A family of 4-designs on 26 points

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Abstract. Using the Kramer-Mesner method, 4-$(26,6,\lambda)$ designs with $PSL(2,25)$ as a group of automorphisms and with λ in the set $\{30,51,60,81,90,111\}$ are constructed. The search uses specific partitioning of columns of the orbit incidence matrix, related to so-called "quasi-designs". Actions of groups $PSL(2,25)$, $PGL(2,25)$ and twisted $PGL(2,25)$ are being compared. It is shown that there exist 4-$(26,6,\lambda)$ designs with $PGL(2,25)$, respectively twisted $PGL(2,25)$ as a group of automorphisms and with λ in the set $\{51,60,81,90,111\}$. With λ in the set $\{60,81\}$, there exist designs which possess all three considered groups as groups of automorphisms. An overview of t-$(q+1,k,\lambda)$ designs with $PSL(2,q)$ as group of automorphisms and with $(t,k)\in\{(4,5),(4,6),(5,6)\}$ is included.

Keywords: block designs, orbits, projective linear group, projective special linear group, twisted projective linear group, Kramer-Mesner method

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The paper is organized as follows: Introductory notions and constructions are described in Section 1. New designs as the main result of the paper, are presented in Section 2. An exhaustive analysis of derived designs respecting the three groups of automorphisms is included in Section 3. A partition of columns of orbit incidence matrices on the basis of so-called quasi-designs is introduced in Section 4; this partition can be viewed as an addition to the Kramer-Mesner method and has played an essential role in deriving the results. Finally, an overview of some related results can be found in Section 5.

Using notions of the reduced orbit (see 1.4) and of the quasi-design (cf. Section 4) we were able to implement the Kramer-Mesner method efficiently enough to make — with respect to the group $PSL(2,q)$ — an exhaustive search for all prime odd powers q up to $q=47$. Table 7 summarizes results of the research performed.

1. Basic facts related to the constructions

An n-set is a set of cardinality n. Given a group G acting upon a ground-set, an *n-G-orbit* is an orbit of n-subsets of the ground-set, arising from action of G. If G is known, an n-G-orbit will be called just an n-orbit. A t-(v,k,λ) *design* ([5]) is an incidence structure on the v-ground-set, which consists of some k-subsets

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(called *blocks*) of the ground-set, without repetitions, satisfying the property that each t-subset of the ground-set is contained in exactly λ blocks.

A natural idea for design constructions would be using the action of highly homogeneous projective linear groups upon the projective line; it is well known ([5]) that the sets within an h-G-orbit of an h-homogeneous group G are blocks of an h-design with G as a group of automorphisms.

A computer aided construction begins with considering action of the linear group $GL(2,q)$ upon the vector space $V(2,q)$ over the field $GF(q)$. This action is implemented as a multiplication of a 2-dimensional row vector from $V(2,q)$ with a 2×2 matrix from $GL(2,q)$. Next step is to introduce projectivity in this action; this requires replacement of matrices by their representatives of homotethy classes and transition from vectors to their corresponding points on the projective line.

1.1 The Kramer-Mesner method.

The well-known Kramer-Mesner method [15] for constructing t-(v,k,λ) designs with a prescribed group of automorphisms, further denoted G, works as follows:

Let λ_{ij} ([5, p. 185]) denote the number of elements of the j-th k-G-orbit, that contain a fixed arbitrary element of the i-th t-G-orbit, $t < k$. This notion is well-defined, since each t-set of a t-G-orbit is contained in the same number of k-sets on a k-G-orbit.

The matrix (λ_{ij}) will be denoted here as $\Lambda(G;t,k)$; the same matrix was denoted as $A(G;H;t,k)$ in [15] and as $A_{t,k}$ in [16]; it can be called the *orbit incidence matrix* for t-orbits and k-orbits by action of G. If $n(G,i)$ denotes the number of i-G-orbits, then the size of $\Lambda(G;t,k)$ is $n(G,t)\times n(G,k)$. The row sums in $\Lambda(G;t,k)$ are uniform and are equal to $\lambda_{max} = \binom{v-t}{k-t}$.

The key idea of the method is to find a *proper* subset S (if exists) of the columns of $\Lambda(G;t,k)$ with uniform row sums λ. Blocks of the required design are exactly all those k-subsets of the v-ground-set that belong to the k-G-orbits corresponding to columns of S. In other words, a t-(v,k,λ) design with G as a group of automorphisms can be recognized as a *proper* submatrix D of Λ that consists of whole columns and also has uniform row sums λ in all t rows. One can easily conclude by using complementary submatrices that it suffices to search λ for $\lambda \le \frac{1}{2}\cdot\lambda_{max}$.

If G is an h-homogeneous group, then $n(G,h)=1$ and each k-G-orbit ($k>h$) corresponds to an h-(v,k,λ) design. The Kramer-Mesner method allows blocks of t-(v,k,λ) design to be obtained as k-sets belonging to the union of *several* k-G-orbits, instead to a single one. Such an approach opens the possibility for obtaining t-designs with groups of automorphisms G, that have a degree of homogenicity smaller than t.

The essential about the Kramer-Mesner method is that it gives designs with a prescribed group as a group of automorphisms. This follows from the facts that orbits are preserved under action of a group and that the family of design blocks is composed of whole orbits. Note, however, that the prescribed group need not be the *full* automorphism group (see, e.g., [16]).

The following observations should be made:

(1) Columns have to be chosen in such a way that every row has a non-zero entry in at least one of the chosen columns.
(2) Kramer-Mesner method gives *all* designs with a prescribed group of automorphisms.
(3) Therefore, if $H \leq G$ and we know all H-designs, we have a set from which we can choose all G-designs.

1.2 Construction of groups $PSL(2,25)$, $PGL(2,25)$ and $TW(2,25)$.

It is well known that the mappings $x \longrightarrow \frac{ax+b}{cx+d}$, where $a,b,c,d \in GF(q)$ and $ad-bc \neq 0$, constitute the *projective linear group* $PGL(2,q)$.

