A note on regular points for solutions of parabolic systems

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Abstract. A vector valued function $u = u(x,t)$, solution of a quasilinear parabolic system cannot be too close to a straight line without being regular.

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1. Introduction

The aim of this paper is to extend the result of [1] to nonlinear parabolic systems of partial differential equations:

$$u_t^\alpha - (a_{ij}^{\alpha\beta}(z,u)u_{x_j}^\beta)_{x_i} = f^\alpha(z,u,u_x),\ 1 \le \alpha \le N \tag{1.1}$$

in a domain $A = \Omega \times (0,T) \subset R^{n+1}$, $N > 1$, $n \ge 2$.

Here Ω is an open subset in R_x^n, $z = (x,t)$, $x \in R_x^n$, $t \in R_t$ denotes a generic point in A, $u(z) = (u^1(z), ..., u^N(z))$ is a vector valued function defined in A and $u_x = \{u_{x_i}^\alpha\}$, $i = 1, ..., n$, $\alpha = 1, ..., N$ denotes the spatial gradient of u.

In the paper the summation convention is used.

We shall suppose that the coefficients $a_{ij}^{\alpha\beta}$ are continuous in $\overline{A} \times R^N$ and bounded

$$|a_{ij}^{\alpha\beta}(z,u)| \le L \tag{1.2}$$

and satisfy the ellipticity condition

$$a_{ij}^{\alpha\beta}(z,u)\xi_i^\alpha \xi_j^\beta \ge \lambda|\xi|^2,\ \xi \in R^{nN},\ (z,u) \in \overline{A} \times R^N \tag{1.3}$$

with a uniform constant $\lambda > 0$.

Finally we shall assume that $f(z,u,p)$ is a Caratheodory function satisfying the growth condition

$$|f(z,u,p)| \le L(1 + |u| + |p|). \tag{1.4}$$

To define the concept of a weak solution to (1.1) let us denote by $W_2^{1,0}(A, R^N)$ the completion of $C^1(A, R^N)$ with respect to the norm:

$$|u|_{2,A} = (\int_A |u|^2\, dz + \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \int_A |u^\alpha_{x_i}|^2\, dz)^{\frac{1}{2}}.$$

Then a weak solution of (1.1) by definition is a function $u \in W_2^{1,0}(A, R^N)$ such that for any smooth function $\varphi \in C_0^\infty(A, R^N)$ we have

$$-\int_A u^\alpha \varphi^\alpha_t\, dz + \int_A a^{\alpha\beta}_{ij}(z,u) u^\beta_{x_j} \varphi^\alpha_{x_i}\, dz = \int_A f^\alpha(z,u,u_x)\varphi^\alpha\, dz. \tag{1.5}$$

In above mentioned hypotheses we shall prove that the weak solution of (1.1) cannot be too close to a straight line without being regular.

If $z_0 = (x_0, t_0) \in R^{n+1}$ and $R > 0$ we define

$$\begin{aligned} B(x_0, R) &= \{x \in R^n : |x - x_0| < R\},\\ \Lambda(t_0, R) &= \{t \in R : |t - t_0| < R^2\},\\ Q(z_0, R) &= B(x_0, R) \times \Lambda(t_0, R). \end{aligned}$$

If we introduce in R^{n+1} the metric

$$\delta(z_1, z_2) = \max\{|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{2}}\},$$

then the set $Q(z_0, R)$ is an open ball of radius R, centered at z_0. By $W_2^{1,\frac{1}{2}}(A, R^N)$ we mean the completion of $C^1(A, R^N)$ with respect to the norm

$$|||u|||_{\frac{1}{2},2,A} = \{|u|^2_{2,A} + \int_\Omega dx \iint_{(0,T)\times(0,T)} \frac{|u(x,t) - u(x,s)|^2}{|t-s|^2}\, dt\, ds\}^{\frac{1}{2}}.$$

The following proposition is well known:

Lemma 1.6. *Let $A = \Omega \times (0,T)$ be bounded and convex. Then the natural imbedding of $W_2^{1,\frac{1}{2}}(A, R^N)$ into $L^2(A, R^N)$ is compact.*

It is well known that for solutions to parabolic systems the partial regularity is only possible to prove. More precisely, referring to the system (1.1) with conditions (1.2), (1.3), (1.4) the following result may be proved (see [4], [2]):

