On finite powers of countably compact groups

Artur Hideyuki Tomita

Abstract. We will show that under $MA_{countable}$ for each $k \in \mathbb{N}$ there exists a group whose k-th power is countably compact but whose 2^k-th power is not countably compact. In particular, for each $k \in \mathbb{N}$ there exists $l \in [k, 2^k)$ and a group whose l-th power is countably compact but the $l+1$-st power is not countably compact.

Keywords: countable compactness, $MA_{countable}$, topological groups, finite powers
Classification: 54D20, 54H11, 54B10, 54A35, 22A05

1. Introduction

In 1966, Comfort and Ross [4] proved that the product of pseudocompact groups is pseudocompact. This motivated the question whether the same would be true for countably compact groups, as every countably compact space is pseudocompact and normal pseudocompact spaces are countably compact. We recall that there are countably compact spaces whose square is not even pseudocompact.

The first counterexample was obtained by van Douwen [5] in 1980. He showed under MA the existence of two countably compact groups whose product is not countably compact. He also showed that the existence of an initially ω_1-compact group whose square is countably compact is independent of $\mathfrak{c} = \aleph_2$. In 1991, Hart and van Mill showed under $MA_{countable}$ the existence of a countably compact group whose square is not countably compact.

It is still open whether there exists in ZFC, a family of countably compact groups whose product is not countably compact. We recall (see [2]) that the product of a family of spaces is countably compact if and only if the product of any subfamily of size at most $2^{\mathfrak{c}}$ is countably compact. In particular a group has every power countably compact if and only if the $2^{\mathfrak{c}}$-th power is countably compact.

This motivated the following question in [2]: *For which cardinals $\kappa < 2^{\mathfrak{c}}$ (not necessarily infinite) there exists a group G such that for each $\lambda < \kappa$, G^λ is countably compact but G^κ is not countably compact?*

In [9], we showed that there exists under $MA(\sigma$-centered), a countably compact free abelian group whose square is not countably compact. This construction used ideas from [5] and [8]. We also showed that there is no free abelian group whose ω-th power is countably compact.

This work has been partially supported by the Conselho Nacional de Pesquisa of Brazil – CNPq

In [10], we showed that under $MA_{countable}$ there exists a group whose square is countably compact but whose cube is not countably compact. We also showed under $MA_{countable}$ that for each $k \in \mathbb{N}$ there exists a family $\{ H_n : n \in \omega \}$ of groups such that for each subset F of ω of size k, the group $\prod_{n \in F} H_n$ is countably compact but for each subset F of ω of size $k+1$, the group $\prod_{n \in F} H_n$ is not countably compact. We used the idea from Hart and van Mill [7] of constructing a group which was generated by a countable group and an ω-bounded group but unlike theirs, which was generated by an ω-independent family, we use the group of all functions in $2^{\mathfrak{c}}$ whose support is bounded in $\mathfrak{c}$.

This work is closely related to some results in my Ph.D thesis but were obtained few months after my graduation. I would like to thank my supervisor, Prof. Steve Watson, for his guidance and encouragement through my years at York University.

2. Countable compactness and finite product of groups

During this section we will fix $k \in \mathbb{N}$. Our goal is to show the following:

Example 2.1 ($MA_{countable}$). *There exists a countable subgroup E of $2^{\mathfrak{c}}$ such that the group H generated by E and $G = \{ x \in 2^{\mathfrak{c}} : \operatorname{supp} x \text{ is bounded in } \mathfrak{c} \}$ is such that H^k is countably compact but H^{2^k} is not countably compact.*

Note that for $k = 1$ we have a group as Hart and van Mill's [7], that is, a countably compact group whose square is not countably compact under $MA_{countable}$.

One can ask why there exists such a gap between the power we know it is countably compact and the power we know it is not countably compact. To explain that we go back to our construction (see [10]) of a family $\{H_n : n \in \omega\}$ of groups whose product of k many distinct elements of this family is countably compact but the product of $k+1$ many distinct ones is not. Each H_n was generated by a countable group E_n and G (the same G defined above). To prove the countable compactness we concluded it was sufficient to prove that certain sequences in $\prod_{n \in F} E_n$ had an accumulation point, where F was a subset of ω of size at most k. Those sequences were shown to have an accumulation point with the help of dense subsets of our partial order and the fact that we did not ask the E_n's to be the same were essential to have the density of the sets associated to those sequences. Since all the sequences we had to guarantee an accumulation point were from a product of at most k groups, we could produce for each product of $k+1$ many groups a sequence witnessing that this product is not countably compact.