The *special projective linear group* $PSL(2,q)$ is a subgroup of index 2 of $PGL(2,q)$ and contains all the mappings of $PGL(2,q)$ satisfying that $ad-bc$ is a square. It can be proved by multiplying the coefficients a,b,c,d by suitable factors that the group $PSL(2,q)$ can be equally defined to contain all the mappings of $PGL(2,q)$ that satisfy that $ad-bc=1$ (these definitions lead to the same group).

The *twisted projective linear group* $TW(2,q^2)$ (denoted "twisted $PGL(2,q^2)$" in [5, p. 171]) can merely be defined for odd prime powers of the form q^2 (i.e., which are also squares). A "square" (resp. "non-square") will further be the abbreviation for a mapping of $PGL(2,q)$ with the square (non-square) determinant. The group $TW(2,q^2)$ contains all mappings of the form $x \longrightarrow \frac{ax+b}{cx+d}$, where $ad-bc$ is a non-zero square and all mappings of the form $x \longrightarrow \frac{ax^q+b}{cx^q+d}$, where $ad-bc$ is a non-square. Thus the group $TW(2,q^2)$ also contains the group $PSL(2,q^2)$ as a subgroup of index 2. The "non-square part" of this group is made of compositions of *conjugacy* $x \longrightarrow x^q$ (a special automorphism of $GF(q^2)$) with the projective linear transformation.

The Kramer-Mesner method is applied here to the case $t=4$, $k=6$. Three groups G will be considered: $PSL(2,25)$, $PGL(2,25)$ and $TW(2,25)$. Their common ground-set (projective line) is $\{0,1,\ldots,24\} \cup \{\infty\}$, so $v=26$. When applying the matrices of a projective linear group, points on the projective line are represented by their homogeneous coordinates as row vectors; that is, $x=(x,1)$ for $x \in \{0,1,\ldots,24\}$ and $\infty=(1,0)$.

1.3 Homogenicity of groups $PGL(2,q)$, $TW(2,q^2)$ and $PSL(2,q)$.

Four statements from [11, Beispiel 1.18 c, p. 151, Hilfsatz 6.11, p. 182] and [12, Remark 6.17 b, p. 377, Example 1.3 c, p. 163] can be combined to make the following one:

Proposition 1. *Group $PGL(2,q)$ acts 3-transitively on the ground-set $\Omega(q) = \{0,1,\ldots,q-1\} \cup \{\infty\}$, while $PSL(2,q)$ acts 2-transitively for all odd prime powers q and 3-homogeneously for $q \equiv 3 \pmod 4$. Group $TW(2,q^2)$ acts 3-transitively on the ground-set $\Omega(q) = \{0,1,\ldots,q^2-1\} \cup \{\infty\}$ for all odd prime powers q.*

The same four statements can be found in [5, Lemma 6.6., p. .169, Exercise 6.11, p. 171, Proposition 6.12, p. 171 and Observation 6.13, p. 171].

1.4 Reduced orbits.

Given an h-homogeneous group G ($h < t < k$), the matrix $\Lambda(G;t,k)$ can be computed by using only those t-subsets of the ground-set which contain a fixed h-subset $FS(h)$. In the case of projective linear groups G acting on $\Omega(q)$, one can take $FS(h)$ to be $\{\infty\} \cup \{0,\ldots,h-2\}$.

When constructing k-G-orbits, it suffices to consider only those $\binom{q+1-h}{k-h}$ k-subsets of $\Omega(q)$, which are supersets of $FS(h)$; we call these k-subsets "special". "Special" k-subsets are proportionally distributed among k-G-orbits; the number of "special" k-subsets within a k-G-orbit is obtained by multiplying the total number of its k-subsets by $\binom{q+1-h}{k-h}/\binom{q+1}{k}$; the result of this multiplication must be an integer. "Special" k-subsets within a k-G-orbit constitute a *reduced k-G-orbit*. An analogous reduction is applied to t-G-orbits.

Reduced k-G-orbits are constructed by applying elements of G to their representative k-subsets; the image k-subsets are recorded iff they are "special".

Reduced t-G-orbits and reduced k-G-orbits are sufficient for construction of the matrix $\Lambda(G;t,k)$, since the set-inclusion preserves "speciality"; that is, all k-supersets of a "special" t-subset are "special" k-subsets.

Table 1 includes some data related to the groups considered in this paper and their reduced orbits. Besides the names of groups and necessary parameters, the table contains the total number of "special" subsets, as well as the reduction factors (the quotients of binomial coefficients cited above) obtained when the transition from all subsets to "special" is performed.

Group		h	number of "special" subsets	reduction factor
$PSL(2,25)$	$k=6$	2	10626	65/3
$PSL(2,25)$	$t=4$	2	276	325/6
$PGL(2,25)$ and $TW(2,25)$	$k=6$	3	1771	130
$PGL(2,25)$ and $TW(2,25)$	$t=4$	3	23	650

Table 1.

Thus $\binom{24}{4} = 10626$ "special" 6-subsets in the 2-homogeneous case constitute only $\frac{3}{65}$ of all $\binom{26}{6}$ 6-subsets of the 26-ground-set.

Note that transition from the 2-homogeneous $PSL(2,25)$ to the other two 3-homogeneous groups increases the reduction factor by factor 6 in the case of 6-subsets and by factor 12 in the case of 4-subsets. This increase of reduction factor is useful for constructions of reduced orbits. However, it turns out that the main computational advantage of using $PGL(2,25)$ and $TW(2,25)$, in comparison with when using $PSL(2,25)$, is due to the *smaller number* of orbits.

2. Designs

The main result of this paper reads:

Theorem 1. *There exist* 4-$(26, 6, \lambda)$ *designs with* $PSL(2, 25)$ *as a group of automorphisms and with each* λ *in the set* $\{30, 51, 60, 81, 90, 111\}$.

Proof: The proof will be given by exhibiting six 4-$(26, 6, \lambda)$ designs with $PSL(2, 25)$ as a group of automorphisms and with the six values of λ above, accompanied with data necessary to document the constructed designs. These data include:

(a) data for identification of 4- and 6-orbits under action of $PSL(2, 25)$; (Tables 2, 3)
(b) matrix $\Lambda(PSL(2, 25); 4, 6)$; (Table 4)
(c) column combinations (sets of columns) of $\Lambda(PSL(2, 25); 4, 6)$ corresponding to the designs.

