The G_δ-topology and incompactness of ω^α

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Abstract. We establish a relation between covering properties (e.g. Lindelöf degree) of two standard topological spaces (Lemmas 4 and 5). Some cardinal inequalities follow as easy corollaries.

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The present note is a contribution into the study of the Lindelöf degree in powers of topological spaces. It answers a question of W.A.R. Weiss.

In what follows, $\kappa^* \subset \beta\kappa$ is the space of all free ultrafilters over a discrete set of κ points, $\mu\kappa \subset \beta\kappa$ is the space of all *uniform* ultrafilters over κ, ω^α denotes the α-th power of the discrete set of integers, $(\mu\kappa, G_\delta)$ is $\mu\kappa$ with the finer G_δ topology. $L(X)$ denotes the Lindelöf degree of X, and $e(X) := \sup\{A \subset X : A$ is closed and discrete$\}$ — its *extent.*

Kenneth Kunen ([1]) proves $L\big(\mu(2^\kappa)^+, G_\delta\big) \geq \kappa^+$, for the same κ's as in our Corollary 6. Our Corollary 6 gives here

$$L\big(\mu(2^\kappa)^+, G_\delta\big) \geq (2^\kappa)^+.$$

J. Mycielski proved ([2]), by inductive "stepping up", that, for α less than the 1^{st} weakly inaccessible cardinal,

$$e(\omega^\alpha) = \alpha.$$

Our Corollary 9 is a weaker statement for a larger class of cardinals. This Corollary was obtained first by Łoś [3] in 1959 using group-theoretic methods. See also Juhàsz [4].

Stevo Todorcevič ([5]) proves, assuming the combinatorial statement $\square_\kappa$,

$$L(\omega^\kappa) = \kappa.$$

1. If $\mathcal{A} = \{A_n : n < \omega\}$ is a countable disjoint partition of the cardinal κ, then

$$\begin{aligned}
\mu\kappa &= \Big(\dot{\bigcup_{n<\omega}} S_n^{\mathcal{A}}\Big)\dot{\cup}\Big(\tilde{S}^{\mathcal{A}}\Big), \quad \text{where}\\
S_n^{\mathcal{A}} &= \{u \in \mu\kappa : A_n \in u\} \ \text{ and}\\
\tilde{S}^{\mathcal{A}} &= \{u \in \mu\kappa : \{\bigcup_{n\geq i} A_n : i < \omega\} \subset u\}.
\end{aligned}$$

Note that $\tilde{S}^{\mathcal{A}}$ is a G_δ set in $\mu\kappa$.

2. We say that a cover of κ^* or of $\mu\kappa$ is a *proper G_δ-cover* if every set in it is of the form $\tilde{S}^{\mathcal{A}}$ for some countable partition $\mathcal{A}$ of κ.

3. Lemma. *If κ is a regular cardinal and $\mu\kappa$ has a proper G_δ-cover of size α, then ω^α has a subset of size κ without a CAP (complete accumulation point).*

PROOF: Suppose $\mu\kappa = \bigcup\{\tilde{S}^{\mathcal{A}^\gamma} : \gamma < \alpha\}$ for some collection $\mathcal{C} = \{\mathcal{A}^\gamma : \gamma < \alpha\}$ of countable partitions $\mathcal{A}^\gamma = \{A_n^\gamma : n < \omega\}$ of κ. For every point $p \in \kappa$ define its history in $\mathcal{C}$ $\bar{p} : \alpha \longrightarrow \omega$ by setting $\bar{p}(\gamma) := n$ such that $p \in A_n^\gamma$. Let $P = \{\bar{p} : p < \kappa\} \subset \omega^\alpha$.

Claim 1. $|P| = \kappa$, moreover, for every $p \in \kappa$, $K_p := \{q \in \kappa : \bar{q} = \bar{p}\}$ has size $|K_p| < \kappa$. Indeed, if not, then no $v \ni K_p$ is covered:

$$\forall\gamma < \alpha \ \ v \notin \tilde{S}^{A^\gamma},$$

because

$$v \in S_{\bar{p}(\gamma)}^{A^\gamma}.$$

And $|P| = \kappa$ follows from the regularity of κ. □

Claim 2. P has no CAP in ω^α. If not, let $\varphi \in \omega^\alpha$ be a CAP of P. Then for every finite $F \subset \alpha$

$$|\{p < \kappa : \bar{p} \restriction F = \varphi \restriction F\}| = \kappa,$$

by Claim 1.