Here, we do want the same countable group, so the sequences which we will have to guarantee an accumulation point with the help of a partial order will be in some E^l, where $l \leq 2^k - 1$. Thus a power we could find a closed discrete sequence was the 2^k-th one. Doing a more careful examination of the sequences which suffices to guarantee an accumulation point to obtain the countable compactness of H^k, it is very likely that we can obtain a smaller interval than $[k, 2^k)$. For instance, for $k = 2$ we had obtained a group whose square is countably compact but whose cube is not ([10]). For $k = 3$, the Example 2.1 tells us there exists H such that H^3 is countably compact but H^8 is not. A more careful examination could give us a group H whose cube is countably compact but the fifth power is not.

The construction is divided in a few lemmas. First we will show which are the sequences that will suffice to guarantee an accumulation point. We follow up with the sketch of how the inductive construction of the set of generators for the countable group E is done. We will then give some definitions and enumerations needed to define precisely the inductive construction. Finally we will define the partial order and the dense sets.

Lemma 2.1. *For each sequence $\{\{F_n^i\}_{i\in k} : n \in \omega\}$, where $F_n^i \in [\omega]^{<\omega}$ there exists a sequence $\{\{S_n^j\}_{j\in 2^k-1} : n \in \omega\}$ and $\{A_i : i \in k\}$ satisfying the following properties:*

(1) *for each $j \in 2^k - 1$ and for each $n \in \omega$ the set S_n^j is a finite subset of ω and for each $n \in \omega$ the set $\{S_n^j : j \in 2^k - 1\}$ is a family of pairwise disjoint sets;*
(2) *for each $i \in k$ the set A_i is a finite subset of $2^k - 1$ and for each $n \in \omega$ we have $F_n^i = \bigcup_{j\in A_i} S_n^j$.*

Proof: The proof is easy and is left to the reader □

Lemma 2.2. *Let G be an ω-bounded subgroup of $2^{\mathfrak{c}}$. Let E be a countable subgroup of $2^{\mathfrak{c}}$ and let $\{x_m : m \in \omega\}$ be a set of generators for E. Suppose that every sequence $\{\{\sum_{m\in E_n^i} x_m\}_{i\in l} : n \in \omega\}$ has an accumulation point in G^l, where $\{\{E_n^i\}_{i\in l} : n \in \omega\}$ is a sequence satisfying the following properties:*

(a) *the natural number l belongs to the interval $[1, 2^k)$;*
(b) *for each $j \in l$ and for each $n \in \omega$ the set E_n^j is a finite subset of ω and for each $n \in \omega$ the set $\{E_n^j : j \in l\}$ is a family of pairwise disjoint sets;*
(c) *for each $j \in l$ and for each $n, m \in \omega$ distinct we have $E_n^j \neq E_m^j$.*

Then the group $H = E + G$ has the k-th power countably compact.

Proof: Let $\{\{h_n^i\}_{i\in k} : n \in \omega\}$ be an arbitrary sequence in H^k. For each $i \in k$ and for each $n \in \omega$ let $e_n^i \in E$ and $g_n^i \in G$ be such that $h_n^i = e_n^i + g_n^i$. Since X generates E, for each $i \in k$ and for each $n \in \omega$, there exists $F_n^i \in [\omega]^{<\omega}$ such that $e_n^i = \sum_{m\in F_n^i} x_m$.

Applying Lemma 2.1, there exist a family $\{\{S_n^j\}_{j\in 2^k-1} : n \in \omega\}$ and a family $\{A_i : i \in k\}$ satisfying the properties (1) and (2) from Lemma 2.1.

It is easy to see that by $2^k - 1$ many refinements of ω, one can find an infinite subset N of ω such that for each $j \in 2^k - 1$ the sequence $\vec{S}^j = \{S_n^j : n \in N\}$ is either constant or its elements are pairwise distinct. Let $J = \{j \in 2^k - 1 : \vec{S}^j$ is not constant$\}$. The sequence $\{\{S_n^j\}_{j\in J} : n \in N\}$ satisfies the conditions (a)-(c), therefore, the sequence $\{\{\sum_{m\in S_n^j} x_m\}_{j\in J} : n \in N\}$ has an accumulation point in G^J. Thus, the sequence $\{\{\sum_{m\in S_n^j} x_m\}_{j\in 2^k-1} : n \in N\}$ has an accumulation point $\{g^j\}_{j\in 2^k-1}$ in H^{2^k-1}. Let p be a free ultrafilter on N such that $\{g^j\}_{j\in 2^k-1}$ is a p-limit of $\{\{\sum_{m\in S_n^j} x_m\}_{j\in 2^k-1} : n \in N\}$.