Throughout this section, "n-orbits" will be an abbreviation for n-G-orbits, where $G = PSL(2, 25)$. It turns out that there are 7 4-orbits and 45 6-orbits. In accordance with discussion in Subsection 1.4, 2-homogenicity of the group $PSL(2, 25)$ ([5]) enables the representatives of all 4-orbits and 6-orbits to be "special" supersets of a fixed 2-set, say $\{0, \infty\}$.

In order to enable identification of 4-orbits and 6-orbits, associated with rows and columns of the matrices, the following data will be given in Tables 2 and 3:

— the ordinal number of an orbit, which is associated to the corresponding row (column) of the matrix $\Lambda(PSL(2, 25); 4, 6)$;
— the elements of the lexicographically the first "special" representative, apart from the compulsory elements 0 and ∞;
— the number of "special" subsets within the orbit.

Example. The denotations $\boxed{2 \mid 1\ \ 5 \mid 72}$ in Table 2 and $\boxed{10 \mid 1\ \ 2\ \ 5\ \ 13 \mid 60}$ in Table 3 mean that the 2nd 4-orbit contains the representative $\{0, 1, 5, \infty\}$ and the total of 72 "special" 4-subsets, while the 10th 6-orbit contains the representative $\{0, 1, 2, 5, 13, \infty\}$ and the total of 60 "special" 6-subsets.

1	1 2	18	2	1 5	72	3	1 6	72	4	1 7	72
5	1 8	12	6	5 10	18	7	5 17	12			

Table 2. Data describing 4-orbits of $PSL(2, 25)$

1	1	2	3	4	3	2	1	2	3	5	360	3	1	2	5	6	360
4	1	2	5	7	360	5	1	2	5	8	360	6	1	2	5	9	360
7	1	2	5	10	360	8	1	2	5	11	360	9	1	2	5	12	360
10	1	2	5	13	60	11	1	2	5	15	360	12	1	2	5	16	180
13	1	2	5	17	360	14	1	2	5	18	180	15	1	2	5	19	180
16	1	2	5	20	180	17	1	2	5	21	360	18	1	2	5	22	180
19	1	2	5	23	180	20	1	2	5	24	360	21	1	2	6	8	180
22	1	2	6	9	180	23	1	2	6	10	360	24	1	2	6	11	90
25	1	2	6	12	360	26	1	2	6	16	180	27	1	2	6	21	180
28	1	2	6	23	180	29	1	2	10	12	90	30	1	2	10	15	90
31	1	5	7	10	360	32	1	5	7	11	360	33	1	5	7	12	120
34	1	5	7	19	180	35	1	5	8	12	360	36	1	5	9	10	360
37	1	5	9	17	180	38	1	5	10	11	90	39	1	5	10	15	360
40	1	5	11	15	360	41	1	5	11	17	180	42	1	5	15	18	180
43	1	5	15	22	60	44	1	6	7	15	120	45	5	10	15	20	3

Table 3. Data describing 6-orbits of $PSL(2, 25)$

The existence of a 4-$(26, 6, \lambda)$ design will be proved in each particular case by exhibiting a proper subset P of the column-set (a combination of columns) of matrix $\Lambda(PSL(2, 25); 4, 6)$, which satisfies that the sum of elements of any row within the columns of P is equal to λ.

Let C denote the set of ordinal numbers of those columns of $\Lambda(PSL(2, 25); 4, 6)$ that constitute required proper subsets P. Six possible sets C are listed below, for λ in the set $\{30, 51, 60, 81, 90, 111\}$:

$$\lambda = 30: \quad C = \{4, 14, 21, 24, 28, 30, 37, 44\};$$
$$\lambda = 51: \quad C = \{1, 9, 10, 16, 18, 23, 24, 26, 28, 29, 30, 33, 37, 38, 43, 44, 45\};$$
$$\lambda = 60: \quad C = \{4, 7, 12, 14, 16, 23, 25, 26, 28, 33, 34, 44\};$$
$$\lambda = 81: \quad C = \{1, 3, 10, 11, 12, 14, 15, 19, 21, 25, 26, 28, 29, 36, 38, 40, 41, 42, 43, 45\};$$
$$\lambda = 90: \quad C = \{5, 7, 8, 9, 11, 18, 19, 20, 23, 24, 26, 27, 30, 31, 32\};$$
$$\lambda = 111: \quad C = \{1, 3, 4, 6, 10, 11, 12, 13, 14, 20, 21, 22, 24, 27, 28, 29, 30, 37, 38, 39, 40, 41, 42, 43, 44, 45\}.$$

□

The existence of the constructed six new 4-designs implies the existence of the accompanying 3-$(26, 6, \lambda)$ designs with $\lambda \in \{230, 391, 460, 621, 690, 851\}$ and 2-$(26, 6, \lambda)$ designs with $\lambda \in \{1380, 2346, 2760, 3726, 4140, 5106\}$. According to data contained in [6], it turns out that these twelve designs are also new.

1	1	0	0	0	0	0	0
2	40	6	6	6	12	0	0
3	16	8	8	8	12	0	0
4	8	8	8	10	0	8	0
5	8	8	8	6	12	8	12
6	8	10	4	10	24	0	0
7	8	6	12	8	0	8	0
8	8	8	8	6	12	8	12
9	8	8	8	10	0	8	0
10	4	1	1	2	0	0	0
11	8	12	6	8	0	8	0
12	8	5	2	5	6	0	0
13	8	10	10	6	0	0	12
14	4	3	3	4	0	8	12
15	4	5	2	6	0	4	0
16	4	2	2	6	0	4	18
17	8	4	10	10	24	0	0
18	4	2	5	6	0	4	0
19	8	2	5	5	6	0	0
20	16	8	8	8	12	0	0
21	4	6	3	3	12	0	0
22	4	3	6	3	12	0	0
23	8	10	10	2	12	8	12
24	4	2	2	1	6	2	0
25	8	10	10	2	12	8	12
26	4	2	2	6	18	4	0
27	4	2	2	5	12	4	12
28	8	3	3	4	12	4	0
29	4	3	3	0	3	0	0
30	2	2	2	1	0	4	6
31	0	8	8	8	0	16	12
32	0	10	4	10	0	8	24
33	0	6	0	4	0	0	0
34	0	5	2	5	0	8	6
35	0	10	10	6	12	8	0
36	0	8	8	8	0	16	12
37	0	6	3	3	0	4	12
38	0	3	3	0	0	4	3
39	0	6	6	6	0	40	12
40	0	4	10	10	0	8	24
41	0	3	6	3	0	4	12
42	0	2	5	5	0	8	6
43	0	1	1	2	0	4	0
44	0	0	6	4	0	0	0
45	0	0	0	0	0	1	0

