On automorphisms of digraphs without symmetric cycles

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Abstract. A digraph is a symmetric cycle if it is symmetric and its underlying graph is a cycle. It is proved that if D is an asymmetric digraph not containing a symmetric cycle, then D remains asymmetric after removing some vertex. It is also showed that each digraph D without a symmetric cycle, whose underlying graph is connected, contains a vertex which is a common fixed point of all automorphisms of D.

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A graph (digraph) G is called *asymmetric* if the only automorphism of G is an identity, and *symmetric* otherwise. An asymmetric graph G is *minimally asymmetric* if every subgraph of G on at least two vertices is symmetric, and it is *critically asymmetric* if for every vertex v of G, the graph $G - v$, obtained from G by removing the vertex v together with all incident edges, is symmetric. During the Oberwolfach Seminar in 1988 Professor Nešetřil conjectured that there are only finite number of minimally asymmetric undirected graphs and that every critically asymmetric directed graph contains a directed cycle of length two. The first, undirected part of this problem, was almost completely settled in papers of Sabidussi [3] and Nešetřil and Sabidussi [2]. In this note we deal with the second part of Nešetřil's conjecture. Although, we are not able to show that each critically asymmetric directed graph contains a directed cycle of length two, we prove that in the underlying graph of such a digraph there exists a cycle which is symmetrically oriented.

In order to state our results we need to introduce some definitions (note that our notation, based mainly on properties of the underlying undirected graph of D, is rather non-standard). A digraph D is *connected* if the underlying graph of D (obtained from D by replacing each arc by an undirected edge) is connected. By a *component* of D we mean a maximum connected subdigraph of D. A *neighbor* of a vertex v of D is a vertex adjacent to v in the underlying graph of D. We call D a *path* (*cycle*) if the underlying graph of D is a path (cycle). Furthermore, we say that D is a *symmetric cycle* if it is a cycle and it is symmetric, and D is an *alternating path* if it is a path and it contains no directed path with two arcs. For a digraph D, by D^s we denote the digraph obtained from D by removing all loops of D and reducing each multiple arc to a single one. Finally, a vertex v is a *fixed point* of an automorphism σ, if $\sigma(v) = v$.

We shall prove that every critically asymmetric digraph contains a symmetric cycle. In fact, our result is even slightly stronger.

Theorem 1. *Let D be a digraph with at least two vertices. If*

(i) *the underlying graph of D is connected,*
(ii) *D^s contains no symmetric cycle having an automorphism without fixed points,*
(iii) *D^s contains no induced symmetric subdigraph being a cycle with at most one diagonal arc, having an automorphism with two fixed points,*
(iv) *D^s is not an alternating path of odd length,*

then there exists a vertex v of D such that the underlying graph of $D - v$ is connected and each neighbor of v is a fixed point of every automorphism of $D - v$.

Clearly, if D or D^s does not contains a symmetric cycle, then assumptions (ii) and (iii) hold.

Corollary 1. *Each asymmetric digraph D on at least two vertices satisfying assumptions* (ii) *and* (iii) *of Theorem* 1 *contains a vertex v such that $D - v$ is asymmetric.*

In particular, Theorem 1 and Corollary 1 hold when D is an oriented tree. (Note that analogous results are *not* valid for undirected trees: a detailed analysis of this case was given by Nešetřil in [1].) Nonetheless, there are also some dense digraphs satisfying assumptions of Theorem 1, as, for example, a transitive tournament, i.e. the digraph with the vertex set $\{1, \dots, n\}$ in which (i, j) is an arc if and only if $i < j$.

Let us introduce some further definitions. For any digraph D and for any vertex set $V_1 \subseteq V(D)$, $D[V_1]$ is the subdigraph of D induced by V_1. A *neighborhood* of V_1 in D, denoted by $N_D(V_1)$, is the set of all neighbors of vertices from V_1 which do not belong to V_1, and $N_D(v) = N_D(\{v\})$. For an automorphism σ of D we define the set of all *movable points* $Mov(\sigma)$ of σ setting

$$Mov(\sigma) = \{v \in V(D) \mid \sigma(v) \neq v\}.$$

Finally, for a subdigraph H of D, let $\sigma(H)$ denote the digraph which is the image of H in σ.