Then, clearly for each $i \in k$ the sequence $\{e_n^i : n \in N\} = \{\sum_{m\in F_n^i} x_m : n \in N\} = \{\sum_{j\in A_i}(\sum_{m\in S_n^j} x_m) : n \in N\}$ has $\sum_{j\in A_i} g^j \in H$ as p-limit. Therefore $\{\{e_n^i\}_{i\in k} : n \in N\}$ has a p-limit in H^k. Since G^k is ω-bounded, the sequence $\{\{g_n^i\}_{i\in k} : n \in N\}$ has a p-limit. Thus $\{\{h_n^i\}_{i\in k} : n \in N\} = \{\{e_n^i\}_{i\in k} + \{g_n^i\}_{i\in k} : n \in N\}$ has a p-limit, since the sum of the p-limits of sequences is the p-limit of the sum of sequences. Therefore $\{\{h_n^i\}_{i\in k} : n \in N\}$ has an accumulation point in H^k and we are done. □

Let us see now how we obtain a sequence in H^{2^k} which does not have an accumulation point in H^{2^k}:

Lemma 2.3. *Let l be a positive integer. Let E be a countable subgroup of $2^{\mathfrak{c}}$ and let $\{x_n : n \in \omega\}$ be a subset of E. Let G be the subgroup of all the elements in $2^{\mathfrak{c}}$ whose support is bounded in $\mathfrak{c}$ and let H be the subgroup generated by E and G. Suppose that for each l-uple $\{e^i\}_{i\in l}$ in E^l there exist $\mathfrak{c}$ many β's in $\mathfrak{c}$ such that $\{n \in \omega : \forall i \in l\ e^i(\beta) = x_{ln+i}(\beta)\}$ is finite. Then the sequence $\{\{x_{ln+i}\}_{i\in l} : n \in \omega\}$ does not have an accumulation point in H^l.*

Proof: A similar proof appeared in [10], where l was 3. In this work we will be interested in $l = 2^k$. For completeness sake we will give the proof.

Let $\{h^i\}_{i\in l}$ be any element of H^l. It is sufficient to show that there exists $\beta \in \mathfrak{c}$ such that $\{n \in \omega : \forall i \in l\ h^i(\beta) = x_{ln+i}(\beta)\}$ is finite, since $\{\{y^i\}_{i\in l} \in H^l : \forall i \in l\ h^i(\beta) = y^i(\beta)\}$ is an open neighbourhood of $\{h^i\}_{i\in l}$.

Since $H = E + G$, for each i in l there exists $e^i \in E$ and $g^i \in G$ such that $h^i = e^i + g^i$. Since $\bigcup\{\text{supp } g^i : i \in l\}$ is bounded in $\mathfrak{c}$, by hypothesis there exists $\beta > \sup \bigcup\{\text{supp } g^i : i \in l\}$ such that $\{n \in \omega : \forall i \in l\ e^i(\beta) = x_{ln+i}(\beta)\}$ is finite. We are done, since for each $i \in l$ we have $h^i(\beta) = e^i(\beta) + g^i(\beta) = e^i(\beta)$. □

Instead of constructing the countable group directly, we will construct a set of generators $X = \{x_n : n \in \omega\}$ for E. The construction of X is by induction and at each stage $\gamma + 1 \in [\omega, \mathfrak{c})$ we will define, for each $n \in \omega$, the value of x_n at γ. To be more precise, for each $n \in \omega$, we will construct a family $\{x_{\alpha,n} : \omega \le \alpha \le \mathfrak{c}\}$ such that $x_{\alpha,n} \in 2^\alpha$ and for each $\alpha < \beta$, $x_{\alpha,n} = x_{\beta,n} \restriction \alpha$. We then define x_n to be $x_{\mathfrak{c},n}$. Let us see now what are the properties we want the family $\{x_{\alpha,n} : n \in \omega, \omega \le \alpha \le \mathfrak{c}\}$ to satisfy during the inductive construction.