Table 4. 7×45 matrix $\Lambda(PSL(2,25);4,6)$
(columns 1, ... , 45 are listed as row vectors)

We have checked that the matrices $\Lambda(PSL(2,25);4,5)$ and $\Lambda(PSL(2,25);5,6)$ have no proper submatrices with uniform row sums. Consequently, one cannot obtain 4-$(26,5,\lambda)$ or 5-$(26,6,\lambda)$ designs with $G = PSL(2,25)$ as group of automorphisms by using the Kramer-Mesner method.

A more detailed classification of the constructed designs w.r.t. the three groups of automorphisms they might possess will be given in Section 3.

3. About orbits and designs arising from three groups

Actions of groups $PGL(2,25)$, $PSL(2,25)$ and $TW(2,25)$ will be compared in this section. Some general considerations that are concerned with inclusion relationships of the orbits of groups $PSL(2,q)$, $PGL(2,q)$ and $TW(2,q^2)$ will be primarily established in subsection 3.1. These relationships will be further applied in subsection 3.2 to the cases $q = 25$, $q = 25$ and $q^2 = 25$, respectively.

When considering existence and uniqueness of designs, one can use the following hierarchy:

parameter level — designs determined up to the parameters

This term is most generally known. The existence question for designs corresponding to the quadruples t-(v, k, λ) is by far the most interesting one.

isomorphism level — designs determined up to an isomorphism

This term is related to the design enumeration problem, which has been solved with a very small number of the known design parameters.

column combination level —

This term is a speciality of the Kramer-Mesner method. Each design obtained with this method corresponds to a column combination with uniform row sums λ within the orbit incidence matrix.

Parameters of a design that corresponds to a column combination are immediately known. However, the isomorphism question for some two designs with the same parameters, that correspond to some two distinct column combinations, remains a hard one.

In this section we shall obtain some results pertaining to the column combination method. For example, Table 6 contains a number of constructed designs with each one belonging to the considered three groups of automorphisms.

3.1 Relationships among the orbits.

By a *PSL-orbit* (resp. *PGL-orbit*) we shall mean a k-orbit under action of $PSL(2, q)$ (resp. $PGL(2, q)$) and by a *TW-orbit* we shall mean a k-orbit under action of $TW(2, q^2)$ for an odd prime power q.

By a *PSL-design* (resp. *PGL-design*) we shall understand a set of PSL-orbits (PGL-orbits) that corresponds to a t-$(q+1, k, \lambda)$ design (blocks of the design are exactly all the k-subsets that belong to the union of orbits of the set) and by a *TW-design* we shall understand a set of TW-orbits corresponding to a t-(q^2+1, k, λ) design.

Remark. The relationships established for PSL- and PGL-orbits, also for PSL- and PGL-designs are valid for any prime power q. However, whenever TW-orbits and TW-designs take part in a discussion, it is restricted to odd prime powers q^2; groups $PGL(2, q^2)$ and $PSL(2, q^2)$ have to be considered in these cases.

A general lemma for orbits of any size will first be proved:

Lemma 1. *If H is a subgroup of index k of a group G, then a G-orbit includes at most k H-orbits.*

Proof: Let G act upon Ω and choose $\alpha \in \Omega$. If $g, h \in G$ map α to different H-orbits, then $Hg \neq Hh$. Hence the number of H-orbits within a G-orbit cannot exceed the number of right cosets of H in G. $\square$

The group $PSL(2, q)$ is a subgroup of index 2 for $PGL(2, q)$ of prime power q and the group $PSL(2, q^2)$ is a subgroup of index 2 for the group $TW(2, q^2)$, for each odd prime power q. By using Lemma 1, these two facts immediately imply the following:

Lemma 2. *Each PGL-orbit, also each TW-orbit, consists of either one or two PSL-orbits.*

We cite three useful statements that follow from general properties of Kramer-Mesner method:

Statement 1. Each PGL-design, also each TW-design, is a PSL-design.

Statement 2. All designs that have $PGL(2,q)$ or $TW(2,q^2)$ as a group of automorphisms can be obtained by applying the Kramer-Mesner method to the group $PSL(2,q)$, respectively $PSL(2,q^2)$.

Statement 3. The application of the Kramer-Mesner method to the group $PSL(2,q)$ produces 3-designs for all prime powers q.

Let $PSL(X)$, $PGL(X)$ and $TW(X)$ respectively denote the PSL-orbit, PGL-orbit and TW-orbit determined by a set X. Further, let $S(X)$ and $N(X)$ respectively denote the set of all images of set X under (bilinear) squares, respectively non-squares and let X^q denote the conjugacy image of set X. Then obviously $PSL(X) = S(X)$; $PGL(X) = S(X) \cup N(X)$ and $TW(X) = S(X) \cup N(X^q)$.

The next lemma is essential for a classification of possible relationships of the three groups:

Lemma 3. *Both $N(X)$ and $N(X^q)$ are PSL-orbits.*

Proof: The proof will be given for $N(X)$ within $PGL(X)$, the proof for $N(X^q)$ within $TW(X)$ being analogous; observe that q should be replaced by odd q^2 in the second case.