Theorem 2. *Let D be a symmetric digraph, let σ be a non-identity automorphism of D and let H be a component of $D[Mov(\sigma)]$. If $\sigma(V(H)) \cap V(H) \neq \emptyset$, then $\sigma(H) = H$ and H contains a symmetric cycle having an automorphism without fixed points.*

The above theorem implies that if a connected digraph D has an automorphism σ such that $Mov(\sigma) = V(D)$, then D contains a symmetric cycle having an automorphism without fixed points. In other words, if D satisfies assumptions (i) and (ii) of Theorem 1, then each automorphism of D has a fixed point. Using a lemma based on Theorem 2 we shall prove the following.

Corollary 2. *Each digraph D satisfying assumptions* (i)–(iii) *of Theorem* 1 *contains a vertex which is a common fixed point of all automorphisms of D.*

Proof of Theorem 2: In order to show that $\sigma(H) = H$ let $\sigma' = \sigma|_{Mov(\sigma)}$. Since $\sigma(Mov(\sigma)) = Mov(\sigma)$, σ' is an automorphism of $D[Mov(\sigma)]$, hence $\sigma'(H)$ is a component of $D[Mov(\sigma)]$. But $\sigma'(H) = \sigma(H)$. Thus, by assumption, we obtain the equality of components $\sigma(H) = H$. This implies that $\sigma|_{V(H)}$ is an automorphism of H.

Let

$$\mathcal{P} = \{(P, v, t) \mid t \in \{1, 2, \dots\} \text{ and } P \text{ is a } (v, \sigma^t(v))\text{-path in } H \text{ such that for every } u \in V(P),\ \sigma^t(u) \neq u\}.$$

The set $\mathcal{P}$ is non-empty since for any $(v, \sigma(v))$-path P_1 in H we have $V(P_1) \subseteq Mov(\sigma)$ and $(P_1, v, 1) \in \mathcal{P}$. Let l be the minimum length of a path P such that for some v and t, $(P, v, t) \in \mathcal{P}$. Define

$$\nu(v, t) = \min\{i \geq 1 \mid \sigma^{it}(v) = v\}.$$

Let n be the smallest value of $\nu(v, t)$ such that $(P, v, t) \in \mathcal{P}$ for some path P of length l. Finally, fix $(P_0, v_0^0, k) \in \mathcal{P}$, where $P_0 = (v_0^0, v_1^0, \dots, v_l^0)$ is a path of length l in H, $k \geq 1$ and $\nu(v_0^0, k) = n$. Then $v_0^l = \sigma^k(v_0^0)$ and $\sigma^{nk}(v_0^0) = v_0^0$. Denote by C the following union of paths:

$$C = P_0 \cup \sigma^k(P_0) \cup \dots \cup \sigma^{(n-1)k}(P_0).$$

We shall show that C is a cycle. Set $v_r^i = \sigma^{ik}(v_r^0)$. Since for $i \geq 0$, $v_l^i = \sigma^{ik}(\sigma^k(v_0^0)) = v_0^{i+1}$ and $v_l^{n-1} = v_0^n = \sigma^{nk}(v_0^0) = v_0^0$, it is enough to show that paths generating C are not intersecting, i.e. that

($*$) for every $0 \leq r \leq l-1$ and $0 \leq i < j \leq n-1$, we have $v_r^i \neq v_r^j$,

($**$) for every $r, s \in \{0, \dots, l-1\}$, $r \neq s$, and $0 \leq i < j \leq n-1$, we have $v_r^i \neq v_s^j$.

Suppose first that ($*$) does not hold. Consider a (v_r^i, v_r^{i+1})-path P' with $V(P') \subseteq \{v_r^i, \dots, v_{l-1}^i, v_0^{i+1}, \dots, v_r^{i+1}\}$. If m and i' are such that $v_m^{i'} \in V(P')$, then, since $(P_0, v_0^0, k) \in \mathcal{P}$, $\sigma^k(v_m^0) \neq v_m^0$. Consequently, by the injectivity of $\sigma^{i'k}$, we have $\sigma^{i'k}(\sigma^k(v_m^0)) \neq \sigma^{i'k}(v_m^0)$ and $\sigma^k(v_m^{i'}) \neq v_m^{i'}$. Thus, since $v_r^{i+1} = \sigma^k(v_r^i)$, $(P', v_r^i, k) \in \mathcal{P}$. It follows that P' is of length l (by definition, P' is of length at most l), so due to the minimality of n, $\nu(v_r^i, k) \geq n$. But, from the negation of ($*$), $v_r^i = v_r^j = \sigma^{(j-i)k}(v_r^i)$, which gives $\nu(v_r^i, k) \leq j - i < n$, a contradiction. Hence ($*$) holds.

