation $\mathcal{L}u = f$ in B_ϱ^+ such that $\mathcal{B}_0 u = 0$ on $B_\varrho \cap \{x_n = 0\}$. Then*

$$\begin{aligned} &D_{ij}u(x) = \\ &= P.V.\int_{B_\varrho^+} \Gamma_{ij}(x, x-y)\left\{\left(a^{hk}(x) - a^{hk}(y)\right) D_{hk}u(y) + f(y)\right\} dy \\ &+ c_{ij}(x) f(x) - I_{ij}(x,x) + J_{ij}(x,x) \quad \forall\, x \in B_\varrho^+, \end{aligned} \tag{4.1}$$

where

$$c_{ij}(x) = \int_{|\xi|=1} \Gamma_i(x,\xi)\xi_j \, d\sigma_\xi,$$

$$I_{ij}(x,z) = \int_{B_\varrho^+} \Gamma_{ij}(z, T(x,z) - y)\left\{\left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y)\right\} dy \qquad i,j < n,$$

$$I_{in}(x,z) = \int_{B_\varrho^+} \Gamma_{ij}(z, T(x,z) - y)\left\{\left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y)\right\} A_j(z)\, dy \qquad i < n,$$

$$\begin{aligned} &I_{nn}(x,z) = \\ &= \int_{B_\varrho^+} \Gamma_{ij}(z, T(x,z) - y)\left\{\left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y)\right\} A_i(z) A_j(z)\, dy, \end{aligned}$$

and

$$J_{ij}(x,z) = \int_{B_\varrho^+} \theta_{ij}(z, T(x,z)-y) \left\{ \left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y) \right\} dy \qquad i,j<n,$$

$$J_{in}(x,z) = \int_{B_\varrho^+} \theta_{ij}(z, T(x,z)-y) \left\{ \left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y) \right\} A_j(z)\, dy \qquad i<n,$$

$$J_{nn}(x,z) = \\ = \int_{B_\varrho^+} \theta_{ij}(z, T(x,z)-y) \left\{ \left(a^{hk}(z) - a^{hk}(y)\right) D_{hk}u(y) + f(y) \right\} A_i(z) A_j(z)\, dy.$$

Moreover there exist constants $\varrho_0 > 0$ *and* c *such that*

$$\|D_{ij}u\|_{L^p(B_\varrho^+)} \le c\|\mathcal{L}u\|_{L^p(B_\varrho^+)} \quad \forall\, \varrho \in (0, \varrho_0). \tag{4.2}$$

PROOF: Let $x_0 \in B_\varrho^+$ and suppose $u \in C^\infty(B_\varrho)$. Rewriting the equation $\mathcal{L}u = f$ as

$$\mathcal{L}_0 u \equiv a^{ij}(x_0) D_{ij}u(x) = \left(a^{ij}(x_0) - a^{ij}(x)\right) D_{ij}u(x) + f(x)$$

we have

$$u(x) = \int_{B_\varrho^+} G(x_0, x, y) \left\{ \left(a^{hk}(x_0) - a^{hk}(y)\right) D_{hk}u(y) + f(y) \right\} dy \tag{4.3}$$

where

$$G(x_0, x, y) = \Gamma(x_0, x-y) - \Gamma(x_0, T(x,x_0) - y) + \theta(x_0, T(x,x_0) - y)$$

by virtue of Lemma 3.1.

The difference above represents the half space Green function for $\mathcal{L}_0$ and therefore the first two integrals in (4.3) can be differentiated as in the proof of [CFL2, Theorem 3.2]. As it concerns the last term in (4.3) that includes $\theta(x_0, T(x,x_0) - y)$ it is possible to differentiate it inside the integral. Hence

$$D_{ij}u(x) = P.V. \int_{B_\varrho^+} \Gamma_{ij}(x_0, x-y) \left\{ \left(a^{hk}(x_0) - a^{hk}(y)\right) D_{hk}u(y) + f(y) \right\} dy \\ + c_{ij}(x_0) f(x) - I_{ij}(x, x_0) + J_{ij}(x, x_0) \quad \forall\, x \in B_\varrho^+.$$

Taking $x = x_0$ above we obtain (4.1) if $u \in C^\infty(B_\varrho)$.

