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Relative symmetrizability and metrizability

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Abstract. Relative metrizability is defined and connections with other relative properties are established.

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§1. Introduction

Location properties of a subset in a topological space play an important role in topology: it is enough to refer to the notions of an open set, closed set, G_δ-subset, C-embedded subset, and so on. In fact, a great variety of location properties based on very different approaches is encountered in topology.

The following general idea was introduced in [4]: every topological property can be naturally transformed (often, in many ways) into a location property of a subspace Y in a space X. Such properties are called *relative properties*, or *properties of Y in X*. The principal conjecture is that *the majority of results of classical topology remain true for relative properties.* This provided a guideline in [4], where the first steps in investigation of many relative properties were taken. In particular, different versions of relative normality were studied in [4] to some depth; also, three versions of relative paracompactness were introduced in [4], and some relative compactness type properties were considered there.

Clearly, in a systematic approach along these lines relative metrizability has to play a central role. Interestingly enough, it turned out that to find an appropriate definition of relative metrizability is not so easy as to define relative normality or relative paracompactness, where the definitions are obvious. It took some time, and an important step towards that goal was made in [5], where the notion of relative symmetrizability was introduced.

In this paper, we continue the study of relative symmetrizability; this provides us with a basis for introduction of relative metrizability properties and their systematic study. Let us note that some ideas in the direction of relative metrizability have appeared in papers [9] of H.H. Corson and E.A. Michael and [6], [7] of C.E. Aull. In the first paper the notion of a subspace Y metrically embedded into a space X was introduced. Our approach to relative metrizability is quite different from Michael and Corson's approach. On the other hand, Aull uses the notion of a relative metric on a pair (Y, X) of spaces, which is a part of our technique as well; extension operators for open sets, generated by relative metrics, is another piece of technique used in both papers. Yet on the whole, Aull's definitions and results are also different from ours.

We follow notation and terminology in [10]. No separation axioms are assumed beforehand.

§2. Relative symmetrics

In what follows Y is a subset of a set X. *A symmetric d on* (Y, X) is a non-negative real valued function d defined on $X \times X$ which satisfies the following two conditions for all x in X and y in Y:

s1) $d(x, y) = 0$ if and only if $x = y$;

s2) $d(x, y) = d(y, x)$.

Of course, if $Y = X$, then d is a symmetric on (Y, X) if and only if d is a symmetric on X in the usual sense [3].

If d is a symmetric on (Y, X), and $x \in X$, $A \subset X$, then the distance $d(x, A)$ is defined in the usual way: $d(x, A) = inf\{d(x, y) : y \in A\}$.

Let us now assume that Y is a subspace of a topological space X. The next definition, in a slightly different form, was introduced in [5]. A symmetric d on (Y, X) *defines* Y *in* X if the next three conditions are satisfied:

o1) for every closed subset P of X, and each $y \in Y \setminus P$, $d(y, P) > 0$;

o2) if $A \subset Y$ and $d(x, A) > 0$, for each $x \in X \setminus A$, then A is closed in X; and

o3) for every closed subset P of X, and for each point x in $X \setminus P$, $d(x, P \cap Y) > 0$.

Clearly, *if a symmetric d on* (Y, X) *defines Y in X, and Z is a subspace of Y, then d also defines Z in X.*

We say that Y *is symmetrizable in* X, if there is a symmetric d on X, which defines Y in X. From condition o2) it follows that *if Y is symmetrizable in X, then $\{y\}$ is closed in X, for each $y \in Y$.* The next assertion is obvious.

Proposition 1 [5]. *If d is a symmetric on X, generating the topology of X, then d defines Y in X, for every subspace Y of X.*

Note 1. From Proposition 1 it follows that if a symmetric d on (Y, X) defines Y in X, then *the restriction of d to Y need not generate the topology of Y*; indeed, a subspace of a symmetrizable space need not be symmetrizable, or even, sequential (see [3], [11]). *If, in addition, Y is closed in X, then the restriction of d to Y generates the topology of Y* (see Proposition 3 below). Observe also that *a symmetric d on X defines X in X if and only if d generates the topology of X.*

We use throughout the following notation: if ϱ is a symmetric on (Y, X), then $[A]_\varrho = \{x \in X : \varrho(x, A) = 0\}$, for each $A \subset X$. The closure of A in X is denoted by $\bar{A}$.