Assume now that ($**$) does not hold. Then $v_r^i = v_s^j = \sigma^{(j-i)k}(v_s^i)$. Let P'' be the (v_s^i, v_r^i)-path contained in $\sigma^{ik}(P_0)$. By ($*$), $\sigma^{(j-i)k}(v_m^i) \neq v_m^i$ for $m = s, \dots, r$, and so $(P'', v_s^i, (j-i)k) \in \mathcal{P}$. But the length of P'' is equal $|r-s| < l$, which contradicts the minimality of l. Therefore, $(**)$ holds and C is a cycle. Now one can define a non-identity automorphism of C setting for $0 \le r \le l-1$ and $0 \le i \le n-1$

$$\rho(v_r^i) = \begin{cases} v_r^{i+1} & \text{if } 0 \le i \le n-2, \\ v_r^0 & \text{if } i = n-1. \end{cases}$$

□

In the proof of Theorem 1 we shall need the following lemma.

Lemma. *Let D be a symmetric digraph satisfying assumptions* (i)–(iii) *of Theorem* 1 *and let σ be a non-identity automorphism of D. Furthermore, let M_1 be the vertex set of a component of $D[Mov(\sigma)]$ and $M_2 = \sigma(M_1)$. Then $M_1 \cap M_2 = \emptyset$ and there exists a fixed point $r_D(M_1)$ of σ such that*

$$N_D(M_1) = N_D(M_2) = \{r_D(M_1)\}.$$

Proof of the Lemma: By (ii), Theorem 2 implies that $M_1 \cap M_2 = \emptyset$. This gives $V(D) \setminus M_1 \ne \emptyset$, and since D is connected, $N_D(M_1) \ne \emptyset$. We shall show that $|N_D(M_1)| = 1$. Indeed, suppose to the contrary that $r, r' \in N_D(M_1)$, $r \ne r'$. Note that since M_1 is a component of $D[Mov(\sigma)]$, we have $\sigma(r) = r$ and $\sigma(r') = r'$. Let P be the shortest (r, r')-path with

$$\emptyset \ne V(P) \setminus \{r, r'\} \subseteq M_1.$$

Then each of digraphs $D^s[V(P)]$ and $D^s[\sigma(V(P))]$ consists of a (r, r')-path and at most one arc connecting r and r' (by (ii) there are no pairs of arcs running in opposite directions). Let $H = D^s[V(P) \cup \sigma(V(P))]$. Since $\sigma(V(P) \setminus \{r, r'\}) \subseteq M_2$ and

$$N_D(M_1) \cap M_2 \subseteq N_D(M_1) \cap Mov(\sigma) = \emptyset,$$

no arcs connect sets $V(P) \setminus \{r, r'\}$ and $\sigma(V(P)) \setminus \{r, r'\}$. Thus, H is a symmetric cycle with at most one diagonal arc, which contradicts assumption (iii).

Hence, $|N_D(M_1)| = 1$, and, due to the fact that $N_D(M_2) = N_D(M_1)$, the result follows. □

Proof of Theorem 1: For any vertex set $V_1 \subseteq V = V(D)$ we define

$$V_1^* = \{v \in V_1 \mid D - v \text{ is connected}\}.$$

Furthermore, for a subdigraph H of D and for $v \in V(H)$ let $d_H^+(v)$ and $d_H^-(v)$ denote the in-degree and the out-degree of v in H, respectively.