Theorem 1.7. *For every $M_0 > 0$ there exist constants ε_0, R_0 such that if $u = u(z)$ is a weak solution of the system* (1.1) *in A and if for some $z_0 \in A$ and $R < \min\{R_0, \delta(z_0, \partial A)\}$*

$$\int^\star_{Q(z_0,R)} |u|^2\, dz \le M_0^2, \tag{1.8}$$

$$U(z_0,R)=\int^{\star}_{Q(z_0,R)}|u-u_{z_0,R}|^2\,dz\le\varepsilon_0^2, \tag{1.9}$$

then u is hölder continuous in a neighborhood of z_0 (with respect to the metric δ mentioned above, see [3]*).*

In (1.8), (1.9) we have used the notation $\int^\star$ to indicate average

$$\int^\star_A f\,dz=\frac{1}{meas(A)}\int_A f\,dz$$

and we have denoted $u_{z_0,R}=\int^\star_{Q(z_0,R)}u\,dz$. Roughly speaking, the theorem asserts that if $u=u(z)$ is sufficiently close to a constant vector in a sufficiently small ball then it is regular near the center of the ball.
The situation is completely different in the case of parabolic equations ($N=1$) whose solutions are regular everywhere.
We have the following theorem: (see [5, Chapter III]).

Theorem 1.10. *Let $g\in W_2^{1,0}(Q,R)$ be a weak solution of the equation*

$$g_t-(b_{ij}(z)g_{x_j})_{x_i}=0 \tag{1.11}$$

in the unit ball $Q=Q(0,1)$ of R^{n+1} (with respect to the metric δ) with bounded, measurable coefficients b_{ij} satisfying

$$|b_{ij}(z)|\le L, \tag{1.12}$$

$$b_{ij}(z)\xi_i\xi_j\ge\lambda|\xi|^2,\ \xi\in R^n,\ z\in Q(0,1). \tag{1.13}$$

Then there exist constants β and K, $\beta=\beta(n,L,\lambda)$, $K=K(n,L,\lambda)$ such that g is β-hölder continuous in $Q(0,\frac{1}{2})$ and

$$||g||_{\beta,\frac{\beta}{2},Q(0,\frac{1}{2})}= \tag{1.14}$$

$$=\sup_{z\in Q(0,\frac{1}{2})}|g(z)|+\sup_{z_1\neq z_2 z_1,z_2\in Q(0,\frac{1}{2})}\frac{|g(z_1)-g(z_2)|}{(\delta(z_1,z_2))^\beta}\le K||g||_{L^2(Q(0,1))}.$$

The above theorem implies that if a solution $u=u(z)$ of the system (1.1) lies on a straight line

$$u(z)=\nu g(z)+\pi,\ \pi\in R^N,\ \nu\in S^{N-1}=\{x:|x|=1\},$$

then u is regular, since in this case g satisfies an parabolic equation for which the conclusion of Theorem 1.10 holds. On the other hand, we may state the following regularity result:

Theorem 1.15. *For each $M_1 > 0$ there exist constants ε_1 and R_1 such that if $u = u(z)$ is a weak solution of system* (1.1) *with conditions* (1.2), (1.3), (1.4) *and if for some $z_0 \in A$, $R < \min\{R_1, \delta(z_0, \partial A)\}$, $\nu \in S^{N-1}$, $\pi \in R^N$, $|\pi| \le M_1$ we have*

$$\int^{\star}_{Q(z_0,R)} |u|^2\, dz \le M_1^2 \tag{1.16}$$

and

$$\int^{\star}_{Q(z_0,R)} |u - \pi|\, dz - \int^{\star}_{Q(z_0,R)} |(u - \pi, \nu)|\, dz \le \varepsilon_1, \tag{1.17}$$

then u is regular in a neighborhood of z_0.