To proceed further, let us note that each $\Gamma_{ij}(x, x-y)$ is a Calderón-Zygmund kernel with respect to $x-y$ as the first derivative of a function that is homogeneous of degree $1-n$. Therefore the principal value integral in (4.1) is bounded from $L^p(B_\varrho^+)$ into itself by virtue of Theorems 2.1, 2.2. Further, it follows from Theorems 2.3, 2.4, and Remarks 2.1 and 3.1 that the integral operators I_{ij} and J_{ij} are bounded too. Thus taking L^p norms of the both sides in (4.1) we have

$$\|D_{ij}u\|_{L^p(B_\varrho^+)} \le c\left(\|a\|_*\|D_{ij}u\|_{L^p(B_\varrho^+)} + \|f\|_{L^p(B_\varrho^+)}\right)$$

where $\|a\|_* \equiv \sum_{ij}\eta_{ij}$. Since $\|a\|_* \to 0$ as $\varrho \to 0$ ($a^{ij} \in VMO$), if we choose ϱ small enough we obtain (4.2).

Finally, a density argument based on Theorem 2.1 gives the statement of Lemma 4.2 in the case $u \in W^{2,p}(B_\varrho^+)$. □

For later purposes we need the following regularity result that is in fact a refinement of the a priori estimate (4.2).

Theorem 4.3. *Suppose $1 < q \le p < +\infty$ and let $u \in W^{2,q}(B_\varrho^+)$ be such that $\mathcal{L}u \in L^p(B_\varrho^+)$ and $\mathcal{B}_0 u = 0$ on $B_\varrho \cap \{x_n = 0\}$.*

Then $u \in W^{2,p}(B_\varrho^+)$ and there exists a constant c such that

$$\|D_{ij}u\|_{L^p(B_\varrho^+)} \le c\|\mathcal{L}u\|_{L^p(B_\varrho^+)}. \tag{4.4}$$

Proof: Let us set for $i,j,h,k = 1,\dots,n$

$$S_{ijhk}(f)(x) = P.V.\int_{B_\varrho^+}\Gamma_{ij}(x,x-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}dy,$$

$$\tilde S_{ijhk}(f)(x) = \begin{cases} \int_{B_\varrho^+}\Gamma_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}dy & i,j<n,\\ \displaystyle\int_{B_\varrho^+}\Gamma_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}A_j(x)\,dy & i<n,\\ \displaystyle\int_{B_\varrho^+}\Gamma_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}A_i(x)A_j(x)\,dy & i=j=n,\end{cases}$$

$$\tilde{\tilde S}_{ijhk}(f)(x) = \begin{cases} \int_{B_\varrho^+}\theta_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}dy & i,j<n,\\ \displaystyle\int_{B_\varrho^+}\theta_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}A_j(x)\,dy & i<n,\\ \int_{B_\varrho^+}\theta_{ij}(x,T(x)-y)\left\{\left(a^{hk}(x)-a^{hk}(y)\right)f(y)\right\}A_i(x)A_j(x)\,dy & i=j=n\end{cases}$$

(recall $T(x) = T(x,x)$). By means of Theorems 2.2, 2.3 and 2.4 all these operators are bounded from $L^p(B_\varrho^+)$ into itself. Moreover, condition $a^{ij} \in VMO$ ensures that we can choose $\varrho_0 > 0$ so small that

$$\sum_{i,j,h,k=1}^{n} \|S_{ijhk} + \tilde{S}_{ijhk} + \tilde{\tilde{S}}_{ijhk}\| < 1,$$

where the operator norm is in the space of linear operators from $L^r(B_\varrho^+)$ into itself, $r \in [q,p]$, $\varrho \in (0,\varrho_0)$.

Let us consider the mapping $\mathcal{F}$: $\left[L^r(B_\varrho^+)\right]^{n^2} \to \left[L^r(B_\varrho^+)\right]^{n^2}$, $r \in [q,p]$, $0 < \varrho < \varrho_0$, defined by

$$\mathcal{F}w = ((\mathcal{F}w)_{ij})_{i,j=1,\dots,n}$$

where

$$(\mathcal{F}w)_{ij} = \sum_{hk} \left(S_{ijhk} + \tilde{S}_{ijhk} + \tilde{\tilde{S}}_{ijhk}\right) w_{ij} + h_{ij}$$

and

$$h_{ij} = P.V. \int_{B_\varrho^+} \Gamma_{ij}(x, x-y)\mathcal{L}u(y)\,dy + c_{ij}(x)\mathcal{L}u(x) + \tilde{I}_{ij}(x) + \tilde{J}_{ij}(x),$$