Observe first that if Theorem 1 holds for D^s, then it holds for D. Thus, we may and shall assume that $D^s = D$. Now, suppose, contrary to Theorem 1, that for every $v \in V^*$, there exists an automorphism σ_v of $D - v$ such that some neighbor of v is a movable point of σ_v. For every $v \in V^*$ consider the digraph

$(D-v)[Mov(\sigma_v)]$ induced in $D-v$ by all movable points of σ_v, and denote by $M_1(v)$ the vertex set of some component of $(D-v)[Mov(\sigma_v)]$ containing a neighbor of v, i.e. $v \in N_D(M_1(v))$. Let us apply the Lemma for $D-v$, σ_v and $M_1(v)$. Then, if we set $M_2(v) = \sigma_v(M_1(v))$, $M(v) = M_1(v) \cup M_2(v)$ and $r(v) = r_{D-v}(M_1(v))$, the Lemma implies that for every $v \in V^*$,

$$M_1(v) \cap M_2(v) = \emptyset,$$

(1) $$N_D(M_1(v)) = \{v, r(v)\},$$

(2) $$\{r(v)\} \subseteq N_D(M_2(v)) \subseteq \{v, r(v)\}.$$

Clearly, $|M_1(v)| = |M_2(v)|$.

Note that both $D[M_1(v)]$ and $D[M_2(v)]$ are components of $D-\{v, r(v)\}$. Thus, for every $v \in V^*$, $i = 1, 2$, and $U \subseteq V$, the following property holds.

> $p(v, i, U)$: *If $v, r(v) \notin U$, $D[U]$ is connected and $U \cap M_i(v) \neq \emptyset$, then $U \subseteq M_i(v)$.*

This elementary observation shall be used in the proof of Theorem 1 many times.

Let B be the smallest subset of V such that $B^* \neq \emptyset$ and $\bigcup_{v \in B^*} M(v) \subseteq B$. We split the proof into two cases, with respect to the two possible choices for the set $N_D(M_2(v))$, specified by (2).

Case 1. There exists $v_1 \in B^*$ such that $N_D(M_2(v_1)) = \{r_1\}$, where $r_1 = r(v_1)$.

We can choose v_1 in such a way that, among all vertices satisfying the assumption of Case 1, the set $M_1(v_1)$ has the maximal size, i.e. if $v \in B^*$ and $N_D(M_2(v)) = \{r(v)\}$, then $|M_1(v)| \leq |M_1(v_1)|$.

Note that $D[M_2(v_1)]$ is a component of $D - r_1$. Furthermore, since $v_1 \in B^*$, $M_2(v_1) \subseteq B$. Denote by S the vertex set of the smallest component of $D - r_1$ with $S \subseteq B$. Then

(3) $$|S| \leq |M_2(v_1)|.$$

From (1) applied with $v = v_1$, the set $M_1(v_1) \cup \{v_1\}$ induces in $D - r_1$ a connected subdigraph. Hence $M_1(v_1) \cup \{v_1\}$ is contained in some component of $D - r_1$. Since, by (3), $|S| \leq |M_2(v_1)| < |M_1(v_1) \cup \{v_1\}|$, this component must be different from $D[S]$, i.e.

(4) $$S \cap (M_1(v_1) \cup \{v_1\}) = \emptyset.$$

Observe that the set S^* is non-empty: for example, if $u \in S$ is the ending vertex of the longest (r_1, u)-path contained in $D[S \cup \{r_1\}]$, then clearly $u \in S^*$. Since $S^* \neq \emptyset$ and, by (3), $|S| < |B|$, from the minimality of B there exists a vertex $v_2 \in S^*$ such that for some w,

(5) $$w \in M(v_2) \setminus S.$$

In next steps of the proof we shall often use the following simple observation: if $r_1 \notin M_1(v_2)$, then

$$M_1(v_2) \subseteq S. \tag{6}$$

In order to verify (6) suppose that $r_1 \notin M_1(v_2)$. (1) taken for $v = v_2$ implies that there is a vertex $u \in M_1(v_2) \cap N_D(v_2)$. Since $v_2 \in S$ and $D[S]$ is a component of $D - r_1$, we have

$$N_D(v_2) \subseteq S \cup N_D(S) = S \cup \{r_1\}.$$

Hence, $u \in M_1(v_2) \cap (S \cup \{r_1\})$, and, due to the assumption that $r_1 \notin M_1(v_2)$, we get $M_1(v_2) \cap S \neq \emptyset$. But, since $r_1 \notin M_1(v_2)$, $M_1(v_2)$ induces in $D - r_1$ a connected subdigraph. Thus (6) holds.