Proposition 2. *If ϱ is a symmetric on (Y, X) which defines Y in X, then $[A]_\varrho \subset \bar{A}$, for each $A \subset Y$.*

Proof: This follows from condition o3). $\square$

Proposition 2 implies the next result:

Proposition 3 [5]**.** *If a symmetric d on (Y, X) defines Y in X, then d also defines Y in the closure $\bar{Y}$ of Y in X.*

The next simple assertion is especially interesting in view of Note 1. It also trivially follows from Proposition 2.

Proposition 4. *Let d be a symmetric on (Y, X) which defines Y in X. Then the restriction of d to Y generates a topology $\mathcal{T}(d, Y)$ on Y, which contains the (original) topology of Y.*

We write $a_n \to b$, if the sequence $\{a_n : n \in \omega\}$ converges to b. The next lemma allows to strengthen considerably Proposition 4 in the important case when X is a Hausdorff space.

Lemma 1. *Assume that X is Hausdorff and d is a symmetric on (Y, X), which defines Y in X, and let $x \in X$, $y_n \in Y$ for each $n \in \omega$ be such that $y_n \to x$. Then $d(y_n, x) \to 0$.*

Proof: Indeed, let B be any infinite subset of ω. Obviously, we may assume that the point x is not in $P = \{y_n : n \in \omega\}$. Then the set $P_B = \{y_n : n \in B\}$ is not closed in X, since $x \in \overline{P_B} \setminus P_B$. Therefore, by condition o2), there is $z \in X$ such that $z \in X \setminus P_B$ and $d(z, P_B) = 0$. Then by Proposition 2, $z \in \overline{P_B}$. Since X is Hausdorff, and $y_n \to x$, there is only one point in X which is in the closure of P_B and not in P_B, that is, x. Therefore, $x = z$ and $d(x, P_B) = 0$. Since B was any infinite subset of ω, it follows, that $d(y_n, x) \to 0$. $\square$

From Lemma 1 and Proposition 4 it follows that if d is a symmetric on (Y, X) which defines Y in X, then the topology $\mathcal{T}(d, Y)$ generated by the restriction of d to Y has the same convergent sequences in Y as the original topology of Y. Since the topology $\mathcal{T}(d, Y)$ is sequential, it is generated by the convergent sequences (see [3]). Therefore, $\mathcal{T}(d, Y)$ is *the sequential coreflection of the topology $\mathcal{T}$ of Y*, that is, the strongest topology on Y with the same convergent sequences as $\mathcal{T}$. Thus, the following result is established:

Theorem 1. *If X is Hausdorff, and d is a symmetric on (Y, X) which defines Y in X, then the topology $\mathcal{T}(d, Y)$ generated by the restriction of d to Y coincides with the strongest topology on Y which has the same convergent sequences as the (original) topology of Y.*

Corollary 1. *If X is Hausdorff and Y is sequential, and a symmetric d on (Y, X) defines Y in X, then the restriction of d to Y generates the topology of Y.*

Corollary 2. *If a sequential space Y is symmetrizable in a Hausdorff space X, then Y is symmetrizable (in itself).*

Though Y is not necessarily sequential in itself when Y is symmetrizable in X, Y is in this case relatively sequential in X in a certain sense, as we are now going to show.

Let us say that Y *is sequential in* X, if for each subset A of Y such that A is not closed in X there is a sequence $\{y_n : n \in \omega\} \subset A$ converging to a point in $X \setminus A$. Clearly, *if X is sequential, then every subspace Y of X is sequential in X* (though it need not be sequential in itself).