$$\begin{aligned}
\tilde{I}_{ij}(x) &= \int_{B_\varrho^+} \Gamma_{ij}(x, T(x)-y)\mathcal{L}u(y)\,dy && i,j<n,\\
\tilde{I}_{in}(x) &= \int_{B_\varrho^+} \Gamma_{ij}(x, T(x)-y)A_j(x)\mathcal{L}u(y)\,dy && i<n,\\
\tilde{I}_{nn}(x) &= \int_{B_\varrho^+} \Gamma_{ij}(x, T(x)-y)A_i(x)A_j(x)\mathcal{L}u(y)\,dy, &&\\
\tilde{J}_{ij}(x) &= \int_{B_\varrho^+} \theta_{ij}(x, T(x)-y)\mathcal{L}u(y)\,dy && i,j<n,\\
\tilde{J}_{in}(x) &= \ ds \int_{B_\varrho^+} \theta_{ij}(x, T(x)-y)A_j(x)\mathcal{L}u(y)\,dy && i<n,\\
\tilde{J}_{nn}(x) &= \int_{B_\varrho^+} \theta_{ij}(x, T(x)-y)A_i(x)A_j(x)\mathcal{L}u(y)\,dy. &&
\end{aligned}$$

By virtue of Lemma 4.2 we have $h_{ij} \in L^p(B_\varrho^+)$. Moreover, $\mathcal{F}$ is a contraction mapping from $\left[L^r(B_\varrho^+)\right]^{n^2}$ into itself as a consequence of the choice of ϱ_0. Therefore $\mathcal{F}$ has a unique fixed point $w \in \left[L^r(B_\varrho^+)\right]^{n^2}$ and this fixed point must be the same for all $r \in [q,p]$. On the other hand $(D_{ij}u)_{i,j=1,\dots,n} \in \left[L^q(B_\varrho^+)\right]^{n^2}$ is also a fixed point of $\mathcal{F}$ as it follows from (4.1). Therefore $D_{ij}u(x) = w_{ij}(x) \in L^p(B_\varrho^+)$ for all $i, j = 1, \dots, n$.

The estimate (4.4) is a consequence of (4.2). $\square$

Proof of Theorem 1.1: It will be done in three steps.

Step 1. Let $u \in W^{2,q}(B_\varrho^+)$, $1 < q \le p < \infty$, solves the nonhomogeneous oblique derivative problem with constant coefficients boundary operator, i.e. $\mathcal{L}u = f \in L^p(B_\varrho^+)$ and $\mathcal{B}_0 u \equiv \partial u/\partial \ell(x_0) + \sigma(x_0)u = \varphi(x) \in W^{1-1/p,p}(B_\varrho \cap \{x_n = 0\})$ where $x_0 \in B_\varrho^+$. We want to prove that $u \in W^{2,p}(B_\varrho^+)$ and

$$\|D_{ij}u\|_{L^p(B_\varrho^+)} \le c\left(\|f\|_{L^p(B_\varrho^+)} + \|\varphi\|_{W^{1-1/p,p}(B_\varrho \cap \{x_n=0\})}\right). \tag{4.5}$$

In fact, let $v \in W^{2,p}(B_\varrho^+)$ be such that $\partial v/\partial \ell(x_0) + \sigma(x_0)v = \varphi(x)$ on $B_\varrho \cap \{x_n = 0\}$. For example, we may take the function v in such a way that $v|_{\{x_n=0\}} = 0$, $\partial v/\partial x_n|_{\{x_n=0\}} = \varphi(x)/\ell_n(x_0)$ (see [AR] for the details). Then $\mathcal{L}(u-v) = f - \mathcal{L}v \in L^p(B_\varrho^+)$ and $\mathcal{B}_0(u-v) = 0$ on $B_\varrho \cap \{x_n = 0\}$. Applying Theorem 4.3 and using

$$\|v\|_{W^{2,p}(B_\varrho^+)} \le c\|\varphi\|_{W^{1-1/p,p}(B_\varrho \cap \{x_n=0\})}$$

we obtain (4.5).

