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On finite loops whose inner mapping groups have small orders

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Abstract. We investigate the situation that the inner mapping group of a loop is of order which is a product of two small prime numbers and we show that then the loop is soluble.

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1. Introduction

If Q is a loop then the left and right translations are permutations of Q and they generate the multiplication group $M(Q)$ of the loop Q. The stabilizer of the neutral element of Q is denoted by $I(Q)$ and is called the inner mapping group of Q. One of the interesting problems here is to study how the structure of $I(Q)$ influences the structure of $M(Q)$ and Q. In this short note we give a partial answer to the following general problem (stated in [8]): If $|I(Q)| = pq$, where p and q are two different prime numbers, does it then follow that $M(Q)$ is a soluble group? We have already answered this problem in the positive in the case that $|I(Q)| = 6$ (for the details, see [5]) and now we are able to show that $M(Q)$ is soluble provided that $pq \leq 21$. We emphasize that all groups and loops in this paper are finite. For the background material about loops and their multiplication groups the reader is advised to consult [6], [7], [8].

2. H-connected transversals

Now many properties of loops can be reduced to the properties of connected transversals in the multiplication group. Thus we first consider some properties of these transversals in this section. If G is a group, $H \leq G$ and A and B are two left transversals to H in G such that the commutator subgroup $[A, B]$ is a subgroup of H then we say that A and B are H-connected in G. For the properties of connected transversals the reader is advised to consult [6]. In the following lemmas we assume that A and B are H-connected transversals.

Lemma 1. *If $C \subseteq A \cup B$ and $T = \langle H, C\rangle$ then $C \subseteq T_G$ (here T_G denotes the core of T in G).*

Lemma 2. *If H is cyclic then G is soluble.*

For the proofs, see [6, Lemma 2.5] and [4, Corollary 2.3].

Lemma 3. *If $|H| = k$ then $[G : C_G(d)] \leq k^2$ for any $d \in A \cup B$.*

Proof: Let $d \in A$ and h be a fixed element from H and write $F(d,h) = \{b \in B : d^{-1}b^{-1}db = h\}$. If $b, c \in F(d,h)$ then $bc^{-1} \in C_G(d)$ and $b \in C_G(d)c$. Thus $F(d,h) \subseteq C_G(d)b_h$ where b_h is a fixed element from $F(d,h)$. Clearly, $B = \cup F(d,h)$ where h goes through all the elements of H. Thus $G = BH$ and $BH \subseteq C_G(d)\{b_h : h \in H\}H$, hence our proof is complete. $\square$

Lemma 4. *If $H_G = 1$ and $[A,B] = 1$ then A and B are isomorphic subgroups of G.*

For the proof, see [7, Lemma 2.3].

After these preparations we are ready to prove our main theorem.

Theorem 1. *Let G be a finite group, $H \leq G$, $|H| = pq$ (where $p > q$ are two different prime numbers) and let A and B be two H-connected transversals such that G is generated by A and B. Then G is soluble at least in the following cases:*

(a) *q is not a factor of $p - 1$;*
(b) *$q = 2$ and $p = 3, 5, 7$;*
(c) *$q = 3$ and $p = 7$.*

Proof: If (a) holds then H is cyclic and we are ready by Lemma 2. Clearly, we may assume that H is noncyclic and therefore we can write $H = PQ$ where P is the normal subgroup of order p and Q is nonnormal of order q. Let G be a minimal counterexample. We first assume that G has a proper subgroup T such that H is a maximal subgroup of T. If P is normal in T then it is easy to see that T/P has H/P-connected transversals. Since H/P is cyclic, T/P is soluble by Lemma 2, hence T is soluble. Thus we can assume that P is not normal in any subgroup of G which properly contains H, hence $N_G(P) = H$. Then consider the group T_G from Lemma 1. If $H \leq T_G$ then $T = T_GH$ is normal in G. Now P is a Sylow p-subgroup of T and by Frattini lemma, $G = N_G(P)T = T$, a contradiction. Now $T_G \cap H$ is normal in H and thus $T_G \cap H \leq P$. If $E = T_GP$ then $N_E(P) = P$ and we conclude that E is a Frobenius group with P as the Frobenius complement. Since the Frobenius kernel is always nilpotent ([3, p. 499]) we conclude that E and T_G are soluble. Then consider the factor group G/T_G. Now HT_G/T_G is cyclic or of order pq and G/T_G is generated by connected transversals. Then from the minimality of G or from Lemma 2, it follows that G/T_G is soluble. Since T_G is soluble we conclude that G is soluble. Thus we can assume that H is a maximal subgroup of G.

