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# A note on finite sets of terms closed under subterms and unification

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*Abstract.* The paper contains two remarks on finite sets of groupoid terms closed under subterms and the application of unifying pairs.

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By a term we shall mean a groupoid term. Let us write $u+v \sim t$ if $t = f(u) = g(v)$ for a unifying pair $f, g$ of the terms $u$ and $v$, i.e., if $t$ is a substitution instance of both $u$ and $v$ and any term that is a substitution instance of both $u$ and $v$, is a substitution instance of $t$. (A survey of unification theory is contained, for example, in Dershowitz and Jouannaud [1].)

Let us call a set $S$ of terms SU-closed if it is closed with respect to subterms and whenever $u + v \sim t$ for two terms $u, v \in S$, then $t \sim t' \in S$ for some $t'$.

**Theorem 1.** *There is no finite, SU-closed set of terms containing the following three terms:*

$$(xy \cdot z)x, \qquad x(yz \cdot u), \qquad x \cdot yx.$$

Proof: Let us define an infinite sequence $a_0, a_1, \dots$ of terms as follows: $a_0$ is a variable; $a_{i+1} = a_i x$ for a variable $x$ not occurring in $a_i$. So, $a_i = (((x_0 x_1)x_2)\dots)x_i$, where $x_0, \dots, x_i$ is a sequence of pairwise distinct variables. Also, put $b_i = y a_i$, where $y$ is a variable not occurring in $a_i$. Hence $b_2 \sim x(yz \cdot u)$. It is easy to see that

$$(xy \cdot z)x + b_i \sim ((a_i x)y)a_i \supseteq a_{i+2}$$

for $i \geq 2$ (where $x, y$ are two distinct variables not occurring in $a_i$, and

$$x \cdot yx + a_{i+1} \sim a_i \cdot x a_i \supseteq b_i$$

for $i \geq 3$. □

The depth of a term is defined inductively as follows: the depth of a variable is 0; the depth of $t_1 t_2$ is $1 + \max(d_1, d_2)$, where $d_1$ is the depth of $t_1$ and $d_2$ is the depth of $t_2$. So, $xy \cdot zu$ is of depth 2.

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**Theorem 2.** *There exists a finite, SU-closed set of terms containing* (*up to similarity*) *all terms of depth at most* 2.

Proof: The set consists of the terms of depth 2, plus the twelve terms

$$\begin{array}{lcl}
xx \cdot (xx \cdot xx) & \sim & x \cdot xx \ + \ xx \cdot y \\
xy \cdot (xy \cdot xy) & \sim & x \cdot xx \ + \ xy \cdot z \\
xx \cdot (xx \cdot y) & \sim & x \cdot xy \ + \ xx \cdot y \\
xy \cdot (xy \cdot z) & \sim & x \cdot xy \ + \ xy \cdot z \\
xx \cdot (xx \cdot x) & \sim & x \cdot xy \ + \ xx \cdot yx \\
xy \cdot (xy \cdot x) & \sim & x \cdot xy \ + \ xy \cdot zx \\
xy \cdot (xy \cdot y) & \sim & x \cdot xy \ + \ xy \cdot zy \\
xx \cdot (y \cdot xx) & \sim & x \cdot yx \ + \ xx \cdot y \\
xy \cdot (z \cdot xy) & \sim & x \cdot yx \ + \ xy \cdot z \\
xx \cdot (x \cdot xx) & \sim & x \cdot yx \ + \ xx \cdot xy \\
xy \cdot (x \cdot xy) & \sim & x \cdot yx \ + \ xy \cdot xz \\
xy \cdot (y \cdot xy) & \sim & x \cdot yx \ + \ xy \cdot yz
\end{array}$$

and their duals. □

## References

[1] Dershowitz N., Jouannaud J.-P., *Rewrite systems*, Chapter 6, 243–320 in J. van Leeuwen, ed., Handbook of Theoretical Computer Science, B: Formal Methods and Semantics, North Holland, Amsterdam, 1990.

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