Step 2. Suppose that $u \in W^{2,q}(B_\varrho^+)$, $1 < q \le p < \infty$, is such that $\mathcal{L}u = f \in L^p(B_\varrho^+)$ and $\mathcal{B}u = \varphi(x) \in W^{1-1/p,p}(B_\varrho \cap \{x_n = 0\})$. Then $u \in W^{2,p}(B_\varrho^+)$ and there exist constants c, $\varrho_0 > 0$ such that

$$\|u\|_{W^{2,p}(B_\varrho^+)} \le c\left(\|u\|_{L^p(B_\varrho^+)} + \|f\|_{L^p(B_\varrho^+)} + \|\varphi\|_{W^{1-1/p,p}(B_\varrho \cap \{x_n=0\})}\right) \tag{4.6}$$

for all $\varrho \in (0, \varrho_0)$.

Let the ball B_ϱ to be centered at $x_0 \in \{x_n = 0\}$. Then rewriting $\mathcal{B}u = \varphi$ as

$$\begin{aligned}\mathcal{B}_0 u \equiv \frac{\partial u(x)}{\partial \ell(x_0)} + \sigma(x_0)u(x) = \varphi(x) &+ \Big(\ell_i(x_0) - \ell_i(x)\Big)D_i u(x) \\ &+ \Big(\sigma(x_0) - \sigma(x)\Big)u(x) = \tilde{\varphi}(x)\end{aligned}$$

we have $\tilde{\varphi} \in W^{1-1/p,p}(B_\varrho \cap \{x_n = 0\})$ and therefore $u \in W^{2,p}(B_\varrho^+)$ as it was proved above. Moreover,

$$\begin{aligned}&\|\tilde{\varphi}\|_{W^{1-1/p,p}(B_\varrho \cap \{x_n=0\})} \\ &\qquad \le c\left(\|\varphi\|_{W^{1-1/p,p}(B_\varrho \cap \{x_n=0\})} + \|u\|_{W^{1,p}(B_\varrho^+)} + \varrho\|D_{ij}u\|_{L^p(B_\varrho^+)}\right).\end{aligned}$$

Here we used the fact that the coefficients of $\mathcal{B}$ are Lipschitz functions and the Rademacher theorem asserts that they admit L^∞ first derivative almost everywhere. Thus (4.5) implies

$$\begin{aligned}
&\|u\|_{W^{2,p}(B_\varrho^+)}\\
&\quad\le c\left(\|f\|_{L^p(B_\varrho^+)}+\|u\|_{W^{1,p}(B_\varrho^+)}+\|\varphi\|_{W^{1-1/p,p}(B_\varrho\cap\{x_n=0\})}+\varrho\|D_{ij}u\|_{L^p(B_\varrho^+)}\right)\\
&\quad\le c\left(\|f\|_{L^p(B_\varrho^+)}+\|u\|_{L^p(B_\varrho^+)}+\|\varphi\|_{W^{1-1/p,p}(B_\varrho\cap\{x_n=0\})}+\varrho\|D_{ij}u\|_{L^p(B_\varrho^+)}\right)
\end{aligned}$$

after applying the Gagliardo-Nirenberg interpolation inequality (see [NL, p. 125]). Since the constant c above is independent of ϱ, if we choose ϱ_0 to be sufficiently small the term $\|D_{ij}u\|_{L^p(B_\varrho^+)}$ may be moved on the left and the estimate (4.6) follows.

STEP 3. Standard arguments based on suitable partition of unity, covering and local flattening of $\partial\Omega$, as well as the application of the interior and boundary estimates (Theorem 4.1 and (4.6)) complete the proof of Theorem 1.1. The last thing we have to point out is that the local $C^{1,1}$-diffeomorphisms that straighten the boundary preserve the VMO character of the coefficients $a^{ij}(x)$.

5. The oblique derivative problem

First of all we deduce uniqueness result for the strong solutions of (1.1).

Theorem 5.1. *Suppose* (1.2) *and* (1.3) *to be fulfilled and let* $u, v \in W^{2,p}(\Omega)$, $1 < p < \infty$, *be strong solutions of the oblique derivative problem* (1.1).

Then $u = v$ *in* Ω.

PROOF: The difference $u - v \in W^{2,p}(\Omega)$ solves the homogeneous boundary value problem (1.1) ($f = 0$ and $\varphi = 0$). The regularity assertion of Theorem 1.1 implies that $u - v \in W^{2,q}(\Omega)$ for all $q \in (1, \infty)$. Then the statement of the theorem follows from a variant of the Aleksandrov-Pucci-Bakelman maximum principle for the oblique derivative problem proved by Luo and Trudinger [LT, Theorem 3.1] (see [L, Theorem 1.5] also). □

PROOF OF THEOREM 1.2: There exists a function $v \in W^{2,p}(\Omega)$ (see [AR]) such that $\mathcal{B}v = \varphi$ in the sense of trace on $\partial\Omega$. Thus, subtracting v of u we may consider the problem

$$\begin{cases} \mathcal{L}u \equiv a^{ij}(x)D_{ij}u = f(x) \quad \text{a.e. } \Omega \\ \mathcal{B}u \equiv \dfrac{\partial u}{\partial \ell} + \sigma(x)u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.1}$$

instead of (1.1). In this setting, the estimate (1.5) has the form

$$\|u\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)} \tag{5.2}$$

with a constant c that does not depend on u.