Proposition 5. *If Y is symmetrizable in X, then Y is sequential in X.*

Proof: Let d be a symmetric on (Y, X) defining Y in X, and let A be a subset of Y which is not closed in X. By condition o2), there is $x \in X \setminus A$ such that $d(x, A) = 0$. For each positive $n \in \omega$, fix $y_n \in A$ such that $d(x, y_n) < 1/n$. Let us show that the sequence $\{y_n : n \in \omega\}$ converges to x. Assume the contrary. Then there is an infinite subset B of $\{y_n : n \in \omega\}$ such that x is not in the closure of B. Condition o3) now implies that $d(x, B) > 0$. On the other hand, it is obvious from the definition of B that $d(x, B) = 0$. This contradiction completes the proof. $\square$

Studying relative symmetrizability, it is natural to consider when a space Y is symmetrizable in a larger space X which has better properties than Y — for example, is more compact than Y, or is complete in some sense. In particular, *when a metrizable space Y is symmetrizable in a Hausdorff compactification of Y?* Here is an example, based on Proposition 5.

Example 1. The discrete space ω is not symmetrizable in its Stone-Čech compactification $\beta(\omega)$. Indeed, ω is not closed in $\beta(\omega)$, while no sequence of elements of ω converges to a point in $\beta(\omega) \setminus \omega$. It remains to apply Proposition 5. Thus, even an open metrizable dense subspace of a compact space need not be symmetrizable in this space.

Let us say that *the extent of Y in X is countable* if every closed in X discrete subspace of Y is countable (see [5]). Clearly, *if the extent of X is countable, then the extent of Y in X is countable.*

Theorem 2. *If Y is symmetrizable in X and the extent of Y in X is countable, then the extent $e(Y)$ of Y is also countable.*

Proof: Let d be a symmetric on (Y, X), which defines Y in X. Take any discrete subspace Z of X, and put $T = \bar{Z}$. Then the restriction of d to T defines Z in T (see Proposition 3), Z is dense in T, and each point of Z is isolated in T. Therefore, the set $F_z = T \setminus \{z\}$ is closed in T, and by condition o1), $\varepsilon_z = d(z, F_z) > 0$, for each $z \in Z$. Let $Z_n = \{z \in Z : d(z, F_z) \geq 1/n\}$, for positive $n \in \omega$. By condition o2), Z_n is closed in T. Since Z_n is discrete and the extent of Y in X is countable, Z_n is countable, for each $n \in \omega$. Clearly, $Z = \cup\{Z_n : n \in \omega\}$. Therefore, Z is countable, and $e(Y) \leq \omega$. $\square$

Corollary 3. *If Y is symmetrizable in X and X is Lindelöf, then the extent of Y (in itself) is countable.*

Example 2. Let X be a Lindelöf space with an uncountable dense discrete subspace Y. Then Y is not symmetrizable in X, by Corollary 3. Observe, that Y is an open dense metrizable subspace of X. In particular, *an uncountable discrete space is not symmetrizable in any of its compactifications.*

§3. Proper relative symmetrics

Theorem 1 and its corollaries can be considerably strengthened if we impose one more natural condition on a symmetric d on (Y, X).

Let us say that a symmetric d on (Y, X) *properly defines* Y *in* X, or that it is *a proper symmetric on* (Y, X), if d satisfies conditions o1), o2), o3), and the next condition o4):

o4) if $y \in Y$ and $A \subset X$ are such that $d(y, A) > 0$, then y is not in the closure of A.

The next assertion is obvious, in view of Proposition 1.

Proposition 6. *A symmetric d on X properly defines X in X if and only if $\bar{A} = [A]_d$, for each $A \subset X$.*

From Proposition 6 we immediately get the next result:

Proposition 7. *Let d be a symmetric on (Y, X), properly defining Y in X. Then the original topology of Y is generated by the restriction of d to Y.*

If d is a symmetric on (Y, X), $x \in X$, and ε is a positive number, then we put: $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. Similarly, we define the set $B(A, \varepsilon)$ for any subset A of X. We also denote by $O(x, \varepsilon)$ the interior of the set $B(x, \varepsilon)$ in X. It is well known that even if d is a symmetric on X which generates the topology of X, the set $O(x, \varepsilon)$ may be empty ([3]).