$S(X) \cup N(X)$ is the PGL-orbit of X. If there are a square s and a non-square n with $s(X) = n(X)$, then $S(X) = PSL(2,q)s(X) = PSL(2,q)n(X) = N(X)$. If $S(X) \cap N(X)$ is empty, then $PGL(2,q)X \setminus PSL(2,q)X = (S(X) \cup N(X)) \setminus S(X) = N(X)$ is the second PSL-orbit contained in the PGL-orbit of X. □

Corollary 1. *If a k-subset Y belongs to $PGL(X)$ and $PGL(X) = PSL(X)$, then exactly one half of mappings from $PGL(X)$, which map X onto Y, are squares; non-squares constitute the other half.*

Corollary 2. *If a PGL-orbit consists of two PSL-orbits, then these two orbits have the same cardinality.*

We are now able to prove the following

Lemma 4. *Given an odd prime power q and a k-subset X of the ground-set, there exist five possible relationships for the orbits $PSL(X)$, $PGL(X)$ and $TW(X)$:*

(a) $PSL(X) = PGL(X) = TW(X)$; (b) $PSL(X) = PGL(X) \neq TW(X)$;

(c) $PSL(X) = TW(X) \neq PGL(X)$; (d) $PSL(X) \neq PGL(X) = TW(X)$;

(e) $PSL(X) = PGL(X) \cap TW(X)$ *and both sets* $PGL(X) \setminus PSL(X)$ *and* $TW(X) \setminus PSL(X)$ *are non-empty.*

Proof: There exist five possible relationships among the PSL-orbits $PSL(X)$, $N(X)$ and $N(X^q)$: they can be all equal, two equal and third different (three possibilities), all three pairwise different. The cases (a)–(e) follow from the relationships:

$(S(X) = N(X) = N(X^q))$ (Case (a)), $(S(X) = N(X) \neq N(X^q))$ (Case (b)),

$(S(X) = N(X^q) \neq N(X))$ (Case (c)), $(S(X) \neq N(X) = N(X^q))$ (Case (d)),

$(S(X) \neq N(X))$ and $(S(X) \neq N(X^q))$ and $(N(X) \neq N(X^q))$ (Case (e)); in this case $PGL(X) \backslash PSL(X) = N(X)$ and $TW(X) \backslash PSL(X) = N(X^q)$ are the further two PSL-orbits. □

3.2 6-orbits over $GF(25)$ and related 4-$(26, 6, \lambda)$ designs.

The meanings of denotations “PSL-orbits”, “PGL-orbits” and “TW-orbits” in this subsection will be restricted to 6-orbits (under action) of $PSL(2, 25)$, $PGL(2, 25)$ and $TW(2, 25)$, respectively. The 45 6-orbits of $PSL(2, 25)$ are included into 28 6-orbits of $PGL(2, 25)$ and 23 6-orbits of $TW(2, 25)$. Table 5 contains data about the inclusion relationships of 6-orbits of these three groups.

The 6-orbits under action of $PSL(2, 25)$, $PGL(2, 25)$ and $TW(2, 25)$ are numerated by natural numbers $1, 2, \ldots, 45$; $1, 2, \ldots, 28$ and $1, 2, \ldots, 23$, respectively.

When comparing different kinds of orbits, it is suitable to use denotations $PSL/n/$, $PGL/n/$ and $TW/n/$ for single orbits, as well as $PSL/n_1, n_2/$ and $PGL/n_1, n_2/$ for unions of two orbits where n, n_1, n_2 are the associated natural numbers. Using the orbit inclusion map given in Table 5, one can recognize that four of the Cases (a)–(e) of Lemma 4 are present with $q^2 = 25$; there exist four different inclusion relationships among 6-orbits of groups $PSL(2, 25)$, $PGL(2, 25)$ and $TW(2, 25)$.

PSL	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
PGL	1	2	3	4	5	6	7	8	4	9	10	11	12	13	14	15	16	17	18	19	20	21	22
TW	1	2	3	4	5	6	7	5	4	8	7	9	10	11	12	13	14	12	15	16	17	18	19

PSL	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
PGL	23	24	15	25	13	26	23	3	6	27	11	12	19	20	26	2	16	21	18	9	28	1
TW	20	19	13	21	11	22	20	16	14	23	15	10	3	18	22	2	6	17	9	8	23	1

Table 5. Inclusion map of orbits of the three groups

A detailed list of orbit relationships is given in the sequel; denotations of cases are in accordance with Lemma 4:

(a) PSL-orbit = PGL-orbit = TW-orbit

$PSL/27/ = PGL/25/ = TW/21/$;

(b) TW-orbit = two PGL-orbits that coincide with PSL-orbits

$TW/5/ = PGL/5, 8/ = PSL/5, 8/$; $TW/7/ = PGL/7, 10/ = PSL/7, 11/$;
$TW/12/ = PGL/14, 17/ = PSL/15, 18/$;

$TW/19/ = PGL/22, 24/ = PSL/23, 25/;$
$TW/23/ = PGL/27, 28/ = PSL/33, 44/;$

(d) PGL-orbit = TW-orbit = two PSL-orbits

$PGL/1/ = TW/1/ = PSL/1, 45/;$ $PGL/2/ = TW/2/ = PSL/2, 39/;$
$PGL/4/ = TW/4/ = PSL/4, 9/;$ $PGL/9/ = TW/8/ = PSL/10, 43/;$
$PGL/12/ = TW/10/ = PSL/13, 35/;$ $PGL/13/ = TW/11/ = PSL/14, 28/;$
$PGL/15/ = TW/13/ = PSL/16, 26/;$ $PGL/23/ = TW/20/ = PSL/24, 30/;$
$PGL/26/ = TW/22/ = PSL/29, 38/;$

(e) Two PGL-orbits = four PSL-orbits = two TW-orbits

$PGL/3/ = PSL/3, 31/; PGL/19/ = PSL/20, 36/;$
$TW/3/ = PSL/3, 36/; TW/16/ = PSL/20, 31/;$

$PGL/6/ = PSL/6, 32/; PGL/16/ = PSL/17, 40/;$
$TW/6/ = PSL/6, 40/; TW/14/ = PSL/17, 32/;$

$PGL/20/ = PSL/21, 37/; PGL/21/ = PSL/22, 41/;$
$TW/17/ = PSL/21, 41/; TW/18/ = PSL/22, 37/;$

$PGL/11/ = PSL/12, 34/; PGL/18/ = PSL/19, 42/;$
$TW/9/ = PSL/12, 42/; TW/15/ = PSL/19, 34/;$

Remark. The case (c) from Lemma 4.2 cannot be found with $q^2 = 25$.

As a consequence of these inclusion relationships for the orbits, one can derive three propositions given in the sequel. Proposition 2 gives the conditions for a TW-design to be a PGL-design, Proposition 3 gives the conditions for a PGL-design to be a TW-design, whilst Proposition 4 gives the conditions for a PSL-design to be either a TW-design, or a PGL-design, or both.

