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On positive operator-valued continuous maps

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Abstract. In the paper the geometric properties of the positive cone and positive part of the unit ball of the space of operator-valued continuous space are discussed. In particular we show that

$\mathsf{ext\text{-}ray}\ \mathsf{C}_+(K,\mathcal{L}(H)) = \{\mathbb{R}_+\mathbf{1}_{\{k_0\}}\mathbf{x}\otimes\mathbf{x} : \mathbf{x}\in\mathbf{S}(H), k_0 \text{ is an isolated point of } K\}$

$\mathsf{ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H))) = \mathsf{s\text{-}ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H)))$
$= \{f\in\mathsf{C}(K,\mathcal{L}(H) : f(K)\subset \mathsf{ext}\ \mathbf{B}_+(\mathcal{L}(H))\}.$

Moreover we describe exposed, strongly exposed and denting points.

Keywords: exposed point, denting point, Hilbert space, positive operator

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1. Introduction

The paper is devoted to the geometric properties of the space of continuous functions from a compact Hausdorff space K with values in the space of operators acting on a Hilbert space H. Namely, we deal with the positive part of the unit ball and the cone of positive operators in $\mathcal{L}(H)$. We consider such points as strongly extreme, exposed, strongly exposed and denting points.

For a Banach space E we denote by $\mathbf{B}(E)$ and $\mathbf{S}(E)$ respectively the unit ball and the unit sphere of E. A subset P of E is called a *convex cone* (of vertex 0) if P is convex ($\mathbf{x},\mathbf{y}\in P$, $\alpha\in[0,1] \Rightarrow \alpha\mathbf{x}+(1-\alpha)\mathbf{y}\in P$) and invariant under multiplication by positive constant ($\mathbf{x}\in P, \lambda\in\mathbb{R}_+ \Rightarrow \lambda\mathbf{x}\in P$). A ray $R=\{\lambda\mathbf{x}_o : \lambda\in\mathbb{R}_+\} = \mathbb{R}_+\mathbf{x}_o$, $0\neq\mathbf{x}_o\in P$, is called an *extreme ray* ($R\in$ **ext-ray** P) if $\mathbf{x}\in R$, $\mathbf{y}\in P$, and $\mathbf{x}-\mathbf{y}\in P$ imply $\mathbf{y}\in R$.

A point $\mathbf{q}$ of a convex set $Q\subset E$ is *extreme* ($\mathbf{q}\in$ **ext** Q) if it is not the midpoint of any segment of positive length contained in Q; *strongly extreme* ($\mathbf{q}\in$ **s-ext** Q) if $\|\frac{\mathbf{x}_n+\mathbf{y}_n}{2}-\mathbf{q}\|\to 0$ for $\mathbf{x}_n$, $\mathbf{y}_n\in Q$ implies $\|\mathbf{x}_n-\mathbf{q}\|\to 0$ (or equivalently $\|\mathbf{x}_n-\mathbf{y}_n\|\to 0$, since $\mathbf{x}_n-\mathbf{q}=\frac{\mathbf{x}_n-\mathbf{y}_n}{2}+(\frac{\mathbf{x}_n+\mathbf{y}_n}{2}-\mathbf{q})$); *exposed* ($\mathbf{q}\in$ **exp** Q) if there exists $\xi\in Q^*$ such that $\xi(q)=\sup\xi(Q)>\xi(\mathbf{x})$ for all $\mathbf{x}\in Q\setminus\{\mathbf{q}\}$; *strongly exposed* ($\mathbf{q}\in$ **s-exp** Q) if it is exposed and if $\xi(x_n)\to\xi(q)$ for $\mathbf{x}_n\in Q$ then $\|\mathbf{x}_n-\mathbf{q}\|\to 0$; and *denting* ($\mathbf{q}\in$ **dent** Q) if for all $\varepsilon>0$ we have $\mathbf{q}\notin \overline{\mathbf{conv}}\,(Q\setminus\{\mathbf{q}+\varepsilon\mathbf{B}(E)\})$. Note that in general this classes of points do not coincide. We have **s-exp** $Q\subset$ **dent** $Q\subset$ **s-ext** $Q\subset$ **ext** Q and **s-exp** $Q\subset$ **exp** $Q\subset$ **ext** Q.