2. Proof of Theorem 1.15

We shall reduce to Theorem 1.7. For that let
$M_0 = 2^{\frac{3}{2}} M_1 (1 + K^2 |Q(0,1)|)^{\frac{1}{2}}$ and let $\varepsilon_0 = \varepsilon_0(M_0)$ and $R_0 = R_0(M_0)$ be the constants in Theorem 1.7.
Let $\tau = \min\{\frac{1}{2}, (\frac{\varepsilon_0}{M_0})^{\frac{1}{\beta}}\}$, ($K$, $\beta -$ constants from Theorem 1.10).
We shall prove that for every $M_1 > 0$ there exist constants ε_1 and $R_1 < R_0(M_0)$ such that if u is a solution of (1.1) satisfying (1.16), (1.17), then

$$\int^{\star}_{Q(z_0,\tau R)} |u|^2\, dz \le {M_0}^2 \tag{2.1}$$

and

$$\int^{\star}_{Q(z_0,\tau R)} |u - u_{z_0,\tau R}|^2\, dz \le \varepsilon_0^2, \tag{2.2}$$

from which the conclusion follows applying Theorem 1.7.

Suppose that our assertion is false. Then it would exist

(i) Sequences $\{z_k\} \subset A$, $\{\pi_k\} \subset R^N$, $\{\nu_k\} \subset S^{N-1}$.

(ii) Two infinitesimal sequences $\{\varepsilon_k\}$ and $\{R_k\}$.

(iii) A sequence $\{u_k\}$ of solutions of the system (1.1) such that

$$\int^{\star}_{Q(z_k,R_k)} |u_k|^2\, dz \le M_1^2, \tag{2.3}$$

$$\int^{\star}_{Q(z_k,R_k)} |u_k - \pi_k|\, dz - \int^{\star}_{Q(z_k,R_k)} |(u - \pi_k, \nu_k)|\, dz \le \varepsilon_k \tag{2.4}$$

but either

$$\int^{\star}_{Q(z_k,\tau R_k)} |u_k|^2 \, dz > M_0^2 \tag{2.5}$$

or

$$\int^{\star}_{Q(z_k,\tau R_k)} |u_k - u_{kz_k,\tau R_k}|^2 > \varepsilon_0^2. \tag{2.6}$$

The functions $v_k(z) = u_k(x_k + R_k x, t_k + R_k^2 t)$ are solutions in $Q = Q(0,1)$ of the system

$$\begin{aligned} -\int_Q v_k^\alpha(z)\varphi_t^\alpha(z) \, dz + \int_Q a_{ijk}^{\alpha\beta}(z, v_k(z)) v_{k,x_j}^\beta(z)\varphi_{x_i}^\alpha(z) \, dz = \\ = \int_Q f_k^\alpha(z, v_k(z), v_{k,x}(z))\varphi^\alpha(z) \, dz, \ \varphi \in C_0^\infty(Q, R^N), \end{aligned}$$

where

$$a_{ijk}^{\alpha\beta}(z, v_k(z)) = a_{ij}^{\alpha\beta}(x_k + R_k x, t_k + R_k^2 t, v_k(z)) \tag{2.7}$$

and

$$f_k^\alpha(z, v_k(z), p) = f^\alpha(x_k + R_k x, t_k + R_k^2 t, v_k(z), R_k^{-1} p) R_k^2. \tag{2.8}$$

We have

$$\int^{\star}_{Q} |v_k|^2 \, dz \le M_1^2, \tag{2.9}$$

$$\int^{\star}_{Q} |v_k - \pi_k| - \int^{\star}_{Q} |(v_k - \pi_k, \nu_k)| \, dz \le \varepsilon_k \tag{2.10}$$

but either

$$\int^{\star}_{Q(0,\tau)} |v_k(z)|^2 \, dz > M_0^2 \tag{2.11}$$

or

$$\int^{\star}_{Q(0,\tau)} |v_k(z) - (v_k)_{0,\tau}|^2 \, dz > \varepsilon_0^2. \tag{2.12}$$

Let now $k \to \infty$. Passing possibly to a subsequence we may suppose that $z_k \to z_0 \in \overline{A}$, $\nu_k \to \nu \in S^{N-1}$, $\pi_k \to \pi$ and $v_k \rightharpoonup v$ weakly in $L^2(Q, R^N)$.
On the other hand, from [4], [6], [5] it follows that

$$|||v_k|||_{\frac{1}{2},2,Q(0,\varrho)} \leq c(\varrho)||v_k||_{L^2(Q)}, \ 0 < \varrho < 1.$$

From this it follows that $v_k \rightharpoonup v$ in the weak topology of $W^{1,\frac{1}{2}}_{2,loc}(Q, R^N)$ and by Lemma 1.6 in the strong topology of $L^2_{loc}(Q, R^N)$.