First, note that we can code every element of E^l, $l \in \omega$, using a sequence in $([\omega]^{<\omega})^l$ and the restriction of such function will have the same coding. More precisely, if $e_n^i \in E$, then there exists $F_n^i \in [\omega]^{<\omega}$ such that $\sum_{m\in F_n^i} x_m = e_n^i$ and for each $\gamma \le \mathfrak{c}$, we have $\sum_{m\in F_n^i} x_m \restriction \gamma = e_n^i \restriction \gamma$. We will deal a lot with restrictions of functions so it is reasonable to introduce the following

Definition 2.1. Let $\{x_{\alpha,n} : \omega \leq \alpha,\, n \in \omega\}$ be a family defined as above, then for each $F \in [\omega]^{<\omega}$ and for each $\gamma < \mathfrak{c}$, we define $\sigma_\gamma(F) = \sum_{m \in F} x_{\gamma,m}$.

To obtain the countable compactness of the k-th power, it suffices to obtain accumulation points for the sequences in the hypothesis of Lemma 2.2. As we mentioned above at stage γ, we only know the restriction of each x_n to γ so we only know the sequence restricted to γ.

What will make possible to guarantee an accumulation point for a sequence $\{\{e^i_n\}_{i\in l} : n \in \omega\}$ by induction is the fact that $\{g^i\}_{i\in l}$ is an accumulation point for $\{\{e^i_n\}_{i\in l} : n \in \omega\}$ if and only if for each $\gamma < \mathfrak{c}$ we have that $\{g^i \restriction \gamma\}_{i\in l}$ is an accumulation point for $\{\{e^i_n \restriction \gamma\}_{i\in l} : n \in \omega\}$. In fact, what we said above holds for any limit ordinal, that is, if α is a limit ordinal smaller or equal to $\mathfrak{c}$ then we have that $\{g^i \restriction \alpha\}_{i\in l}$ is an accumulation point for $\{\{e^i_n \restriction \alpha\}_{i\in l} : n \in \omega\}$ if and only if for each $\gamma < \alpha$ we have that $\{g^i \restriction \gamma+1\}_{i\in l}$ is an accumulation point for $\{\{e^i_n \restriction \gamma+1\}_{i\in l} : n \in \omega\}$.

Therefore we have only to worry about making sure the restriction of the sequence coded has the restriction of the fixed function as an accumulation point in the successor ordinals. We will use a dense set from a partial order to guarantee an accumulation point for the sequences at these stages.

In order to apply $MA_{countable}$, we must have less than $\mathfrak{c}$ many dense sets at a time, but the number of sequences that need our attention to have an accumulation point is $\mathfrak{c}$. For that we use ideas which appeared in van Douwen [5]: we list the code of all sequences we have to worry about in length $\mathfrak{c}$ and at each stage γ we only worry about the sequences which are indexed by an ordinal smaller than γ. Once it comes the time to worry about the sequence we fix, as in Hart and van Mill [7], an accumulation point for it (the accumulation point for the sequence can be completely defined at this point, though the whole sequence will only be known at the end of the construction. We recall that in van Douwen's construction he would fix the coding of the accumulation point, not the accumulation point). From that stage on, we have to keep the promise about the restriction of the accumulation point fixed being actually an accumulation point for the sequence associated to the coding in that stage.

Let us discuss now how to make by induction the 2^k-th power not countably compact. It suffices to enumerate every 2^k-uple $\{F^j\}_{j\in 2^k}$ of finite subsets of ω such that each of them appears $\mathfrak{c}$ many times in the enumeration. Then at stage $\gamma+1$ we pick the $\{F^j\}_{j\in 2^k}$ whose index is γ in the fixed the enumeration and we will make the set $\{n \in \omega : \forall\, j \in 2^k\ x_{\gamma+1,2^k n+j}(\gamma) = \sigma_{\gamma+1}(F^j)(\gamma)\}$ finite. Clearly E generated by $\{x_{\mathfrak{c},n} : n \in \omega\}$ satisfies the conditions in Lemma 2.3 for $l = 2^k$.

Let us resume now what we have concluded so far. Before, we will fix the two enumerations we mentioned above.

Fixing two enumerations

Let $\{\vec{S}_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ be an enumeration of all sequences satisfying conditions (a)-(c) from Lemma 2.2. For each $\alpha \in [\omega, \mathfrak{c})$ the sequence $\vec{S}_\alpha$ will also be denoted by $\{\{S^j_{\alpha,n}\}_{j\in l_\alpha} : n \in \omega\}$.