Let $r_2 = r(v_2)$. We shall show that $r_2 \neq r_1$. Indeed, suppose that $r_1 = r_2$. Then $r_1 \notin M_1(v_2) \cup M_2(v_2)$, so we can use (6). By (5) and (6) we have $M_2(v_2) \setminus S \neq \emptyset$. As $r_1 \notin M_2(v_2)$, the digraph induced in $D - r_1$ by $M_2(v_2)$ is connected. Consequently, $D[M_2(v_2)]$ is a component of $D - r_1$ different from $D[S]$. Furthermore, since $v_2 \in S^* \subseteq B^*$, by the definition of B we have $M_2(v_2) \subseteq B$. But $|M_2(v_2)| = |M_1(v_2)| < |S|$, due to (6) and the fact that $v_2 \in S \setminus M_1(v_2)$. This contradicts the minimality of S. Hence,

$$r_2 \neq r_1. \tag{7}$$

We shall show now that $r_1 \in M_1(v_2)$. Suppose to the contrary that $r_1 \notin M_1(v_2)$. Note that (5), together with (1) and (2) taken for $v = v_2$, implies that there is a (v_2, w)-path P_1 with

$$V(P_1) \setminus \{v_2\} \subseteq M(v_2) \cup \{r_2\}.$$

Since $v_2 \in S$ and $w \notin S$, the path $P_1 - v_2$ must intersect the neighborhood $N_D(S) = \{r_1\}$. Hence $r_1 \in V(P_1) \setminus \{v_2\}$. Thus, using (7), we get $r_1 \in M(v_2)$, and, since by our assumption $r_1 \notin M_1(v_2)$, we arrive at

$$r_1 \in U \cap M_2(v_2),$$

where $U = M_1(v_1) \cup \{r_1\}$. From (1) and (6) it follows that

$$r_2 \in N_D(M_1(v_2)) \subseteq S \cup N_D(S) = S \cup \{r_1\},$$

which together with (7) gives $r_2 \in S$. By (4) we have $S \cap U = \emptyset$. Thus, $v_2, r_2 \notin U$. Moreover, from (1), $D[U]$ is connected. Consequently, because of $p(v_2, 2, U)$ we get $M_1(v_1) \cup \{r_1\} \subseteq M_2(v_2)$. This fact, (6) and (3) yield

$$|M_1(v_1) \cup \{r_1\}| \leq |M_2(v_2)| = |M_1(v_2)| \leq |S| \leq |M_2(v_1)| = |M_1(v_1)|.$$

which is impossible, so

$$r_1 \in M_1(v_2). \tag{8}$$

Now we split our argument into three subcases.

Subcase 1a. $r_2 \notin M_1(v_1)$.

Set $U = M_1(v_1) \cup \{r_1\}$. Then, by (7), $r_2 \notin U$. Furthermore, by (4) we have $v_2 \notin U$, and (8) implies that $r_1 \in U \cap M_1(v_2)$. Thus, $p(v_2, 1, U)$ gives $M_1(v_1) \cup \{r_1\} \subseteq M_1(v_2)$, contradicting the maximality of $|M_1(v_1)|$.

Subcase 1b. $r_2 \in M_1(v_1)$ and $S \neq M_2(v_1)$.

Set $U = M_2(v_1) \cup \{r_1\}$. Then $r_2 \notin U$. Since due to the assumption of Case 1 both S and $M_2(v_1)$ induce components of $D - r_1$, the fact that $S \neq M_2(v_1)$ implies that $S \cap M_2(v_1) = \emptyset$. Hence $v_2 \notin U$. By (8), $r_1 \in U \cap M_1(v_2)$. Therefore, $p(v_2, 1, U)$ yields $M_2(v_1) \cup \{r_1\} \subseteq M_1(v_2)$, which leads to a contradiction like the one above.

Subcase 1c. $r_2 \in M_1(v_1)$ and $S = M_2(v_1)$.