To prove (5.2) we suppose the opposite. It means that for fixed values of the parameters n, p, M, $\partial\Omega$, λ, η and for every $h \in \mathbb{N}$ there exists a sequence of operators $\mathcal{L}_h \equiv a_h^{ij}(x)D_{ij}$ satisfying (1.2), and a sequence of functions $\{u_h(x)\}$, $u_h \in W^{2,p}(\Omega)$ such that $\mathcal{B}u_h = 0$ on $\partial\Omega$ and

$$\|u_h\|_{W^{2,p}(\Omega)} = 1, \qquad \lim_{h\to\infty} \|\mathcal{L}_h u_h\|_{L^p(\Omega)} = 0.$$

By means of the VMO results already stated in Section 2, we may choose a_h^{ij} in such a way that their VMO moduli η_h and L^∞ norms are uniformly bounded by $\eta(r)$ and $\|a^{ij}\|_{L^\infty(\Omega)}$, respectively.

Let $B \subset \mathbb{R}^n$ be a ball of radius r and set $\Omega_r = B \cap \Omega$. The sequence $\{a_h^{ij} - (a_h^{ij})_{\Omega_r}\}$ is compact in $L^1(\Omega_r)$. In fact,

$$\|a_h^{ij} - (a_h^{ij})_{\Omega_r}\|_{L^\infty(\Omega_r)} \leq 2\lambda^{-1} \qquad \forall\, h \in \mathbb{N}$$

and moreover

$$\frac{1}{|\Omega_r|} \left\| \left(a_h^{ij}(x-y) - (a_h^{ij})_{\Omega_r}\right) - \left(a_h^{ij}(x) - (a_h^{ij})_{\Omega_r}\right)\right\|_{L^1(\Omega_r)}$$
$$\leq \|a_h^{ij} - (a_h^{ij})_{\Omega_r}\|_* \leq c\,\eta_h(r) \leq c\,\eta(r)$$

as consequence of the choice of a_h^{ij}. Thus

$$\|a_h^{ij} - (a_h^{ij})_{\Omega_r}\|_{L^1(\Omega_r)} \leq c|\Omega_r|\eta(r) \to 0 \ \text{ as } \ r \to 0,$$

and therefore the set $\{a_h^{ij} - (a_h^{ij})_{\Omega_r}\}$ is uniformly bounded and equicontinuous in $L^1(\Omega_r)$.

It follows that a subsequence of $\{a_h^{ij}\}$ converges almost everywhere in Ω_r to a function $\alpha^{ij}(x)$. Standard arguments based on a suitable covering of Ω with balls and taking subsequences lead to the conclusion that $\{a_h^{ij}\}$ converges almost everywhere on Ω to $\alpha^{ij}(x)$, and $\alpha^{ij}(x)$ satisfy (1.2).

Denote $\mathcal{L}_\alpha \equiv \alpha^{ij}(x)D_{ij}$.

By virtue of Theorem 1.1 the set $\{u_h\}$ is bounded in $W^{2,p}(\Omega)$ and therefore a subsequence (still denoted $\{u_h\}$) is weakly convergent to a function $u_\alpha \in W^{2,p}(\Omega)$. Furthermore, Rellich theorem implies $\lim_{h\to\infty} \|u_h - u_\alpha\|_{W^{1,p}(\Omega)} = 0$ whence

$$0 = \mathcal{B}u_h = \frac{\partial u_h}{\partial \ell} + \sigma u_h \to \frac{\partial u_\alpha}{\partial \ell} + \sigma u_\alpha = \mathcal{B}u_\alpha \ \text{ on } \ \partial\Omega \ \text{ as } \ h \to \infty.$$