[1] Written while the author was a research fellow of the Alexander von Humboldt-Stiftung at Mathematisches Institut der Eberhard Karls-Universität in Tübingen.

Moreover, if Q is compact then $\textsf{\textbf{dent}}\ Q = \textsf{\textbf{s-ext}}\ Q = \textsf{\textbf{ext}}\ Q$ and $\textsf{\textbf{s-exp}}\ Q = \textsf{\textbf{exp}}\ Q$. Note that if $\mathbf{q} \in \textsf{\textbf{ext}}\ Q$ is a point of continuity for Q ($\mathbf{x}_n \to \mathbf{q}$ weakly, $\mathbf{x}_n \in Q$, implies $\mathbf{x}_n \to \mathbf{q}$ in norm) then $\mathbf{q} \in \textsf{\textbf{dent}}\ Q$ ([14]). For an operator $T : E \to E$ we denote by $\textsf{\textbf{IsDom}}\ T = \{\mathbf{x} \in E : \|T\mathbf{x}\| = \|\mathbf{x}\|\}$ its isometric domain.

Let H be a (real or complex) Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$. By $\mathcal{L}(H)$ we denote the space of bounded operators acting on H. The space $\mathcal{L}(H)$ is equipped with the standard operator norm. Note that $\textsf{\textbf{IsDom}}\ T$ is a closed linear subspace for all $T \in \mathbf{B}(\mathcal{L}(H))$. Moreover, $T(\{\mathbf{x}\}^{\perp}) \subset (T\mathbf{x})^{\perp}$ for $\mathbf{x} \in \textsf{\textbf{IsDom}}\ T$ and $T\left((\textsf{\textbf{IsDom}}\ T)^{\perp}\right) \perp T(\textsf{\textbf{IsDom}}\ T)$, $T \in \mathbf{B}(\mathcal{L}(H))$.

For $\mathbf{y}, \mathbf{z} \in H$ we denote by $\mathbf{y} \otimes \mathbf{z}$ the one dimensional operator defined by $(\mathbf{y} \otimes \mathbf{z})(\mathbf{x}) = \mathbf{y}\langle \mathbf{x}, \mathbf{z} \rangle$, $\mathbf{x} \in H$.

The operator $T \in \mathcal{L}(H)$ is called *positive* ($T \geq 0$) if T is self-adjoint ($T = T^*$) and $\langle T\mathbf{x}, \mathbf{x} \rangle \ \geq 0$ for all $\mathbf{x} \in H$. An operator T is a (orthogonal) projection if $T = T^2$ and $T = T^*$. If T is a projection then $T\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \perp \textsf{\textbf{IsDom}}\ T$.

The cone of all positive operators is denoted by $\mathcal{L}_+(H)$. The positive part of the unit ball is denoted by $\mathbf{B}_+(\mathcal{L}(H))$. Note that $\|T\| = \sup\{\langle Tx, x \rangle : \|x\| \leq 1\}$ for $T \geq 0$. Hence $\|T\| \leq \|T + R\|$ for $T, R \in \mathcal{L}_+(H)$.

Let $T \in \mathbf{B}_+(\mathcal{L}(H)) = \{T \in \mathcal{L}(H) : 0 \leq T \leq I\}$. Then $T^2, (I-T) \in \mathbf{B}_+(\mathcal{L}(H))$. We have $2T - T^2 = T(2I - T) \geq 0$ and $0 \leq (I-T)^2 = I - 2T + T^2$, so $2T - T^2 \leq I$. Thus $2T - T^2 \in \mathbf{B}_+(\mathcal{L}(H))$, too.