In what follows, ϱ is a symmetric on (Y, X), defining Y in X. We are not assuming that ϱ generates the topology of X. We are going to define a mapping associating with every open subset of Y an open subset of X; this mapping in some important cases will turn out to be an extension operator.

Let V be an open subset of the space Y. For each $y \in V$, put $\varepsilon(y, V) = \varrho(y, Y \setminus V)$. Since y is not in $\overline{Y \setminus V}$, $\varepsilon(y, V) > 0$, for each $y \in V$ (by the condition o1)). Finally, put $env(V) = env_\varrho(V) = \cup\{O(y, \varepsilon(y, V)/2) : y \in V\}$. We shall call $env_\varrho(V)$ *an envelope of* V (*with respect to* ϱ). Thus, we have defined *an envelope operator* which associates with each open subset V of Y an open subset $env(V)$ of X. This operator is generated by ϱ. Clearly, $env(V)$ is always an open subset of X; yet, $env(V)$ might be empty for a non-empty open subset V of Y. Of course, $env(\emptyset) = \emptyset$.

Proposition 8. *Let ϱ be a symmetric on (Y, X), properly defining Y in X. Then:*

o5) *for each $y \in Y$ and each $\varepsilon > 0$, $y \in O(y, \varepsilon)$;*

o6) *$env(V) \cap Y = V$, for each $V \subset Y$ open in Y;*

o7) *for any open subsets U and V of Y such that $U \subset V$, $env(U) \subset env(V)$.*

Proof: o5) Put $A = X \setminus B(y, \varepsilon)$. Then $\varrho(y, A) > 0$. Since ϱ properly defines Y in X, y is not in the closure of A. Then $y \in X \setminus \bar{A} \subset B(y, \varepsilon)$. As $X \setminus \bar{A}$ is open in X, it follows that $y \in O(y, \varepsilon)$.

o6) Clearly, $O(y, \varepsilon(y, V)) \cap Y$ is contained in V, for each $y \in V$. This, together with o5) and the definition of $env(V)$, implies that $env(V) \cap Y = V$.

o7) is obvious from the definition of the envelope operator env. $\square$

Proposition 9. *If ϱ is a symmetric on (Y, X), properly defining Y in X, then the space X is first countable at all points of Y.*

Proof: This follows from o1) and Proposition 8 o5). $\square$

We say that Y is *properly symmetrizable in X*, if there is a symmetric ϱ on (Y, X) which properly defines Y in X. Of course, a basic question is: when Y is properly symmetrizable in X?

Indeed, from Propositions 7 and 9 it follows that if Y is not a semi-metrizable space, then Y is not properly symmetrizable in any larger space. The next example helps to clarify the situation further.

Example 3. Let N be the set of all positive integers, and let X be a subset of the coordinate plane defined as follows: $X = A \cup B \cup C$, where $A = \{(0,0)\}$, $B = \{(1/n, 0) : n \in N\}$, and $C = \{(1/n, 1/m) : n, m \in N\}$. Let us define three symmetrics on X, by the following formulae combined with the axioms s1) and s2). For $a = (0,0)$, and any $c \in C$ we put: $d(a, c) = \varrho(a, c) = \theta(a, c) = 1$. Further, for $b_n = (1/n, 0)$, let $d(a, b_n) = 1/n$, and $\varrho(a, b_n) = \theta(a, b_n) = 1$. Also $d(b_n, (1/n, 1/m)) = \varrho(b_n, (1/n, 1/m)) = \theta(b_n, (1/n, 1/m)) = 1/m$, for each $m \in N$. If x, y are any two different points of B, the distance between x and y is 1, with respect to each of the three symmetrics on X. We also put $d(b_n, (1/k, 1/m)) = \varrho(b_n, (1/k, 1/m)) = \theta(b_n, (1/k, 1/m)) = 1$, whenever $k \neq n$, $k, n \in N$. The distance between any two different points of C with respect to d is 1. We also put $\varrho((1/n, 1/m), (1/k, 1/l)) = \theta((1/n, 1/m), (1/k, 1/l)) = 1$, if $n \neq k$. Now the only difference between ϱ and θ is here: for $m \neq l$, $\varrho((1/n, 1/m), (1/n, 1/l)) = |1/n - 1/l|$, while $\theta((1/n, 1/m), (1/n, 1/l)) = 1$. On the other hand, d and ϱ differ only in the distances between a and the points of B.