Proposition 2. *If TW-orbits determining a* 4-$(26, 6, \lambda)$ *TW-design do not include exactly one of the two orbits in some of the 2-subsets* $\{3, 16\}$, $\{6, 14\}$, $\{9, 15\}$ *and* $\{17, 18\}$ *of TW-orbits, then this design is also a PGL-design.*

Proof: The cited 2-subsets of orbits of $TW(2, 25)$ correspond to the above stated Case (e). The union of two orbits in each one of these 2-subsets is equal to the union of some 2-subsets of orbits of $PGL(2, 25)$; however, none of the TW-orbits in the 2-subset is itself a PGL-orbit. On the other hand, TW-orbits in Cases (a), (b) and (d) are also PGL-orbits. □

Proposition 3. *If PGL-orbits determining a* 4-$(26, 6, \lambda)$ *PGL-design do not include exactly one of the two orbits in some of the 2-subsets*
$\{5, 8\}$, $\{7, 10\}$, $\{14, 17\}$, $\{22, 24\}$, $\{27, 28\}$, $\{3, 19\}$, $\{6, 16\}$, $\{11, 18\}$ *and* $\{20, 21\}$ *of PGL-orbits, then this design is also a TW-design.*

Proof: The first five cited 2-subsets of orbits of $PGL(2, 25)$ correspond to Case (c), while the last four correspond to Case (d). None of the PGL-orbits in any of these 2-subsets is itself a TW-orbit. □

These two propositions imply that there exist only two column combinations (for $\lambda = 60$ and $\lambda = 81$) that correspond to both PGL- and TW-designs.

The third statement in this series reads:

Proposition 4. *If PSL-orbits determining a* 4-$(26, 6, \lambda)$ *PSL-design include exactly one of the two orbits in some of the following* 2*-subsets of PSL-orbits:*

- **I.** $\{1, 45\}$, $\{2, 39\}$, $\{4, 9\}$, $\{10, 43\}$, $\{13, 35\}$, $\{14, 28\}$, $\{16, 26\}$, $\{24, 30\}$ *and* $\{29, 38\}$;
- **II.** $\{5, 8\}$, $\{7, 11\}$, $\{15, 18\}$, $\{23, 25\}$, $\{33, 44\}$, $\{3, 36\}$, $\{20, 31\}$, $\{6, 40\}$, $\{17, 32\}$, $\{21, 41\}$, $\{22, 37\}$, $\{12, 42\}$ *and* $\{19, 34\}$;
- **III.** $\{3, 31\}$, $\{20, 36\}$, $\{6, 32\}$, $\{17, 40\}$, $\{21, 37\}$, $\{22, 41\}$, $\{12, 34\}$ *and* $\{19, 42\}$;

then

- **I.** *the design is neither a PGL-design, nor a TW-design*
- **II.** *the design is not a TW-design*
- **III.** *the design is not a PGL-design.*

PROOF: Follows from 2-subsets of PSL-orbits mentioned in Cases: (d) (part I), (b) and (e) (part II), (e) (part III). □

Table 6 summarizes the results of our design enumeration:

λ	$TW \cap PGL$ designs	$TW \setminus PGL$ designs	$PGL \setminus TW$ designs	only PSL designs	total # of designs
30	0	0	0	16	16
51	0	8	4	344	356
60	1	0	28	696	725
81	1	4	54	6024	6083
90	0	22	48	5016	5086
111	0	4	90	10074	10168

Table 6. Number of found 4-$(26, 6, \lambda)$ designs with specific groups as groups of automorphisms

Each one of the PSL-designs listed in Section 2 is neither a PGL- nor a TW-design; in particular, the same holds for all designs with $\lambda = 30$ that we have found.

Each one of non-zero numbers of designs in Table 6 will be illustrated by an example. In order to prove that the exhibited designs do not belong to some of the considered classes of designs, a critical pair of orbits is exhibited in each particular case; with only one of two orbits within a pair being used by a design.

According to Proposition 4/I, six PSL-designs listed in Section 2 are neither PGL-designs, nor TW-designs. It suffices to consider the 2-subsets $(4, 9)$ and $(16, 26)$ of this statement. These six PSL-designs include the PSL-orbits numerated by 4, 9, 4, 26, 9, 4, but do not include the PSL-orbits numerated by 9, 4, 9, 16, 4, 9, respectively.

Let PGL$[x_1, \dots, x_k]$ denote a PGL-design composed of PGL-orbits numerated by $x_1, \dots, x_k$. The analogous denotation is used for TW-designs and PSL-designs. However, when PSL-designs are considered, round brackets containing one or two PSL-orbits are also used; PSL-orbit(s) within a round bracket corresponds (correspond) to a single TW-orbit, respectively single PGL-orbit.

Designs that are both TW-designs and PGL-designs

PSL-orbits are being combined in two different ways in order to show that each one of the considered two PSL-designs is both a TW-design and a PGL-design.

$\lambda = 60$:
TW[3,5,11,16,23] = PSL[(3,36)(5,8)(14,28)(20,31)(33,44)]
PGL[3,5,8,13,19,27,28] = PSL[(3,31)(5)(8)(14,28)(20,36)(33)(44)];

$\lambda = 81$:
TW[1,5,7,8,10,11,12,13,21,22] =
PSL[(1,45)(5,8)(7,11)(10,43)(13,35)(14,28)(15,18)(16,26)(27)(29,38)]
PGL[1,5,7,8,9,10,12,13,14,15,17,25,26] =
PSL[(1,45)(5)(7)(8)(10,43)(11)(13,35)(14,28)(15)(16,26)(18)(27)(29,38)].

TW-designs that are not PGL-designs

Two arguments, based on Proposition 2, respectively on Proposition 4/III, are provided in order to explain why each one of the four exhibited TW-designs is not a PGL-design.