A one dimensional operator $\mathbf{x} \otimes \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbf{S}(H)$, is positive if and only if $\mathbf{x} = \mathbf{y}$.

Let $\textsf{\textbf{C}}(K, E)$ denote the Banach space of all continuous functions from a compact Hausdorff space K into a Banach space E equipped with the supremum norm $\|f\| = \sup_{k \in K} \|f(k)\|_E$.

Obviously for a convex set $Q \subset E$ if $f(K) \subset \textsf{\textbf{ext}}\ Q$ then $f \in \textsf{\textbf{ext}}\ \{f \in \textsf{\textbf{C}}(K, E) : f(K) \subset Q\}$. There is a natural question for which classes of convex sets Q the inverse implication characterize extreme points. Negative example of continuous function $F : K \to Q$ (Q is closed symmetric subset of $\mathbb{R}^4$) was presented in [2]. In fact $f \in \textsf{\textbf{ext}}\ f \in \mathbf{B}(\textsf{\textbf{C}}(K, E))$ with $f(k) \notin \textsf{\textbf{ext}}\ \mathbf{B}(E)$ for all $k \in K$.

Using Michael's selection theorem ([16]) we can prove that $\textsf{\textbf{ext}}\ \{f \in \textsf{\textbf{C}}(K, E) : f(K) \subset Q\} = \{f \in \textsf{\textbf{C}}(K, E) : f(K) \subset \textsf{\textbf{ext}}\ Q\}$ for any stable convex subset Q of E. Recall that a convex set $Q \subset E$ is said to be *stable* if the barycenter map $Q \times Q \ni (\mathbf{x}, \mathbf{y}) \to \frac{\mathbf{x}+\mathbf{y}}{2} \in Q$ is open. Point out that in finite dimensional space a set is stable (see [18]) if and only if all m-skeletons ($m = 0, 1, \ldots, n$) of Q are closed (an m-skeleton of Q is a set of all $\mathbf{x} \in Q$ such that the face generated by $\mathbf{x}$ in Q has dimension less than or equal to m).

We say that a compact Hausdorff space K carries a strict positive measure if there exists a strictly positive Radon measure μ on K (i.e. $\mu(U) > 0$ for all nonempty open $U \subset K$). The problem of characterization of spaces K which carry a strictly positive measure has been studied by many authors (e.g., see [1], [5], [11], [15], [17]). In particular Kelley ([13]) introduced the notion of intersection numbers of a collection of subsets to give the characterization of spaces which carry a strictly positive measure. It should be pointed out that in the case of a compact

Hausdorff space the problem mentioned above is equivalent to the problem of existence of a finitely additive strictly positive measure. Note that $\mathbf{C}(K,\mathbb{R})$ carries a strictly positive functional if and only if its dual $\mathbf{C}(K,\mathbb{R})$ contains a weakly compact total subset ([19, Theorem 4.5b]). We refer the reader [3, Chapter 6], for summary of those and related results. In fact a strictly positive measure on K can be considered as a functional on $\mathbf{C}(K,\mathbb{R})$ which exposes the function $\mathbf{1}_K$. By $\mathbf{1}_A$ we denote the characteristic function of a set $A \subset K$.

We have

$$\textbf{ext } \mathbf{B}_+(\mathcal{L}(H)) = \{T \in \mathcal{L}(H) : T^2 = T, T^* = T\} \qquad ([12], [7])$$

$$\textbf{s-ext } \mathbf{B}_+(\mathcal{L}(H)) = \textbf{ext } \mathbf{B}_+(\mathcal{L}(H)) \quad ([9]).$$

$$\textbf{exp } \mathbf{B}_+(\mathcal{L}(H)) = \begin{cases} \textbf{ext } \mathbf{B}_+(\mathcal{L}(H)) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases} \quad ([7]),$$

$$\textbf{s-exp } \mathbf{B}_+(\mathcal{L}(H)) = \textbf{dent } \mathbf{B}_+(\mathcal{L}(H)) = \begin{cases} \textbf{ext } \mathbf{B}_+(\mathcal{L}(H)) & \text{if } \textbf{dim } H < \infty \\ \emptyset & \text{if } \textbf{dim } H = \infty \end{cases},$$

$$\textbf{ext-ray } \mathcal{L}_+(H) = \{\mathbb{R}_+ \mathbf{x} \otimes \mathbf{x} : \mathbf{0} \neq \mathbf{x} \in H\}.$$