Clearly, d, ϱ, and θ are symmetrics on X. Let us endow the set X with the topology $\mathcal{T} = \mathcal{T}_d$, generated by d, and let $Y = A \cup C$, with the subspace topology. Of course, X is just the well known *Arens space* (see [1], [3]). It is easy to see, that the spaces X and Y are not first countable at the point a. Proposition 9 implies that Y is not properly symmetrizable in X. On the other hand, by Proposition 1, d defines Y in X, and therefore, Y is symmetrizable in X (by d).

It is trivially verified that ϱ and θ also define Y in X, though the topology they generate on X is different from the topology $\mathcal{T}$ generated by d. Note that the restrictions of d, ϱ, and θ to Y are metrics on Y generating the discrete topology on Y which does not coincide with the subspace topology on Y. This perfectly agrees with Lemma 1, since there are no non-trivial convergent sequences in the space Y.

Lemma 2. *If d is a symmetric on (Y, X) which properly defines Y in X, then for each subset A of Y, there is a G_δ-subset P of X such that $A \subset P \subset [A]_d$.*

PROOF: This trivially follows from condition o4). □

Theorem 3. *If Y is properly symmetrizable in X, then every closed in X subset of Y is a G_δ-set in X.*

PROOF: It is enough to refer to Lemma 2 and Proposition 2. □

Corollary 4. *If Y is closed in X and properly symmetrizable in X, then Y is a G_δ-set in X.*

Note 2. Assume that d and ϱ are two symmetrics on (Y, X) such that $d(x, y) = \varrho(x, y)$ for all x in X and y in Y, that is, d and ϱ coincide on the set $Y_X = (Y \times X) \cup (X \times Y)$. Then ϱ properly defines (defines) Y in X if and only if d properly defines (defines) Y in X; if this is the case, the envelope operators generated by ϱ and d coincide. This happens because the sets $B(y, \varepsilon)$ and $O(y, \varepsilon)$ are the same when $y \in Y$. For the purposes of the argument in this section, we could identify symmetrics on (Y, X) which take the same values on Y_X: what really matters are the values of d or ϱ, when at least one point is in Y. The same holds true for the next section.

§4. Relative metrics and 2-metrics

A symmetric ϱ on (Y, X) is called *a metric on* (Y, X), if $\varrho(y, z) \leq \varrho(y, x) + \varrho(x, z)$, for every y, z in Y and x in X. A subspace Y of a space X is *metrizable in* X, if there is a symmetric ϱ on (Y, X) which defines Y in X and is a metric on (Y, X).

Theorem 4. *If Y is closed in X and metrizable in itself, then Y is metrizable in X.*

PROOF: Let ϱ be a metric on Y generating the topology of Y. We may assume that $\varrho(y, z) \leq 1$, for every y, z in Y. Now we extend ϱ to become a metric ϱ^* on X by the following rule: if $x \in X \setminus Y$, and $y \in X$, $x \neq y$, then $\varrho^*(x, y) = \varrho^*(y, x) = 1$, and $\varrho^*(x, y) = \varrho(x, y)$, if x and y are both in Y. Clearly, ϱ^* is a metric on X, and it is routinely verified that ϱ^* defines Y in X (here we rely heavily upon our assumption that Y is closed in X). □

Example 4. Let X be the Niemytzky plane and Y the discrete line at the bottom of it. According to Theorem 4, Y is metrizable in X. On the other hand, the classical argument shows that Y is not normal in X. Thus, metrizability of Y in a Tychonoff space X does not imply in general that Y is normal in X or that Y is 2-paracompact in X.

