A note on Banach spaces with ℓ^1-saturated duals

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Abstract. It is shown that there exists a Banach space with an unconditional basis which is not c_0-saturated, but whose dual is ℓ^1-saturated.

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Let E and F be Banach spaces. We say that E is F-saturated if every infinite dimensional closed subspace of E contains an isomorphic copy of F. In [2], it is shown that there exists a c_0-saturated Banach space with an unconditional basis whose dual contains an isomorphic copy of ℓ^2. In this note, we give an example where the dual situation occurs. It is shown that there is a Banach space with an unconditional basis which contains an isomorphic copy of ℓ^2, and whose dual is ℓ^1-saturated.

We follow standard Banach space terminology as used in [3]. Our example is a certain subspace of the weak L^2 space $L^{2,\infty}[0,\infty)$. Recall that this is the space of all measurable functions f on $[0,\infty)$ such that

$$\|f\| = \sup_{c>0} c(\lambda\{|f| > c\})^{1/2} < \infty, \tag{1}$$

where λ is the Lebesgue measure on $[0,\infty)$. Although equation (1) only defines a quasi-norm on $L^{2,\infty}[0,\infty)$, it is well known that it is equivalent to a norm on $L^{2,\infty}[0,\infty)$, and that $L^{2,\infty}[0,\infty)$ is norm complete. The reader may consult [4] for further information concerning the family of Lorentz spaces, of which $L^{2,\infty}[0,\infty)$ is a member. Finally, for a measurable function f, we let f^* be the decreasing rearrangement of $|f|$, as defined in §2a of [4].

Proposition 1. *For each $n \in \mathbb{N}$, define*

$$f_n(t) = \begin{cases} \min(2^{n/2}, (t-n+1)^{-1/2}) & \text{if } n-1 < t \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $m \in \mathbb{N}$, and any sequence of scalars (a_n),

$$\Big(\frac{1}{2}\sum_{n=1}^{m} a_n^2\Big)^{1/2} \le \Big\|\sum_{n=1}^{m} a_n f_n\Big\| \le \Big(\sum_{n=1}^{m} a_n^2\Big)^{1/2},$$

where $\|\cdot\|$ *refers to the quasi-norm defined by* (1).

PROOF: Given a scalar sequence $(a_n)_{n=1}^m$, let $a = \max\{|a_n| : 1 \le n \le m\}$, and suppose this maximum is attained at n_0. Let N be the set of all natural numbers $\le m$ such that $a_n > 2^{-n/2}a$. If $0 < c < a$, and $n \in N$, then $c < 2^{n/2}a_n$. Hence $\lambda\{|a_n f_n| > c\} = \min(a_n^2/c^2, 1)$. Therefore,

$$\begin{aligned}\Big\|\sum_{n=1}^m a_n f_n\Big\| &\ge \sup_{0<c<a} c\Big(\lambda\Big\{\sum_{n\in N}|a_n f_n| > c\Big\}\Big)^{1/2}\\ &= \sup_{0<c<a}\Big(\sum_{n\in N}\min(a_n^2, c^2)\Big)^{1/2}\\ &= \Big(\sum_{n\in N} a_n^2\Big)^{1/2}.\end{aligned}$$

If $1 \le n \le m$, and $n \notin N$, then $a_n \le 2^{-n/2}a$. Thus

$$\sum_{\substack{1\le n\le m\\ n\notin N}} a_n^2 \le a^2 \sum_{n=1}^m 2^{-n} < a^2.$$

Hence

$$\begin{aligned}\Big\|\sum_{n=1}^m a_n f_n\Big\| &\ge \Big(\sum_{n=1}^m a_n^2 - \sum_{\substack{1\le n\le m\\ n\notin N}} a_n^2\Big)^{1/2}\\ &\ge \Big(\sum_{n=1}^m a_n^2 - a^2\Big)^{1/2}\\ &= \Big(\sum_{\substack{1\le n\le m\\ n\ne n_0}} a_n^2\Big)^{1/2}.\end{aligned}$$

Clearly, $\|\sum_{n=1}^m a_n f_n\| \ge \|a_{n_0} f_{n_0}\| = |a_{n_0}|$ as well. This proves the first half of the inequality. Observe that for any $c > 0$,

$$\lambda\Big\{\Big|\sum_{n=1}^m a_n f_n\Big| > c\Big\} = \sum_{n=1}^m \lambda\{|f_n| > \frac{c}{|a_n|}\} \le \frac{1}{c^2}\sum_{n=1}^m a_n^2.$$

The second half of the inequality follows. $\square$

For $n \in \mathbb{N}$, and $1 \le j \le 2^n$, let $g_{n,j}$ be the characteristic function of the interval $[n-1+(j-1)2^{-n}, n-1+j2^{-n})$. Let E be the closed linear span of the sequence $(g_{n,j})_{j=1}^{2^n}{}_{n=1}^{\infty}$. Clearly $(g_{n,j})_{j=1}^{2^n}{}_{n=1}^{\infty}$ is an unconditional basis of E.

Proposition 2. *The space E contains an isomorphic copy of ℓ^2.*

PROOF: Recall the sequence (f_n) defined in Proposition 1. For each n, let $h_n = \sum_{j=1}^{2^n} \sqrt{2^n/j}\, g_{n,j}$. Then $0 \leq h_n \leq f_n \leq \sqrt{2}\, h_n$ for all n. It follows from Proposition 1 that the subspace $[\{h_n\}]$ of E is isomorphic to ℓ^2. $\square$

It remains to show that E' is ℓ^1-saturated. Let F be the closed subspace of $L^{2,\infty}[0,\infty)$ generated by $L^1 \cap L^\infty$. It is well known that F' is canonically isomorphic to $L^{2,1}[0,\infty)$, where the latter is the space of all measurable functions f on $[0,\infty)$ such that

$$\|f\|_{2,1} = \int_0^\infty \frac{f^*(t)}{\sqrt{t}}\, dt < \infty.$$

Let Σ be the σ-algebra generated by the sets $[n-1+(j-1)2^{-n}, n-1+j2^{-n})$, $1 \leq j \leq 2^n$, $n \in \mathbb{N}$. Then clearly E is the subspace of F consisting of all Σ-measurable functions. It follows easily that E' can be identified canonically with the subspace of $L^{2,1}[0,\infty)$ consisting of all Σ-measurable functions. Now if G is a subspace of E', then it contains a basic sequence equivalent to a normalized disjointly supported sequence (u_n) in $L^{2,1}[0,\infty)$. By [1, Corollary 2.4], $[\{u_n\}]$, and hence G, contains a copy of ℓ^1.

We end this note with the following problem.

Problem. Suppose E is a Banach space (with or without an unconditional basis) such that E' has the Schur property. Is E necessarily c_0-saturated?

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