The aim of this paper is to continue investigation for the space of operator valued continuous functions with values in $\mathcal{L}_+(H)$. We show that

$$\textbf{ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) = \{f \in \mathbf{C}(K,\mathcal{L}(H)) : f(K) \subset \textbf{ext } \mathbf{B}_+(\mathcal{L}(H))\},$$

$$\textbf{s-ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) = \textbf{ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))),$$

$$\textbf{exp } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) = \begin{cases} \textbf{ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) & \text{if } H \text{ is separable and } K \\ & \text{carries a strictly positive} \\ & \text{measure} \\ \emptyset & \text{if } H \text{ otherwise} \end{cases},$$

$$\textbf{s-exp } \mathbf{B}_+(\mathcal{L}(H)) = \textbf{dent } \mathbf{B}_+(\mathcal{L}(H)) = \begin{cases} \textbf{ext } \mathbf{B}_+(\mathcal{L}(H)) & \text{if } \textbf{dim } H < \infty \text{ and} \\ & \textbf{card } K < \infty \\ \emptyset & \text{if } \textbf{dim } H = \infty \end{cases},$$

$$\textbf{ext-ray } \mathbf{C}_+(K,\mathcal{L}(H)) = \{\mathbb{R}_+ \mathbf{1}_{\{k_0\}} \mathbf{x} \otimes \mathbf{x} : \mathbf{0} \neq \mathbf{x} \in H,\ k_0 \text{ is an isolated point of } K\}.$$

The corresponding results for the whole unit ball are presented in [8], [10].

2. Extremality

Theorem 1. *For any Hilbert space H we have*

$$\textbf{ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) = \{f \in \mathbf{C}(K,\mathcal{L}(H)) : f(K) \subset \textbf{ext } \mathbf{B}_+(\mathcal{L}(H))\}.$$

Proof: Fix $f \in \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$ with non extremal value. Let $\mathbf{x}_o \in \mathbf{S}(H)$ be such that $f^2(k)\mathbf{x}_o \neq f(k)\mathbf{x}_o$ for some $k \in K$. Put $f_1 = 2f - f^2$ and $f_2 = f^2$. We have $\frac{f_1+f_2}{2} = f$ and $0 \leq f_i(k) \leq I$. Moreover $f_1(k)\mathbf{x}_o = 2f(k)\mathbf{x}_o - f^2(k)\mathbf{x}_o \neq f^2(k)\mathbf{x}_o = f_2(k)\mathbf{x}_o$, so $f_1 \neq f_2$. $\square$

Theorem 2. *We have* $\textbf{s-ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H))) = \textbf{ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H)))$.

PROOF: Let $f \in \textbf{ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H)))$. Fix $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that $\|\frac{g_n+h_n}{2} - f\| < \delta$, $\mathbf{x},\mathbf{y} \in \mathbf{B}(H)$ implies $\|g_n - h_n\| < \varepsilon$.

From the uniform convexity of H there exists $\delta(\varepsilon)$ such that $\left\|\frac{\mathbf{x}+\mathbf{y}}{2}\right\| > 1-\delta(\varepsilon)$ implies $\|\mathbf{x}-\mathbf{y}\| < \varepsilon/2$. Put $\delta = \min(\frac{\varepsilon}{8}, \delta(\varepsilon))$. Fix $k \in K$. For $\mathbf{x} \perp \textbf{IsDom}\ f(k)$ with $\|\mathbf{x}\| \leq 1$ we have $\|g_n(k)\mathbf{x} - h_n(k)\mathbf{x}\| \leq 2\|g_n(k)\mathbf{x} + h_n(k)\mathbf{x}\| \leq 4\delta \leq \frac{\varepsilon}{2}$.