Arguing as in [4] we conclude that v satisfies

$$\int_Q a^{\alpha\beta}_{ij}(z_0, v) v^\beta_{x_j} \varphi^\alpha_{x_i}\, dz = \int_Q v^\alpha \varphi^\alpha_t\, dz \tag{2.13}$$

where

$$a^{\alpha\beta}_{ij}(z_0, v(z)) = \lim_{k\to\infty} a^{\alpha\beta}_{ijk}(z, v_k(z)), \ \varphi \in C^\infty_0(Q, R^N).$$

Passing to the limit in (2.9), (2.11), (2.12) we obtain

$$\int^\star_Q |v|^2\, dz \leq M_1^2, \tag{2.14}$$

but either

$$\int^\star_{Q(0,\tau)} |v|^2\, dz \geq M_0^2 \tag{2.15}$$

or

$$\int^\star_{Q(0,\tau)} |v(z) - v_{0,\tau}|^2\, dz \geq \varepsilon_0^2. \tag{2.16}$$

For every $\varrho < 1$ we have:

$$\int^\star_{Q(0,\varrho)} [|v_k - \pi_k| - |(v_k - \pi_k, \nu_k)|]\, dz \leq$$

$$\leq \varrho^{-(n+2)} \int^\star_Q [|v_k - \pi_k| - |(v_k - \pi_k, \nu_k)|]\, dz$$

and for $k \to \infty$ we obtain:

$$\int_{Q(0,\varrho)} [|v - \pi| - |(v - \pi, \nu)|]\, dz = 0, \ \varrho < 1 \tag{2.17}$$

so that $v(z)$ lies on straight line

$$v(z) = \pi + (v(z), \nu)\nu =: \pi + g(z)\nu. \tag{2.18}$$

From (2.18) and (2.13) we obtain

$$\int_Q g(z)\varphi_t(z)\,dz = \int_Q A_{ij}(z) g_{x_j}(z)\varphi_{x_i}(z)\,dz,\ \varphi \in C_0^\infty(Q), \tag{2.19}$$

where $A_{ij}(z) = a_{ij}^{\alpha\beta}(z_0, v(z))\nu^\alpha\nu^\beta$ are bounded measurable coefficients satisfying (1.12), (1.13).

From Theorem 1.10 it follows that

$$\int^{\star}_{Q(0,\tau)} |v|^2\,dz \le 2(M_1^2 + \sup_{Q(0,\frac{1}{2})} |g(z)|^2) \le 2M_1^2(1 + K^2|Q|) = \frac{1}{4}M_0^2$$

and

$$\int^{\star}_{Q(0,\tau)} |v - v_{0,\tau}|^2\,dz = \int^{\star}_{Q(0,\tau)} |g - g_{0,\tau}|^2\,dz \le K^2|Q|M_1^2\tau^{2\beta} \le \frac{1}{4}\varepsilon_0^2.$$

These two inequalities contradict (2.15), (2.16). The proof is complete. □

References

[1] Giusti E., Modica G., *A note on regular points for solutions of elliptic systems*, Manusc. Math. **29** (1979), 417–426.

[2] Giusti E., *Regolarita partiale delle soluzioni di sistemi ellittici quasilineari di ordine arbitrario*, Ann. Scuola Norm. Sup. Pisa **23** (1969), 115–141.

[3] Campanato S., *Equazioni paraboliche del secondo ordine e spazi* $\mathcal{L}^{2,\Theta}(\Omega,\delta)$, Ann. Mat. Pura Appl. **73** (1966), 55–102.

[4] Giaquinta M., Giusti E., *Partial regularity for the solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. **47** (1973), 253–266.

[5] Ladyzhenskaya O.A., Solonnikov V.A., Uralceva N.N., *Linear and quasilinear equations of parabolic type*, Translations of Math. Monographs 23, Providence, Rhode Island: AMS 1968.

[6] Giaquinta M., Struwe M., *On the partial regularity of weak solutions of nonlinear parabolic systems*, Math. Z. **179** (1982), 437–451.

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