Set $U = (M_2(v_1) \cup \{r_1\}) \setminus \{v_2\}$. Note that the graph induced in D by U is connected. Indeed, let P be a path in $D - v_2$ connecting two vertices from U (the existence of such a path follows from the fact that $v_2 \in S^*$). Using the assumption of Case 1, we have $V(P) \subseteq M_2(v_1) \cup \{r_1\}$, and so $V(P) \subseteq U$. Hence $D[U]$ is connected. By (8), $r_1 \in U \cap M_1(v_2)$, and by (7) we have $r_2 \notin U$. Thus $p(v_2, 1, U)$ holds and

$$(M_2(v_1) \cup \{r_1\}) \setminus \{v_2\} \subseteq M_1(v_2). \tag{9}$$

Set now $U = (M_2(v_2) \cup \{r_2\}) \setminus \{v_1\}$. To show that $D[U]$ is connected let P' be a path in $D - v_1$ connecting two vertices from U (note that since $v_1 \in B^*$, such a path always exists). As $v_2 \in S$, from (9) and the assumption that $v_2 \in S = M_2(v_1)$ we get

$$N_D(v_2) \subseteq (M_2(v_1) \cup \{r_1\}) \setminus \{v_2\} \subseteq M_1(v_2). \tag{10}$$

Thus, $N_D(v_2) \cap M_2(v_2) = \emptyset$, and using (2) we arrive at

$$N_D(M_2(v_2)) = \{r_2\}. \tag{11}$$

It follows that $V(P') \subseteq M_2(v_2) \cup \{r_2\}$, so $D[U]$ is connected. Due to the assumption of Subcase 1c, $r_2 \in U \cap M_1(v_1)$, and from (8) we have $r_1 \notin U$. Consequently, $p(v_1, 1, U)$ gives

$$(M_2(v_2) \cup \{r_2\}) \setminus \{v_1\} \subseteq M_1(v_1).$$

We obtain

$$\begin{aligned}|M_2(v_1)| &\le |(M_2(v_1)\cup\{r_1\})\setminus\{v_2\}| \le |M_1(v_2)| = |M_2(v_2)|\\ &\le |(M_2(v_2)\cup\{r_2\})\setminus\{v_1\}| \le |M_1(v_1)| = |M_2(v_1)|.\end{aligned}$$

Thus,

$$(M_2(v_1)\cup\{r_1\})\setminus\{v_2\} = M_1(v_2), \tag{12}$$

$$(M_2(v_2)\cup\{r_2\})\setminus\{v_1\} = M_1(v_1), \tag{13}$$

$$|M_2(v_2)| = |M_1(v_1)|. \tag{14}$$

For $u \in V$ define the following assertion.

$q(u)$: *$d^+_D(u) = 2$ and $d^-_D(u) = 0$ or $d^+_D(u) = 0$ and $d^-_D(u) = 2$.*

By (8) and (1) we have $N_D(r_1) \subseteq (M_1(v_2)\cup\{v_2, r_2\})\setminus\{r_1\}$, so, from (12),

$$N_D(r_1) \subseteq M_2(v_1)\cup\{r_2\}.$$

Since $M_1(v_1)\cap M_2(v_1) = \emptyset$, the above fact and the assumption that $r_2 \in M_1(v_1)$ imply that $N_D(r_1)\cap M_1(v_1) = \{r_2\}$ and $N_D(r_1) \subseteq M_1(v_1)\cup M_2(v_1)$. Hence, $N_D(r_1) = \{r_2, \sigma_{v_1}(r_2)\}$ and $q(r_1)$ holds. Similarly, using (13), one can check that $N_D(r_2) = \{r_1, \sigma_{v_2}(r_1)\}$ and $q(r_2)$ holds.

In such a way we have proved that for $k = 1$ the following assertion holds.

$a(k)$: There exist vertices $u_1, \dots, u_k \in M_1(v_1)$ and $w_1, \dots, w_k \in M_1(v_2)$ such that $u_1 = r_2$, $w_1 = r_1$, $P_k = (u_k, \dots, u_1, w_1, \dots, w_k)$ is an alternating path and assertions $q(u_i)$, $q(w_i)$ hold for $i = 1, \dots, k$.

We shall show that $a(|M_1(v_1)|)$ is true using an induction with respect to k. Suppose that $1 \le k < |M_1(v_1)|$ and that $a(k)$ holds. Let u_{k+1} and w_{k+1} be neighbors of u_k and w_k respectively, which do not belong to P_k. $q(u_1), \dots, q(u_k)$, together with the facts that $|M_1(v_1)| > k$ and $M_1(v_1)$ induces in $D - v_1$ a connected subdigraph, give $u_{k+1} \in M_1(v_1)$. Hence v_1 is not a neighbor of u_k and $|N_{D-v_1}(u_k)| = 2$. By $a(k)$, $w_{k+1} = \sigma_{v_1}(u_k)$. Thus, $q(u_k)$ implies $q(w_{k+1})$. Similarly, $|N_{D-v_2}(w_k)| = 2$, $u_{k+1} = \sigma_{v_2}(w_k)$ and $q(u_{k+1})$ holds. Hence $a(k+1)$ is true. We conclude that $a(|M_1(v_1)|)$ holds.