Let $\{\{g^i_\alpha\}_{i<2^k} : \omega \leq \alpha < \mathfrak{c}\}$ be an enumeration of $([\omega]^{<\omega})^{2^k}$ such that each element appears $\mathfrak{c}$ many times.

Lemma 2.4. *Suppose that $\{x_{\alpha,n} : \omega \leq \alpha \leq \mathfrak{c},\ n \in \omega\}$ and $\{\{h^i_\alpha\}_{i\in l_\alpha} : \omega \leq \alpha < \mathfrak{c}\}$ are such that the following are satisfied:*

1. *(1) for each $\alpha \leq \mathfrak{c}$ and $n \in \omega$, the function $x_{\alpha,n}$ is an element of 2^α and for each $\alpha < \beta \leq \mathfrak{c}$ we have $x_{\alpha,n} \subseteq x_{\beta,n}$;*
2. *(2) for each $\alpha < \mathfrak{c}$ and for each $i < l_\alpha$, the function h^i_α is a function in $2^\mathfrak{c}$ whose support is bounded in $\mathfrak{c}$ and for each β such that $\omega \leq \beta < \alpha < \mathfrak{c}$ the l_β-uple $\{h^i_\beta \restriction \alpha\}_{i\in l_\beta}$ is an accumulation point of $\{\{\sigma_\alpha(S^j_{\beta,n})\}_{j\in l_\beta} : n \in \omega\}$;*
3. *(3) for each $\alpha \in [\omega, \mathfrak{c})$ the set $\{n \in \omega : \forall j \in 2^k\ x_{\alpha+1,2^k n+j}(\alpha) = \sigma_{\alpha+1}(g^j_\alpha)(\alpha)\}$ is finite.*

Then the group generated by $\{x_{\mathfrak{c},n} : n \in \omega\}$ and $G = \{x \in 2^\mathfrak{c} :$ supp x is bounded in $\mathfrak{c}\}$ is such that the k-th power is countably compact but the 2^k-th power is not countably compact.

Therefore, we will be done with our construction if we show the following:

Lemma 2.5 ($MA_{countable}$). *There exist a family $\{x_{\alpha,n} : \omega \leq \alpha \leq \mathfrak{c},\ n \in \omega\}$ and a family $\{\{h^i_\alpha\}_{i\in l_\alpha} : \omega \leq \alpha < \mathfrak{c}\}$ satisfying the hypothesis of Lemma 2.4.*

Proof: The construction of the $x_{\alpha,n}$'s is by induction on α.

At stage ω, for each $n \in \omega$ let $x_{\omega,n}$ be any element of 2^ω.

At stage α limit and bigger than ω, let $x_{\alpha,n} = \bigcup_{\omega\leq\beta<\alpha} x_{\beta,n}$.

Clearly in both cases the conditions (1)–(3) are satisfied.

At stage $\alpha = \gamma + 1$:

First, let $\{y^i\}_{i\in l_\gamma} \in (2^\gamma)^{l_\gamma}$ be an accumulation point of the sequence $\{\{\sigma_\gamma(S^i_{\gamma,n})\}_{i\in l_\gamma} : n \in \omega\}$. Note that such accumulation point exists, since the sequence $\{\{\sigma_\gamma(S^i_{\gamma,n})\}_{i\in l_\gamma} : n \in \omega\}$ is contained in the compact space $(2^\gamma)^{l_\gamma}$. Now, define $\{h^i_\gamma\}_{i\in l_\gamma} \in (2^\mathfrak{c})^{l_\gamma}$ so that for each $i \in l_\gamma$, we have $h^i_\gamma = y^i \cup 0 \restriction [\gamma, \mathfrak{c})$.

We will be done in this case if we construct a function ϕ as follows:

Lemma 2.6. *Suppose that there exists a function $\phi : \omega \longrightarrow 2$ such that for each $\beta \leq \gamma$ and for each $F \in [\gamma]^{<\omega}$ the set $\{n \in \omega : \{\sigma_\gamma(S^i_{\beta,n}) \restriction F\}_{i\in l_\beta} = \{h^i_\beta \restriction F\}_{i\in l_\beta}$ and $\forall\, i \in l_\beta\ h^i_\beta(\gamma) = \sum_{m\in S^i_{\beta,n}} \phi(m)\}$ is infinite. Furthermore suppose that $\{n \in \omega : \forall\, i \in 2^k\ \phi(2^k n + i) = \sum_{m\in g^i_\gamma} \phi(m)\}$ is finite.*

Then $x_{\gamma+1,n} = x_{\gamma,n} \cup \{\langle\gamma, \phi(n)\rangle\}$ will satisfy the inductive conditions.