Therefore, the family $\mathcal{F} := \{A_{\varphi(\gamma)}^\gamma : \gamma < \alpha\}$ has the uniform finite intersection property (i.e. $\forall\mathcal{F}_0 \in [\mathcal{F}]^{<\aleph_0}\ |\cap\mathcal{F}_0| = \kappa$). [By $\bar{p} \restriction F = \varphi \restriction F \longleftrightarrow p \in \bigcap_{\gamma\in F} A_{\varphi(\gamma)}^\gamma$].

Pick a $u \in \mu\kappa$ extending $\mathcal{F}$.

Then $u \notin \bigcup_{\gamma<\alpha} \tilde{S}^{A^\gamma} = \mu\kappa$. Contradiction. Hence $P \subset \omega^\alpha$ has no CAP in ω^α, so it is as required. □

4. Lemma. *If κ^* has a proper G_δ-cover of size α, then ω^α has a closed discrete subset of size κ.*

PROOF: Here we, similarly, study the family P of the histories of points $p \in \kappa$ in the family of partitions of κ defining our proper G_δ-cover, in this case of κ^*. Only finitely many points $p \in \kappa$ may have the same history, so $|P| = \kappa$, and, arguing as in Claim 2 of the previous lemma, $P \subset \omega^\alpha$ has no limit points in ω^α whatsoever. $\square$

5. Theorem. *If κ is a regular not Ulam measurable cardinal, then*

$$L\big(\omega^{L(\mu\kappa, G_\delta)}\big) \geq \kappa.$$

6. Corollary. *$L(\mu\kappa, G_\delta) \geq \kappa$, for the same κ's.*

PROOF OF THEOREM 5 AND COROLLARY 6: Since every ultrafilter over κ is countably incomplete, there is a proper G_δ cover of $\mu\kappa$, and so $L(\mu\kappa, G_\delta) = \alpha \Rightarrow$ there is a proper G_δ-cover of size α $\Rightarrow$ (By Lemma 3) ω^α has a subset of size κ without a CAP $\Rightarrow$

(a) $L(\omega^\alpha) \geq \kappa$, and
(b) $\alpha \geq \kappa$ (because $\alpha \geq L(\omega^\alpha)$).

$\square$

7. Corollary. *$L\big(\omega^{2^\kappa}\big) \geq \kappa$, for the same κ's as in Theorem 5.*

PROOF: $(\mu\kappa, G_\delta)$ has a base of size $(2^\kappa)^\omega = 2^\kappa$. Hence $L(\mu\kappa, G_\delta) \leq 2^\kappa$ and so $L(\omega^{2^\kappa}) \geq L(\omega^{L(\mu\kappa, G_\delta)}) \geq \kappa$. $\square$

8. Theorem. . *If κ is not Ulam measurable, then*

$$L(\kappa^*, G_\delta) \geq L\big(\omega^{L(\kappa^*, G_\delta)}\big) \geq e\big(\omega^{L(\kappa^*, G_\delta)}\big) \geq \kappa.$$

PROOF: Immediate from Lemma 4. $\square$

9. Corollary. *If $\kappa < 1^{st}$ measurable cardinal, then*

$$e\big(\omega^{2^\kappa}\big) \geq \kappa,$$

i.e. ω^{2^κ} has a closed discrete subspace of size κ.

PROOF: Same as of Corollary 7. $\square$

10. Corollary. *Let λ be a strong limit cardinal $\leq$ the 1^{st} measurable cardinal. Then the set $\{e(\omega^\alpha) : \alpha < \lambda\}$ is cofinal in λ. Hence, if $cf(\lambda) > \omega$, the set $\{\alpha < \lambda : e(\omega^\alpha) = \alpha\}$ is closed and unbounded in λ.*

Remark. Murray Bell observed that the converses of Lemmas 3 and 4 are also true.

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References

[1] Kunen K., *Box products of ordered spaces*, Topology and its applications **20** (1985).
[2] Mycielski J., *α-incompactness of N^{α}*, Bull. Acad. Pol., vol. XII, no. 8, 1964.
[3] Łoś J., *Linear equations and pure subgroups*, Bull. Acad. Polon. Sci. **7** (1959).
[4] Juhàsz I., *On closed discrete subspaces of product spaces*, Bull. Acad. Pol., vol. XVII, no. 4, 1969.
[5] Todorcevic̀ S., *Incompactness of* $\mathbb{N}^{\theta}$, Handwritten notes, 1990.

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