On the other hand, for each $w \in L^{p/(p-1)}(\Omega)$ we have

$$\int_\Omega |\mathcal{L}_h u_h - \mathcal{L}_\alpha u_\alpha||w|\,dx \le \int_\Omega |\mathcal{L}_h u_h - \mathcal{L}_h u_\alpha||w|\,dx + \int_\Omega |\mathcal{L}_h u_\alpha - \mathcal{L}_\alpha u_\alpha||w|\,dx$$

$$\le \lambda^{-1}\int_\Omega |w||D_{ij}u_h - D_{ij}u_\alpha|\,dx + \int_\Omega |w||a_h^{ij} - \alpha^{ij}|\,|D_{ij}u_\alpha|\,dx \to 0$$

as result of the weak convergence of u_h to u_α and almost everywhere convergence of a_h^{ij} to α^{ij}. Hence

$$\|\mathcal{L}_\alpha u_\alpha\|_{L^p(\Omega)} = \lim_{h\to\infty}\|\mathcal{L}_h u_h\|_{L^p(\Omega)} = 0, \qquad \mathcal{B}u_\alpha = 0 \ \text{ on } \ \partial\Omega$$

and the uniqueness result (Theorem 5.1) implies $u_\alpha = 0$ in Ω that contradicts $1 = \|u_\alpha\|_{L^p(\Omega)} = \lim_{h\to\infty}\|u_h\|_{L^p(\Omega)}$. Thus the estimate (5.2) is proved.

We are now in position to prove the existence part of Theorem 1.2. For this goal the *method of continuity* [GT, Theorem 5.2] will be used.

Assume first that $p > n$ and for $\tau \in [0,1]$ consider the family of oblique derivative problems:

$$\begin{cases} \mathcal{L}_\tau u \equiv \tau\mathcal{L}u + (1-\tau)\Delta u = f \ \text{ a.e. } \Omega \\ \mathcal{B}_\tau u \equiv \tau\mathcal{B}u + (1-\tau)\left(\dfrac{\partial u}{\partial \nu} - u\right) = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{5.3}$$

Obviously $\mathcal{L}_\tau$ and $\mathcal{B}_\tau$ satisfy conditions (1.2) and (1.3) respectively, and the problem (5.3) coincides with (5.1) in the case $\tau = 1$.

Defining the operator

$$\mathcal{P}_\tau = (\mathcal{L}_\tau, \mathcal{B}_\tau) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \times \{0\}$$

by $\mathcal{P}_\tau = (\mathcal{L}_\tau u, \mathcal{B}_\tau u) = (\mathcal{L}_\tau u, 0)$, it is clear that the solvability of (5.3) is equivalent to the fact that $\mathcal{P}_\tau$ is an invertible mapping. To show this, let u_τ be a solution of (5.3) for fixed $f \in L^p(\Omega)$. It follows from Theorem 5.1 and (5.2) that

$$\|u_\tau\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)} = c\|\mathcal{P}_\tau u_\tau\|_{L^p(\Omega)\times\{0\}}$$

and c does not depend on τ. Thus $\mathcal{P}_\tau$ is *one-to-one* mapping for each $\tau \in [0,1]$. Now the invertibility of $\mathcal{P}_\tau$ for $\tau = 1$ (i.e. the solvability of (5.1)) is a consequence of continuity's method and the fact that $\mathcal{P}_0$ maps $W^{2,p}(\Omega)$ *onto* $L^p(\Omega) \times \{0\}$, i.e. the operator $\mathcal{P}_0$ is invertible. Indeed, the last claim is equivalent to the solvability in $W^{2,p}(\Omega)$ of the third boundary value problem

$$\begin{cases} \Delta u = f(x) & \text{a.e. } \Omega \\ \dfrac{\partial u}{\partial \nu} - u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

for each $f \in L^p(\Omega)$ that follows immediately from [C1, Theorem 2] since $p > n$. The only fact we have to point out is that the cited result of M. Chicco remains true in the case $\partial\Omega \in C^{1,1}$ by means of Rademacher theorem.

To prove solvability of (5.1) for $1 < p \le n$ we take a sequence $\{f_h\}$ of L^q-functions, $q > n$, that approximates $f(x)$ in $L^p(\Omega)$. Denoting by $\{u_h(x)\}$ the sequence of solutions of the corresponding oblique derivative problems, similar arguments to those used in proving (5.2) complete the proof of Theorem 1.2. $\square$

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Department of Mathematics, University of Catania, 95125 Catania, Italy

Department of Mathematics, Technological University of Sofia, 1156 Sofia, Bulgaria

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