Now if N is a nontrivial proper normal subgroup of G then $G = NH$. It is easy to conclude that NP is a Frobenius group with P as the Frobenius complement. It follows that N is soluble and therefore also G is soluble. This means that we can assume that G is simple.

Let $a \in A$. If the commutator subgroup $[a,B]$ is not contained in any cyclic subgroup of H then the set $[a,B]$ generates H. Thus we can find elements b and c from B such that $G = \langle a,b,c \rangle$. From Lemma 3 we see that $[G : C_G(d)] \leq (pq)^2$

for any $d \in A \cup B$. Since G is simple and therefore $Z(G) = 1$, we conclude that $|G| \le (pq)^6$.

Then let a_1 and a_2 be elements from A such that the commutator subgroups $[a_1, B]$ and $[a_2, B]$ are both nontrivial and they are contained in two different proper subgroups of H. Then we can find elements b and c from B such that $G = \langle a_1, a_2, b, c\rangle$, hence $|G| \le (pq)^8$ by Lemma 3.

Finally, let $[A, B] \le S$ where S is a proper subgroup of H. If $[A, B] = 1$ then A and B are isomorphic subgroups of G by Lemma 4. If $A \ne B$ then $A \cap B = 1$, because G is generated by $A \cup B$ and $Z(G) = 1$. It follows that $|G| \le (pq)^2$. If $A = B$ then $G = \langle A\rangle = A$ which is not possible. Then assume that $1 \ne [A, B] \le S$. We also assume that no subset of $A \cup B$ containing less than six elements can generate G. Now we can proceed as in the proof Theorem 3.1 in [6] (here S is cyclic of prime order) and it follows that $A = B$ is an abelian group. Since this is not possible we conclude that there exists such a subset of $A \cup B$ containing at most five elements that it generates G. Since $[A, B] \le S$ and $|S| \le p$, we can apply the proof of Lemma 3 and we have that $[G : C_G(d)] \le p^2 q$ for any $d \in A \cup B$. Thus $|G| \le (p^2 q)^5 = p^{10} q^5$.

Now we can sum up what we have obtained so far: G is simple, H is maximal in G and in any case $|G| \le \max\{(pq)^8, p^{10} q^5\}$.

The list of maximal subgroups of finite simple groups is complete up to the order $D = 495766656000$ for the sporadic Conway group (see the *Atlas of Finite Groups*, p. 134). Simple calculations show that $pq \le 21$. By checking the orders of maximal subgroups from *Atlas* we have to investigate the following cases (we list here the order of H and after that the corresponding finite simple group): 6 and 10 for PSL(2,5); 14 for PSL(2,8), PSL(2,13) and for the Suzuki group Sz(8); 21 for PSL(2,7). Now it follows from the results of Vesanen ([9] and [10]) that the special linear groups that appear in our list do not have connected transversals to the corresponding subgroups. Thus we only have to have a closer look at the Suzuki group Sz(8) of order 29120 (for the properties of this group, see [1, pp. 182–195] and [2, p. 28]). We can now apply Lemma 3 (with $G = \mathrm{Sz}(8)$ and $|H| = 14$) and we see that $|C_G(d)| > 148$ for any $d \in A \cup B$. Now $C_G(d) = G$ is not possible and therefore $C_G(d)$ has to be a maximal subgroup of order 448. From [1, Theorem 3.11, p. 193] it follows that this subgroup is nilpotent, a contradiction. Thus we conclude that the group Sz(8) does not have connected transversals to maximal subgroups of order 14. The proof of the theorem is complete. □

3. Soluble loops

The connection between multiplication groups of loops and connected transversals is given in [6, Theorem 4.1]. From this and Theorem 1 we can conclude that if Q is a finite loop and $|I(Q)| = pq$ where p and q are as in Theorem 1 then $M(Q)$ is soluble. Quite recently it has been shown by Vesanen [11] that if Q is a finite loop such that $M(Q)$ is soluble then also Q is soluble. By combining all these results we get

Theorem 2. *If Q is a finite loop such that $|I(Q)| = pq$ where p and q are two different prime numbers as in Theorem 1 then Q is a soluble loop.*

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