Assume now that $k = |M_1(v_1)|$. From $a(k)$ we have $M_1(v_1) = \{u_1, \dots, u_k\}$ and $N_D(v_1)\cap M_1(v_1) = \{u_k\}$. Since by (13) and (14) we have $v_1 \in M_2(v_2)$, (11) and (13) yield $N_D(v_1) \subseteq (M_2(v_2)\cup\{r_2\})\setminus\{v_1\} = M_1(v_1)$. It follows that $N_D(v_1) = \{u_k\}$. Similarly, $M_1(v_2) = \{w_1, \dots, w_k\}$ and $N_D(v_2)\cap M_1(v_2) = \{w_k\}$, and, by (10), we get $N_D(v_2) = \{w_k\}$. Therefore, D is an alternating path with odd length equal $2k+1$, which contradicts assumption (iv) of Theorem 1.

Case 2. For every $v \in B^*$, $N_D(M_i(v)) = \{v, r(v)\}$, $i = 1, 2$.

Let $v_3 \in B^*$ be a vertex with the largest size of set $M_1(v_3)$ and let $r_3 = r(v_3)$. Denote by P the shortest (v_3, r_3)-path with $V(P)\setminus\{v_3, r_3\} \subseteq M_1(v_3)$. Let

$$U_1 = M_1(v_3)\setminus V(P).$$

We shall show first that $V(P) \setminus \{v_3, r_3\} = M_1(v_3)$. Suppose that it is not the case. Then $U_1 \neq \emptyset$ and $U_1^* \neq \emptyset$. Since $v_3 \in B^*$, we have $M(v_3) \subseteq B$. Hence $U_1 \subset B$, $U_1 \neq B$, and, from the minimality of B, there exists $v_4 \in U_1^*$ such that some vertex $w \in M(v_4) \setminus U_1$. Let $r_4 = r(v_4)$. Due to the assumption of Case 2 for $v = v_4$ (which can be used since $v_4 \in U_1^* \subseteq B^*$), the digraph induced in D by the set $M(v_4) \cup \{v_4\}$ is connected, so there is a (v_4, w)-path P' with $V(P') \setminus \{v_4\} \subseteq M(v_4)$. Since $w \notin U_1$ and $v_4 \in U_1$, the path $P' - v_4$ intersects the neighborhood of U_1, and since

$$N_D(U_1) \subseteq V(P), \tag{15}$$

there exists $w' \in V(P) \cap (V(P') \setminus \{v_4\}) \subseteq V(P) \cap M(v_4)$. Let

$$U = (M_2(v_3) \cup V(P)) \setminus \{r_4\}.$$

As $r_4 \notin M(v_4)$, $w' \neq r_4$. It follows that $w' \in U \cap M(v_4)$, and thus, for some $i \in \{1, 2\}$,

$$U \cap M_i(v_4) \neq \emptyset. \tag{16}$$

Note that $D[U]$ is connected. Indeed, suppose to the contrary that r_4 disconnects the digraph $D[M_2(v_3) \cup V(P)]$. Then, since both ends of P belong to the neighborhood of the connected set $M_2(v_3)$, we have $r_4 \notin V(P)$ and $r_4 \in M_2(v_3)$. Hence, by (15),

$$r_4 \notin U_1 \cup N_D(U_1).$$

Since $v_4 \in U_1$, the above fact and (15), together with the assumption of Case 2 for $v = v_4$, imply that both $M_1(v_4)$ and $M_2(v_4)$ contain vertices of path P. This is impossible because $M_1(v_4)$ and $M_2(v_4)$ induce disjoint components of digraph $D - \{v_4, r_4\}$ and $v_4, r_4 \notin V(P)$, so $D[U]$ is connected.