The proof of Lemma 2.6 is just a rewriting of the conditions (1)–(3) for the successor case and it is left for the reader. We will construct ϕ later, as we still need to define a partial order and some dense sets. $\square$

Before we give the precise definition of the partial order and the dense sets we will use, let us give a rough idea.

To construct ϕ, we will use finite approximations, that is, the elements of the partial order will be a function from a finite subset of ω into 2. To keep the promise about the accumulation points, it will be sufficient to use some dense sets. The ordering we give will guarantee that the set $\{n \in \omega : \forall\, i \in 2^k\ \phi(2^k n + i) = \sum_{m \in g^i_\gamma} \phi(m)\}$ is finite. For a technical reason, we will already fix the values which ϕ will have at $\bigcup_{i \in 2^k} g^i_\gamma$.

Let r be a function from $2^k \bar{n}$ into 2, where $2^k \bar{n} \supseteq \bigcup_{i \in 2^k} g^i_\gamma$.

Definition 2.2. Let $\mathbb{P}$ be the set of all functions p from $2^k n$ into 2, where $n \in \omega$, such that $p \supseteq r$.

Given $p \in \mathbb{P}$, we denote by K_p the element of ω such that $2^k K_p = dom\, p$. Then the ordering is defined as follows:

$p < q$ if and only if $p \supseteq q$ and for each $n \in [\,K_q, K_p)$ there exists $i \in 2^k$ such that $p(2^k n + i) \neq \sum_{m \in g^i_\gamma} r(m)$.

Before starting the discussion about the dense sets we will use, we fix the following notation:

Definition 2.3. For each $\beta \leq \gamma$ and for each $F \in [\,\gamma\,]^{<\omega}$ let $S(\beta, F)$ be the set of all n's in ω such that $\{\sigma_\gamma(S^i_{\beta,n}) \restriction F\}_{i \in l_\beta} = \{h^i_\beta \restriction F\}_{i \in l_\beta}$. Note that $S(\beta, F)$ is exactly the set of indexes of the elements of the sequence which are inside the open neighbourhood $\{\{x^i\}_{i \in l_\beta} \in 2^\gamma : \forall\, i \in l_\beta\ x^i \restriction F = h^i_\beta \restriction F\}$ of $\{h^i_\beta\}_{i \in l_\beta}$.

Let $\{S(\beta, F, m) : m \in \omega\}$ be a partition of $S(\beta, F)$ into infinite many pieces of infinite size.

From Lemma 2.6, we will be done if we show that for each $F \in [\,\gamma\,]^{<\omega}$ and for each $\beta \leq \gamma$ the set $\{n \in S(\beta, F) : \forall\, i \in l_\beta\ h^i_\beta(\gamma) = \sum_{m \in S^i_{\beta,n}} \phi(m)\}$ is infinite. For this, it is enough to show that for each $m \in \omega$ the set $\{n \in S(\beta, F, m) : \forall\, i \in l_\beta\ h^i_\beta(\gamma) = \sum_{m \in S^i_{\beta,n}} \phi(m)\}$ is not empty.

Let $\mathcal{S} = \{S(\beta, F, m) : \beta \leq \gamma, F \in [\,\kappa\,]^{<\omega}$ and $m \in \omega\}$. We are now ready to define the dense sets associated to the accumulation points for the sequences:

Definition 2.4. For each $S = S(\beta, F, m) \in \mathcal{S}$ and for each $\vec{v} \in 2^{l_\beta}$, we define $E(S, \vec{v}) = \{p \in \mathbb{P} : \exists\, n \in S$ such that $p \supseteq \bigcup_{i \in l_\beta} S^i_{\beta,n}$ and $\forall\, i \in l_\beta\ \sum_{m \in S^i_{\beta,n}} p(m) = \vec{v}(i)\}$.

Besides these dense sets, we will need some other dense sets to make sure that the domain of ϕ is ω:

Definition 2.5. For each $n \in \omega$ let $D_n = \{p \in \mathbb{P} : n \in dom\, p\}$.

We are close to finish our construction now:

Lemma 2.7. *Suppose that the filter $\mathfrak{G}$ meets each $E(S, \vec{v})$, where $S = S(\beta, F, m)$ $\in \mathcal{S}$ and $\vec{v} \in 2^{l}_{\beta}$ and also meets each D_n, where $n \in \omega$. Then $\phi = \bigcup \mathfrak{G}$ satisfies the conditions from Lemma* 2.6.