For $\mathbf{y} \in \textbf{IsDom}\ f(k)$ with $\|\mathbf{y}\| \leq 1$ we have $\|\frac{g_n(k)\mathbf{y}+h_n(k)\mathbf{y}}{2}\| \geq \|f_n(k)\mathbf{y}\| - \|\frac{g_n(k)\mathbf{y}+h_n(k)\mathbf{y}}{2} - f_n(k)\mathbf{y}\| \geq 1-\delta$. Thus $\|g_n(k)\mathbf{y} - h_n(k)\mathbf{y}\| \leq \frac{\varepsilon}{2}$.

Now let $\mathbf{z} \in \mathbf{B}(H)$. And let $\mathbf{y} \in \textbf{IsDom}\ f(k)$ and $\mathbf{x} \in (\textbf{IsDom}\ f(k))^{\perp}$ be such that $\mathbf{z} = \mathbf{x}+\mathbf{y}$. Obviously $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$. Now we have $\|(g_n(k) - h_n(k))\mathbf{z}\| = \|g_n(k)\mathbf{x} - h_n(k)\mathbf{x} + g_n(k)\mathbf{y} - h_n(k)\mathbf{y}\| \leq \|g_n(k)\mathbf{x} - h_n(k)\mathbf{x}\| + \|g_n(k)\mathbf{y} - h_n(k)\mathbf{y}\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $\|g_n(k) - h_n(k)\| \leq \varepsilon$ and $\|g_n - h_n\| \leq \varepsilon$. □

Theorem 3. $\textbf{ext-ray}\ \mathsf{C}_+(K,\mathcal{L}(H)) = \{\mathbb{R}_+ f : f = \mathbf{1}_{\{k_0\}}\mathbf{x}\otimes\mathbf{x} \in \mathsf{C}(K,\mathcal{L}(H)), \mathbf{0} \neq \mathbf{x} \in H,\ k_0$ *is an isolated point of* $K\}$.

PROOF: Fix $f = \mathbf{1}_{\{k_0\}}\mathbf{x}\otimes\mathbf{x} \in \mathsf{C}(K,\mathcal{L}(H))$. Let $0 \neq g \in \mathsf{C}_+(K,\mathcal{L}(H))$ such that $f - g \in \mathsf{C}_+(K,\mathcal{L}(H))$. Then $g \leq f$, so $g(k) = 0$ for $k \neq k_0$. Moreover $0 \leq g(k_0) \leq \mathbf{x}\otimes\mathbf{x}$. Hence $g(k_0) = \alpha\mathbf{x}\otimes\mathbf{x}$ where $\alpha \in (0,1]$, i.e. $g = \alpha f$ and $f \in \textbf{ext-ray}\ \mathsf{C}_+(K,\mathcal{L}(H))$.

Let k_0 be a not isolated point of K such that $f(k_0) \neq 0$. Then there exists a continuous function $\gamma : K \to [0,1]$ with $\gamma(k_0) = 1$ and $\gamma(k_1) = 0$ for some $k_1 \neq k_0$ such that $f(k_1) \neq 0$. Put $g = \gamma f \in \mathsf{C}_+(K,\mathcal{L}(H))$. Then $f - g \in \mathsf{C}_+(K,\mathcal{L}(H))$ and $g \neq \lambda f$, $\lambda \in \mathbb{R}_+$, i.e. f do not generate the extreme ray.

If for two isolated points k_i, $i = 1,2$, $f(k_i) \neq 0$, then by the analogous arguments $\mathbb{R}_+ f \notin \textbf{ext-ray}\ \mathsf{C}_+(K,\mathcal{L}(H))$.