Since $v_4 \in U_1$, $v_4 \notin U$. Thus, by (16), $p(v_4, i, U)$ yields $U \subseteq M_i(v_4)$, contradicting the maximality of $|M_1(v_3)|$. Therefore,

$$V(P) \setminus \{v_3, r_3\} = M_1(v_3). \tag{17}$$

Finally, we shall show that the cycle

$$C = P \cup \sigma_{v_3}(P)$$

has an automorphism without fixed points. Let $r \in V(C - v_3)$ and let v be the vertex of C lying opposite r. Then $v \in V^*$. We want to show that $r(v) = r$. This clearly holds for $r = r_3$, so let us assume that $r \neq r_3$. Then $v \in V(C) \setminus \{v_3, r_3\}$. Note that (17) and the assumption of Case 2 for $v = v_3$, imply that for every $u \in V(C) \setminus \{v_3, r_3\}$, we have $N_D(u) \subseteq V(C)$ and, by the minimality of P,

$$N_D(u) = N_C(u). \tag{18}$$

Using (18) and the assumption of Case 2 for vertex v, one can find $u_1, u_2 \in V(C)$ such that $N_D(v) = \{u_1, u_2\}$, $u_1 \in M_1(v)$ and $u_2 \in M_2(v)$. Hence, $C - v$ is a (u_1, u_2)-path in $D - v$ connecting $M_1(v)$ and $M_2(v)$. Such a path must contain $r(v)$. Consequently, the facts that for every $u \in V(C) \setminus \{v_3, r_3\}$ (18) holds, $u_2 = \sigma_v(u_1)$ and $D[M_2(v)] = \sigma_v(D[M_1(v)])$, imply that $r(v)$ must lie precisely in the middle of the path $C - v$, i.e. $r(v) = r$. It follows that $d_C^+(r) = 2$ or $d_C^-(r) = 2$. Since it holds for every $r \in V(C - v_3)$, C has an automorphism without fixed points, which contradicts assumption (ii) of Theorem 1.

This completes the proof of Case 2, and, due to the observation that, by (1) and (2), Case 1 is the negation of Case 2, the proof of Theorem 1. □

Proof of Corollary 1: Let D_1 be the smallest component of D. Clearly, if either $V(D_1) = \{v\}$, or D_1^s is an alternating path of odd length and v is a neighbor of an ending vertex of this path, then $D - v$ is asymmetric. Thus, let v be the vertex from Theorem 1 applied for D_1, and suppose that $D - v$ has a non-identity automorphism σ. By the minimality of D_1, $\sigma(V(D_1 - v)) = V(D_1 - v)$, so $\sigma|_{V(D_1 - v)}$ is an automorphism of $D_1 - v$. Thus, by Theorem 1, all neighbors of v are fixed points of $\sigma|_{V(D_1 - v)}$, and so also of σ. Hence, clearly, σ' defined as

$$\sigma'(u) = \begin{cases} \sigma(u) & \text{if } u \in V(D - v) \\ u & \text{if } u = v \end{cases}$$

is a non-identity automorphism of D, which contradicts the assumption that D is asymmetric. □

Proof of Corollary 2: Assume that D has a non-identity automorphism. Let σ be a non-identity automorphism of D having the largest size of the maximum component of the digraph $D[Mov(\sigma)]$ induced in D by all movable points of σ. Denote this component by $M_1(\sigma)$. Let us apply the Lemma for σ and $M_1(\sigma)$. Set $v = r_D(M_1(\sigma))$ and $M_2(\sigma) = \sigma(M_1(\sigma))$. We have $M_1(\sigma) \cap M_2(\sigma) = \emptyset$.

We shall show that v is a fixed point of every automorphism of D. Suppose to the contrary that v is a movable point of some automorphism ρ of D. Let $M_1(\rho)$ be the vertex set of the component of $D[Mov(\rho)]$ containing v. Using the Lemma for ρ and $M_1(\rho)$, set $r = r_D(M_1(\rho))$. Let $i \in \{1, 2\}$ be such that $r \notin M_i(\sigma)$. Then the digraph induced in $D - r$ by the set $M_i(\sigma) \cup \{v\}$ is connected. But, from the Lemma, $M_1(\rho)$ induces a component of $D - r$ and, by assumption, $v \in M_1(\rho)$. Thus, $M_i(\sigma) \cup \{v\} \subseteq M_1(\rho)$, which gives $|M_1(\rho)| > |M_i(\sigma)| = |M_1(\sigma)|$, contradicting the maximality of $|M_1(\sigma)|$. □

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