Proof: Since $\mathfrak{G}$ is a filter, each two members of $\mathfrak{G}$ have a common extension in $\mathfrak{G}$, therefore $\bigcup \mathcal{G} = \phi$ is a function from a subset of ω into 2.

Let n be an element of ω. We want to show that $n \in dom\,\phi$. By hypothesis, there exists $p \in \mathcal{G} \cap D_n$, that is, $n \in dom\, p \subseteq \phi$. Therefore, the domain of ϕ is ω.

We have seen that to show that for each $\beta \leq \gamma$ the sequence in $2^{\gamma+1}$ associated to S_β has $\{h^i_\beta \restriction \gamma + 1\}_{i \in l_\beta}$ as accumulation point, it suffices to show that for each $F \in [\gamma]^{<\omega}$ and for each $m \in \omega$ the set $\{n \in S(\beta, F, m) : \forall\, i \in l_\beta\ h^i_\beta(\gamma) = \sum_{m \in S^i_{\beta,n}} \phi(m)\}$ is not empty. But this follows from the fact that $\mathfrak{G}$ intercepts $E(S(\beta, F, m), \{\, \langle i, h^i_\beta(\gamma)\rangle : i \in l_\beta\})$.

Finally, let us show that $\{n \in \omega : \forall\, i \in 2^k\ \phi(2^k n + i) = \sum_{m \in g^i_\gamma} \phi(m)\}$ is finite. Let q be any element of $\mathfrak{G}$. We claim that $\{n \in \omega : \forall\, i \in 2^k\ \phi(2^k n + i) = \sum_{m \in g^i_\gamma} \phi(m)\}$ is a subset of K_q, where $dom\, q = 2^k K_q$. In fact, let $N \geq K_q$ and let $s \in D_{2^k N} \cap \mathfrak{G} \neq \emptyset$. Since $\mathfrak{G}$ is a filter, there exists $p \in \mathfrak{G}$ which extends q and s. Clearly $N \in [\, K_q, K_p\,)$. Since $p \leq q$, we have that for some $i < 2^k$ we have $\phi(2^k N + i) = p(2^k N + i) \neq \sum_{m \in g^i_\gamma} r(m) = \sum_{m \in g^i_\gamma} \phi(m)$ and we are done. $\square$

Suppose we have shown that the $E(S, \vec{v})$'s and D_n's are dense subsets. Clearly we have less than $\mathfrak{c}$ many dense sets, therefore, we can apply $MA_{countable}$ to obtain a generic filter $\mathfrak{G}$ which intercepts each of them. Therefore, we will be done with our construction if we show that the sets are in fact dense.

Lemma 2.8. *For each $n \in \omega$ the set D_n is dense.*

Proof: Let n be an arbitrary element of ω. Fix $q \in \mathbb{P}$ and let K_q be such that $dom\, q = 2^k K_q$. If $n \in dom\, q$ we have nothing to do so assume that $n \notin dom\, q$. Let $K \in \omega$ such that $2^k K > n$. We will construct $p \in \mathbb{P}$ such that $dom\, p = 2^k K$. For each $i \in l_\beta$ and for each $m \in [\, K_q, K)$ define $p(2^k m + i) = 2 - \sum_{m \in g^i_\gamma} r(m)$. Clearly $p < q$ and we are done. $\square$

Lemma 2.9. *For each $\beta \leq \gamma$, for each $F \in [\,\gamma\,]^{<\omega}$, for each $m \in \omega$ and for each $\vec{v} \in 2^{l_\beta}$, the set $E(S(\beta, F, m), \vec{v})$ is dense.*

Proof: Fix β, F, m and $\vec{v}$. Let q be an arbitrary element of $\mathbb{P}$ and let K_q be such that $2^k K_q = dom\, q$. Since for each $i < 2^k$ the sets in $\{S^i_{\beta,n} : n \in S(\beta, F, m)\}$ are pairwise distinct, there exists $n \in S(\beta, F, m)$ such that for each $i < 2^k$ we have $S^i_{\beta,n} \setminus dom\, q \neq \emptyset$. Let $K \in \omega$ be such that $2^k K \supseteq \bigcup_{i \in l_\beta} S^i_{\beta,n}$.