$\lambda = 51$; TW[1,12,15,16,17,20,21,22] =
PSL[(1,45)(15,18)(19,34)(20,31)(21,41)(24,30)(27)(29,38)]
$TW/16/$ without $TW/3/$; $PSL/31/$ without $PSL/3/$;

$\lambda = 81$; TW[1,3,5,8,9,12,16,18,21,22] =
PSL[(1,45)(3,36)(5,8)(10,43)(12,42)(15,18)(20,31)(22,37)(27)(29,38)]
$TW/18/$ without $TW/17/$; $PSL/22/$ without $PSL/41/$;

$\lambda = 90$; TW[4,8,9,11,13,16,17,19,20,23] =
PSL[(4,9)(10,43)(12,42)(14,28)(16,26)(20,31)(21,41)(23,25)(24,30)(33,44)]
$TW/17/$ without $TW/18/$; $PSL/41/$ without $PSL/22/$;

$\lambda = 111$; TW[1,3,4,5,9,12,13,15,19,20,21,22,23] =
PSL[(1,45)(3,36)(4,9)(5,8)(12,42)(15,18)(16,26)(19,34)(23,25)(24,30)(27)
(29,38)(33,44)]
$TW/3/$ without $TW/16/$; $PSL/3/$ without $PSL/31/$.

PGL-designs that are not TW-designs

Two arguments, based on Proposition 3, respectively on Proposition 4/II, are provided in order to explain why each one of the five exhibited PGL-designs is not a TW-design.

$\lambda = 51$: PGL[1,14,17,18,19,20,23,25,26] =
PSL[(1,45)(15)(18)(19,42)(20,36)(21,37)(24,30)(27)(29,38)]
$PGL/19/$ without $PGL/3/$; $PSL/20/$ without $PSL/31/$;

$\lambda = 60$: PGL[3,5,9,13,19,20,28] =
PSL[(3,31)(5)(10,43)(14,28)(20,36)(21,37)(44)]
$PGL/5/$ without $PGL/8/$; $PSL/5/$ without $PSL/8/$;

$\lambda = 81$: PGL[1,3,7,10,13,14,15,17,21,24,25,26,27] =
PSL[(1,45)(3,31)(7)(11)(14,28)(15)(16,26)(18)(22,41)(25)(27)(29,38)(33)]
$PGL/24/$ without $PGL/22/$; $PSL/25/$ without $PSL/23/$;

$\lambda = 90$: PGL[4,9,13,15,18,19,20,22,23,24,27,28] =
PSL[(4,9)(10,43)(14,28)(16,26)(19,42),(20,36)(21,37)(23)(24,30)(25)(33)(44)]
$PGL/19/$ without $PGL/3/$; $PSL/20/$ without $PSL/31/$;

$\lambda = 111$: PGL[1,4,7,9,11,13,14,15,17,18,20,21,22,23,24,25,26,27] =
PSL[(1,45)(4,9)(7)(10,43)(12,34)(14,28)(15)(16,26)(18)(19,42)
(21,37)(22,41)(23)(24,30)(25)(27)(29,38)(33)]
$PGL/7/$ without $PGL/10/$; $PSL/7/$ without $PSL/11/$.

Open problem 1. Does there exist a k-$PGL(2, q^2)$-orbit for some odd prime power q and some natural number k, that includes two k-$TW(2, q^2)$-orbits ? It would correspond to Case (c) of Lemma 4, the only one of the five cases that does not exist with $q = 5$ and $k = 6$.

Open problem 2. Test isomorphism of the constructed designs within the class of designs with a fixed λ (it is hoped that Table 6 could be of use with such a testing).

4. An addition to the Kramer-Mesner method

This section describes a method for the partitioning of the column-set of $\Lambda(PSL(2, 25); 4, 6)$, based on the notion of so-called "quasi-designs". This method has played an essential role in making the computational time (necessary for full search) feasible. After the application of this method, we are sure about the following fact:

- The only new λ-value that can be obtained with 4-$(26, 6, \lambda)$ designs by using the Kramer-Mesner method, after the groups $PGL(2, 25)$ and $TW(2, 25)$ are replaced by $PSL(2, 25)$ — is $\lambda = 30$.

If the number of k-G-orbits for some group G and some k is equal to 30, then a brute-force search over 2^{29} column combinations, based on Gray code, would last about a week using PC-386; with complementation, it is allowed to exclude one column from the search. Thirty columns seem to be an approximate upper bound for a reasonable brute-force search. Since $n(PGL(2, 25), 6) = 28$ and $n(TW(2, 25), 6) = 23$, a brute force search for submatrices D has been applied. However, the 7×45 matrix $\Lambda = \Lambda(PSL(2, 25); 4, 6)$ has been treated by a specific approach, going to be described. Columns of this matrix Λ will be shortly denoted by numbers $1, \ldots, 45$.

Note that the necessary condition for design existence ([5]): "$\lambda \cdot \binom{v-i}{t-i} \cdot \binom{k-i}{t-i}^{-1}$ must be an integer for $0 \le i < t$", — implies that λ must be divisible by 3; the same conclusion can be easily derived from 7th row of Λ.

The first observation is that any of considered designs must use either both columns in pairs (1,45), (29,38) and (24,30), or none; the first two pairs are used for λ odd and the last one for $\lambda \equiv 2, 3 \pmod 4$. This gives possibility to reduce the number of columns in Λ to 42. In what follows, Λ will denote the 7×42 matrix with columns denoted as before, with exception that the columns 30, 38 and 45 are excluded, while the columns 1, 24 and 29 are replaced by sums of two columns in the corresponding pairs: $1 \longleftarrow 1+45$; $24 \longleftarrow 24+30$; $29 \longleftarrow 29+38$.

It can be noticed that the pairs of 1st and 6th 4-orbit, as well as of 5th and 7th 4-orbit play outstanding roles in the matrix Λ; these two pairs of 4-orbits of $PSL(2,25)$ arise by fusion of two 4-orbits of $PGL(2,25)$. A useful partitioning of columns of Λ is based on this notice. Some special notions are introduced for this purpose:

A general column $c = (c_1, \ldots, c_7)$ of Λ is said to have $(1=6;\ 5=7)$-*property* if $(c_1 = c_6)$ and $(c_5 = c_7)$. Similarly, a set S of columns of Λ is said to have $(1=6;\ 5=7)$-*property* if $(s_1 = s_6)$ and $(s_5 = s_7)$, where $s_i \stackrel{\text{def}}{=} \sum_{c \in S} c_i$.