If k_0 is an isolated point of K and $f(k_0)$ is not of the form $\mathbf{x}\otimes\mathbf{x}$ ($f(k_0)$ do not generate extreme ray in $\mathcal{L}_+(H)$). Then there exists $0 \neq T \in \mathcal{L}_+(H)$ such that $f(k_0) \pm T \in \mathsf{C}_+(K,\mathcal{L}(H))$ and $T \neq \lambda f(k_0)$, $\lambda \in \mathbb{R}_+$. For $g = \mathbf{1}_{\{k_0\}}T \in \mathsf{C}_+(K,\mathcal{L}(H))$ we have $f - g \in \mathsf{C}_+(K,\mathcal{L}(H))$ and $g \neq \lambda f$, $\lambda \in \mathbb{R}_+$, i.e. f do not generate the extreme ray, too. □

Theorem 4. *If H is separable and a compact Hausdorff space K carries a strictly positive measure then*

$$\textbf{exp}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H))) = \textbf{ext}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H))).$$

Otherwise

$$\textbf{exp}\ \mathbf{B}_+(\mathsf{C}(K,\mathcal{L}(H))) = \emptyset.$$

PROOF: Let H be separable and let μ be a strictly positive measure on K with $\mu(K) = 1$. We fix an orthonormal basis $\{\mathbf{e}_i\}_{i\in I}$ and a sequence of strictly positive reals α_i such that $\sum_{i\in I}\alpha_i = 1$. Fix $f_o \in \textbf{ext } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$. We define a functional ξ on $\mathbf{C}(K,\mathcal{L}(H))$ by

$$\xi(g) = \int_K \sum_{i\in I} \alpha_i Re\langle (2g(k)-I)\mathbf{e}_i, (2f_o(k)-I)\mathbf{e}_i\rangle \, d\mu(k),$$

$g \in \mathbf{B}(\mathbf{C}(K,\mathcal{L}(H)))$. We have $\xi(g) \le 1 = \xi(f_o)$ for $g \in \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$. Now suppose that $\xi(g) = 1$ for some $g \in \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$. Note that if $0 \le T \le I$ then $-I \le (2T-I) \le I$ and $\|2T-I\| \le 1$. We get $\langle (2g(k)-I)\mathbf{e}_i, (2f_o(k)-I)\mathbf{e}_i\rangle = 1$ μ-a.e. and $(2g(k)-I)\mathbf{e}_i = (2f_o(k)-I)\mathbf{e}_i$. Hence $(2g(k)-I) = (2f_o(k)-I)$ and $g = f_o$, i.e. $f_o \in \textbf{exp } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$.

Now suppose that a functional ξ_o exposes $\mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$ at f_o belonging to $f_o \in \textbf{exp } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$. Obviously $\|f_o(k)\| = 0$ or 1. Put $K_0 = \{k \in K : f_o(k) = 0\}$ and $K_1 = K \setminus K_0$. The sets K_0, K_1 are clopen.

Fix $\mathbf{x} \in \mathbf{S}(H)$. We define a functional ν on $\mathbf{C}(K,\mathbb{R})$ by

$$\nu(h) = \xi_o(h(f_o - \mathbf{1}_{K_0}\mathbf{x}\otimes\mathbf{x})), \qquad h \in \mathbf{C}(K,\mathbb{R}).$$

We claim that ν is strictly positive. Indeed, suppose to get a contradiction, that there exists $h_o \in \mathbf{C}(K,\mathbb{R})$ such that $0 \le h_o \le 1$, $h_o \ne 0$, and $\nu(h_o) \le 0$. If $\textbf{supp } h_o \subset K_1$ then $h_o f_o \ne 0$, and $\nu(1) \le \nu(1) - \nu(h_o) = \nu(1-h_o) = \xi_o((1-h_o)f_o) - \xi_o(\mathbf{1}_{K_0}\mathbf{x}\otimes\mathbf{x}) < \xi_o(f_o) - \xi_o(\mathbf{1}_{K_0}\mathbf{x}\otimes\mathbf{x}) = \nu(1)$, which is impossible. It follows that K_1 carries a strictly positive measure.