We will define $p \in \mathbb{P}$ such that $dom\, p = 2^k K$. For each $i \in l_\beta$, let m^i be an element of $S^i_{\beta,n} \setminus dom\, q$. Note that the m^i's are distinct, since $\{S^i_{\beta,n} : i \in l_\beta\}$ are pairwise disjoint. For each $t \in [\,K_q, K)$, let $i_t \in 2^k$ be such that $2^k t + i_t \notin \{m^i : i < l_\beta\}$ (this is possible, since $l_\beta < 2^k$).

For each $t \in [\,K_q, K)$, let $p(2^k t + i_t)$ be equal to $2 - \sum_{j \in g^{i_t}_\gamma} r(j)$. Note that no matter how we define $p(u)$ for $u \in [\,2^k K_q, 2^k K) \setminus \{2^k t + i_t : t \in [\,K_q, K)\}$, we will have $p < q$.

For each $u \in [\,2^k K_q, 2^k K) \setminus (\{m^i : i < l_\beta\} \cup \{2^k t + i_t : t \in [\,K_q, K)\})$, define $p(u) = 0$.

For each $i < l_\beta$, we have already defined $p(u)$ for each $u \in S^i_{\beta,n} \setminus \{m^i : i < l_\beta\}$. Let $p(m^i) = a$, where a is such that $a + \sum_{m \in S^i_{\beta,n} \setminus \{m^i\}} p(m) = \vec{v}(i)$. Then clearly $p \in E(S(\beta, F, m), \vec{v})$. We are done, since as noted above, $p < q$. $\square$

3. Final remarks

It is known that a compact group contains a non-trivial convergent sequence. Under MA(σ-centered), one could construct a group without non-trivial convergent sequences whose every finite power is countably compact. I believe the following is an open question:

Is there a p-compact group (for some free ultrafilter p over ω) without non-trivial convergent sequences?

Salvador Garcia has told me the following is still an open question:

Is there a p-compact group and a q-compact group (for some free ultrafilters p and q over ω) whose the product is not countably compact?

He has told me that under Shelah's model without p-points, the product of every p-compact space and a q-compact space is countably compact.

Under MA, he has shown that there exist a p-compact and a q-compact space whose product is not countably compact.

It is also open whether *there exists a group whose every finite power is countably compact but which is not p-compact for any free ultrafilter p over ω.*

However, one can show the following:

Theorem 3.1 ($MA_{countable}$). *There exist two groups whose every finite power is countably compact but whose product is not countably compact.*

The proof would be a modification of what we have done in the previous section. From this we conclude the following:

Corollary 3.1 ($MA_{countable}$). *Either* (i) *there exists a group whose every finite power is countably compact but it is not r-compact for any free ultrafilter r over ω or* (ii) *there exist two free ultrafilters p and q over ω, a p-compact group and a q-compact group whose product is not countably compact.*

References

[1] Comfort W.W., *Topological Groups*, Handbook of Set-Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, 1984, pp. 1143–1263.

[2] Comfort W.W., *Problems on topological groups and other homogeneous spaces*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, 1990, pp. 311–347.

[3] Comfort W.W., Hofmann K.H., Remus D., *Topological groups and semigroups*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), Elsevier Science Publishers, 1992, pp. 57–144.

[4] Comfort W.W., Ross K.A., *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. **16.3** (1966), 483–496.

[5] van Douwen E.K., *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (Dec. 1980), 417–427.

[6] Hajnal A., Juhász I., *A separable normal topological group need not be Lindelöf*, Gen. Top. and its Appl. **6** (1976), 199–205.

[7] Hart K.P., van Mill J., *A countably compact H such that $H \times H$ is not countably compact*, Trans. Amer. Math. Soc. **323** (Feb. 1991), 811–821.

[8] Tkachenko M.G., *Countably compact and pseudocompact topologies on free abelian groups*, Izvestia VUZ. Matematika **34** (1990), 68–75.

[9] Tomita A.H., *Between countable and sequential compactness in free abelian group*, preprint.

[10] Tomita A.H., *A group under $MA_{countable}$ whose square is countably compact but whose cube is not*, preprint.

[11] Tomita A.H., *The Wallace Problem: a counterexample from $MA_{countable}$ and p-compactness*, to appear in Canadian Math. Bulletin.

[12] Weiss W., *Versions of Martin's Axiom*, Handbook of Set-Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, 1984, pp. 827–886.

Department of Mathematics, Universidade de São Paulo, Caixa Postal 66281, CEP 05389-970, São Paulo - SP, Brazil

E-mail: tomita@ime.usp.br

(*Received* February 16, 1996)