Let B denote the submatrix of Λ consisted of all those (whole) columns that satisfy (1=6; 5=7)-property and let A denote the complementary submatrix of Λ. It can be easily checked that the columns of B are 4, 5, 7, 8, 9, 11, 15, 18, 23, 25, 27, 33, 44, 1, 24, 29 (the last three being sums of two original columns), while A contains the remaining 26 columns of Λ. A *quasi-design of* Λ is defined to be a subset of columns of A that has (1=6; 5=7)-property.

The following lemma claims that a necessary condition for the combination of columns of Λ corresponding to a design — is to include a quasi-design of Λ:

Lemma 5. *Let D be a combination of columns of Λ corresponding to a* 4-$(26, 6, \lambda)$ *design. Then $D \cap A$ is a quasi-design of Λ.*

PROOF: Sums $s_{D \cap A}(i)$ of entries in rows of $D \cap A$ are obtained by substracting the sums of the corresponding rows of $D \cap B$ from $\lambda_{max} = 231$. The conclusion follows from the fact that each column of $D \cap B$ has (1=6; 5=7)-property. $\square$

This lemma enables the search for designs D to be broken into two independent stages. A brute-force search for quasi-designs $D \cap A$ of Λ is primarily performed over 2^{25} combinations of columns of submatrix A (one column may be excluded again). It turns out that there are only 175072 quasi-designs of Λ among all these combinations.

Next reduction is achieved by partitioning quasi-designs D of Λ with respect to the difference $dif = s_{D \cap A}(5) - s_{D \cap A}(6)$. Since all entries in A are even, these differences are also even; their range is $[-40, +62]$. The condition of uniform row sums within D implies that the difference $s_{D \cap B}(5) - s_{D \cap B}(6)$ has to be equal to $-dif$. Therefore, the 2^{16} combinations of columns of submatrix B are also partitioned w.r.t. differences of sums of entries in 5th and 6th row. The combinations with odd differences (one half of the total number) are immediately discarded. The range of even differences is $[-42, +24]$. It turns out that partitioning w.r.t. differences dif (resp. $-dif$) reduces the total number of candidates for D from $175072 \cdot 32768 = 5.736.759.296$ to only 209.195.552.

Each candidate for D has uniform row sums λ on 1st, 5th, 6th and 7th row. Three successive tests are used to check whether the sums in the remaining three rows are also equal to λ; if this is the case, then a 4-$(26, 6, \lambda)$ design is recorded.

Using this approach, we obtained that there exist 22434 combinations of columns of $\Lambda(PSL(2, 25); 4, 6)$ that correspond to 4-$(26, 6, \lambda)$ designs. A more detailed classification of these combinations respecting the three groups of automorphisms was given in Table 6, Section 3.

5. An overview of related results

Using the Kramer-Mesner method, we have found some new t-$(q+1, k, \lambda)$ designs with $PSL(2, q)$ as a group of automorphisms and have reconstructed some old ones. Here is the list of these values:

q	t	k	λ_{max}	found λ values $\leq \lambda_{max}/2$
11	5	6	7	1, 2
17	4	5	14	4
19	4	6	120	60
23	5	6	19	1, 2, 3, 4, 5, 6, 7, 8, 9
25	4	6	231	30, 51, 60, 81, 90, 111
27	5	6	23	2, 3, 4, 5, 6, 7, 8, 9, 10, 11
31	5	6	27	6, 12
32	4	5	29	4, 5, 9
37	4	5	34	6, 10, 12, 16
47	4	5	44	8, 12, 16, 20

Table 7. Some data on design parameters related to action of $PSL(2, q)$

The designs with $q = 11$ and $q = 23$ are related to the well-known Steiner systems 5-$(12, 6, 1)$ and 5-$(24, 6, 1)$ ([5]). The first one of these Steiner systems is, as stated in [10, Theorem 2.26], the uniquely determined Steiner system with the automorphism group isomorphic to the famous Mathieu 5-transitive group M_{12}. The design with $q = 17$ is due to Alltop and was described in [5, Example 8.5, pp. 186-187]. The designs with $q = 19$ and $q = 27$ were constructed in [14] and [17], respectively. The designs with $q = 32$, described in [2], are related [13] to the 4-homogeneous group $PGamaL(2, 32)$; moreover, the 4-$(33, 5, 5)$ design is the first member of an infinite class of 4-designs ([8]).

The designs with $q \in \{31, 37\}$ were justified in multiple communications to be new results of our investigations ([3], [11]). The designs with $q = 47$ ([7]) might be related to the Steiner system 5-$(48, 6, 1)$ ([8]).

It was stated in [13] that no 4-designs on 38 points were known. A 4-$(38,5,16)$ design with $PGL(2,37)$ as a group of automorphisms was described in [1]; later on, we realized that the 2-homogeneous group $PSL(2,37)$ gives 4-$(38,5,\lambda)$ designs with three new values of λ.

When $q=25$ is considered, the extensive table of known design parameters in [6] claims that the question of existence of 4-$(26,6,3s)$ designs is open for $1 \le s \le 38$; this paper solves this question to affirmative for $s \in \{10,17,20,27,30,37\}$.

Moreover, data from the same table for [6] say that the only up-to-date known 4-$(26,k,\lambda)$ design was with $k=13$ and $\lambda=84700$; this design is a consequence of the 5-$(24,11,4620)$ design found in [9].

Results of our computer search can be summarized to the following:

Proposition 5. *Design parameters listed in Table 7 are the only ones that can be derived by the Kramer-Mesner method from $\Lambda(PSL(2,q);t,k)$ matrices in the class of*

(a) 4-$(q+1,5,\lambda)$ *designs*
- *arising from $PSL(2,q)$ for $q \le 37$ and $q \in \{43,47\}$,*
- *arising from $PGL(2,q)$ for $q \in \{41,49\}$,*

(b) 4-$(q+1,6,\lambda)$ *designs and* 5-$(q+1,6,\lambda)$ *designs*
- *arising from $PSL(2,q)$ and $q \le 31$.*

where q denotes a prime power.

Open problem 3. Find some t-$(q+1,k,\lambda)$ designs arising from $PSL(2,q)$ with larger values of k and/or larger values of q.

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