If $\textbf{supp } h_o \subset K_0$ then $\nu(1) \le \nu(1) - \nu(h_o) = \nu(1-h_o) = \xi_o(f_o + h_o\mathbf{x}\otimes\mathbf{x}) - \xi_o(\mathbf{1}_{K_0}\mathbf{x}\otimes\mathbf{x}) < \xi_o(f_o) - \xi_o(\mathbf{1}_{K_0}\mathbf{x}\otimes\mathbf{x}) = \nu(1)$, which is impossible. It follows that K_0 carries a strictly positive measure. Therefore if $\textbf{exp } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) \ne \emptyset$ then K carries a strictly positive measure.

Let $\{\mathbf{e}_i\}_{i\in I}$ be an orthonormal basis of H such that, $\{\mathbf{e}_i\}_{i\in J}$, $J \subset I$, is the orthonormal base of $\bigcap_{k\in K} \textbf{Ker } f(k)$. For $L \subset I$ we denote by P_L a projection on $\overline{\textbf{lin}}\ \{\mathbf{e}_i\}_{i\in L}$. Consider now a function m on all subsets of I defined by

$$m(L) = \xi_o(f_o P_{L\cap(I\setminus J)} - P_{L\cap J}).$$

If $i \in J$ then $f_o + \mathbf{e}_i\otimes\mathbf{e}_i \in \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H)))$ and $\xi_o(f_o + \mathbf{e}_i\otimes\mathbf{e}_i) < \xi_o(f_o)$. Thus $\xi_o(\mathbf{e}_i\otimes\mathbf{e}_i) < 0$ and $m(\{i\}) = \xi_o(-\mathbf{e}_i\otimes\mathbf{e}_i) > 0$. If $i \notin J$ then there exists $k \in K$ such that $\mathbf{e}_i \in (\textbf{Ker } f_o(k))^c$ i.e. $f_o(k)\mathbf{e}_i \ne 0$ and $f_o P_{\{i_o\}} \ne 0$. We have $f_o = f_o P_{\{i_o\}} + f_o P_{I\setminus\{i_o\}}$ and $\xi_o(f_o P_{I\setminus\{i_o\}}) < \xi_o(f_o)$. Hence $m(\{i_o\}) = \xi_o(f_o P_{\{i_o\}}) > 0$.

Using the same arguments we get that $m(L) > 0$ if L is a subset of J or L is a subset of $I \setminus J$.

Thus the function m is finitely additive and strictly positive on the family of all subsets of I. Therefore if $\textbf{exp } \mathbf{B}_+(\mathbf{C}(K,\mathcal{L}(H))) \ne \emptyset$ then I is countable and H is separable. $\square$

Theorem 5. *If* $\mathsf{dim}\ H = \infty$ *or* $\mathsf{card}\ K = \infty$ *then*

$$\mathsf{dent}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H))) = \emptyset.$$

Proof: Suppose that $\mathsf{dim}\ H = \infty$. Fix $f \in \mathsf{ext}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$ and $k_0 \in K$. Consider the case when $\mathsf{dim}\ \mathsf{IsDom}\ f(k_0) = \infty$. Let $\{\mathbf{e}_i\}_{i=1}^{\infty}$ be orthonormal system in $\mathsf{IsDom}\ f(k_0)$. Let P_j be a projection on $\{\mathbf{e}_j\}^{\perp}$. Put $f_j = P_j f$. Obviously $\|f_j - f\| \geq \|f_j(k_0) - f(k_0)\| = \|\mathbf{e}_j \otimes \mathbf{e}_j\| = 1$. We have $\|I - \frac{1}{n}\sum_{i=1}^{n} P_i\| = \|\frac{1}{n}\sum_{i=1}^{n} \mathbf{e}_i \otimes \mathbf{e}_i\| = \frac{1}{n}$ and $\|f - \sum_{i=1}^{n} f_i\| \leq \|I - \frac{1}{n}\sum_{i=1}^{n} P_i\| = \frac{1}{n}$, i.e. $f \notin \mathsf{dent}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$.

Consider the case when $\mathsf{dim}\ \mathsf{Ker}\ f(k_0) = \infty$. Then for $g = I - f \in \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$ $\mathsf{dim}\ \mathsf{IsDom}\ g = \infty$, and we can apply the above argument for g.

Now suppose that $\mathsf{card}\ K = \infty$.

Suppose that there exists a sequence $\{k_n\}$ of distinct points of K such that $\lim_n k_n = k_0$ and $f(k_n) \neq 0$. We choose the sequence of continuous functions $\gamma_n : K \to [0, 1]$ such that $\gamma_n(k_n) = 1$ and $\mathsf{supp}\ \gamma_{n_1} \cap \mathsf{supp}\ \gamma_{n_2} = \emptyset$ if $n_1 \neq n_2$. Put $f_j = (1 - \gamma_j) f_o \in \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$. Obviously $\|f_j - f\| \geq \|f_j(k_j) - f(k_j)\| = 1$. We have $\|f - \sum_{i=1}^{n} f_i\| \leq \|\frac{1}{n}\sum_{i=1}^{n} h_i\| = \frac{1}{n}$, i.e. $f \notin \mathsf{dent}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$.

Finally if such sequence $\{k_n\}$ does not exist we can find a closed $K_1 \subset K$ such that $\mathsf{card}\ K_1 = \infty$ and $f(k) = 0$ for all $k \in K_1$. We choose the sequence of continuous functions $\gamma_n : K \to [0, 1]$ such that $\|\gamma_n\| = 1$, $\mathsf{supp}\ \gamma_n \subset K_1$ and $\mathsf{supp}\ \gamma_{n_1} \cap \mathsf{supp}\ \gamma_{n_2} = \emptyset$ if $n_1 \neq n_2$. Put $f_j = f + \gamma_j \mathbf{x} \otimes \mathbf{x} \in \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$, $\mathbf{x} \in \mathbf{S}(H)$. Obviously $\|f_j - f\| \geq \|\mathbf{x} \otimes \mathbf{x}\| = 1$. We have $\|f - \sum_{i=1}^{n} f_i\| \leq \|\frac{1}{n}\sum_{i=1}^{n} h_i\| = \frac{1}{n}$, i.e. $f \notin \mathsf{dent}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$. ☐

Theorem 6. *If* $\mathsf{dim}\ H < \infty$ *and* $\mathsf{card}\ K < \infty$ *then*

$$\mathsf{s\text{-}exp}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H))) = \mathsf{ext}\ \mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H))).$$

Proof: If $\mathsf{dim}\ H < \infty$ or $\mathsf{card}\ K < \infty$ then $\mathsf{C}(K, \mathcal{L}(H))$ is finite dimensional, so $\mathbf{B}_+(\mathsf{C}(K, \mathcal{L}(H)))$ is compact. Hence exposed and strongly exposed coincide. In view of Theorem 4 we finish the proof. ☐

Remark. All the above theorems can be proven using the same arguments for the space of compact operators $\mathcal{K}(H)$ instead of $\mathcal{L}(H)$.

Questions. In [6] and [7] it is shown that the unit ball and the positive part of the unit ball is stable if $\mathsf{dim}\ H < \infty$. Are $\mathbf{B}(\mathcal{L}(H))$ and $\mathbf{B}_+(\mathcal{L}(H))$ stable for infinite dimensional H?

In [8] it is presented an example of the extreme points of the unit ball of continues operator-valued map into l^p, $1 < p < \infty$, $p \neq 2$ with non-extremal values. What about extreme positive continuous maps into $\mathcal{L}(l^p)$, $1 < p < \infty$, $p \neq 2$.

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