Example 5. This is a generalization of Example 3. Let Z be a space with exactly one non-isolated point m, $M = Z \setminus \{m\}$, and $T = Z \times \omega$. We put $a = (m, 0)$, $B = \{(z, 0) : z \in M\}$, and $C = M \times (\omega \setminus \{0\})$. We define a symmetric ϱ on $X = \{a\} \cup B \cup C$ by the following rule and conditions s1), s2). If $x \in B \cup C$, then $\varrho(a, x) = 1$. If b_1, b_2 are two different points of B, then $\varrho(b_1, b_2) = 1$. Let $b = (z_1, 0) \in B$, and $c = (z_2, k) \in C$. If $z_1 = z_2$, we put $\varrho(b, c) = 1/k$; in the other case $\varrho(b, c) = 1$. Similarly, if $c_1 = (z_1, n)$, $c_2 = (z_2, k)$ are in C, and $z_1 = z_2$, then $\varrho(c_1, c_2) = |1/n - 1/k|$; if z_1 and z_2 are different, then $\varrho(c_1, c_2) = 1$. The set $A = \{a\} \cup B$ can be naturally identified with the set Z. Let $\mathcal{T}_1$ be the topology induced on A by this identification of A with the space Z. The restriction of ϱ to $B \cup C$ generates a topology $\mathcal{T}_2$ on $B \cup C$. Let us call a subset U of X open in X, if $U \cap A$ belongs to $\mathcal{T}_1$, and $U \cap (B \cup C)$ is in $\mathcal{T}_2$. This defines a topology $\mathcal{T}$ on X. Put $Y = A \cup C$. It is easy to see that ϱ defines Y in X, and that ϱ is a metric on X. Therefore, Y is metrizable in X. Note, that a is non-isolated in Y. Let us assume that the space Z is chosen in such a way that, in addition, all countable sets in Z are closed (for example, the one-point Lindelöfication of an uncountable discrete space will do). Then, since the natural projection of X onto $Z = A$ is continuous and a is the only non-isolated point of Y, all countable subsets of Y are closed in Y as well. It follows that the tightness of Y is not countable, though Y is sequential in X, according to Proposition 5. Thus, *metrizability of Y in X does not imply, that the tightness of Y (in itself) is countable.*

We say that Y is *properly metrizable in* X, if there is a metric ϱ on (Y, X) properly defining Y in X.

Proposition 10. *Let ϱ be a metric on (Y, X), properly defining Y in X. Then:*
o8) *if U and V are open subsets of Y, and $U \cap V = \emptyset$, then $env(U) \cap env(V) = \emptyset$.*

Proof: Let $z \in env(U) \cap env(V)$. Then there are $y \in U$ and $x \in V$ such that $\varrho(z, y) < \varrho(y, Y \setminus U)/2$ and $\varrho(z, x) < \varrho(x, Y \setminus V)/2$. We may assume that $\varrho(y, Y \setminus U) \leq \varrho(x, Y \setminus V)$. Then from the triangle inequality we have: $\varrho(x, y) < \varrho(x, Y \setminus V)$. It follows that $y \in V$ — a contradiction with $U \cap V = \emptyset$. Hence, $env(U) \cap env(V) = \emptyset$. □

From Proposition 7 we immediately get the next

Proposition 11. *If Y is properly metrizable in X (by a metric ϱ on (Y, X)), then the subspace Y of X is metrizable (by the restriction of ϱ to Y).*

Example 6. Let X, Y, and ϱ be the same as in Example 3. Clearly, ϱ is a metric on (Y, X), and ϱ defines Y in X. Therefore, Y is metrizable in X. On the other hand, Y is not first countable, and the topology generated on Y by the restriction of ϱ to Y is discrete and hence, does not coincide with the topology of Y. It follows that ϱ does not properly define Y in X; we can even make a stronger conclusion: that Y is not properly metrizable in X.

Theorem 5. *If Y is properly metrizable in X, then Y is strongly normal in X.*

Proof: Let us fix a metric ϱ on (Y, X), properly defining Y in X. Let A and B be any two disjoint closed subsets in the space Y. From Proposition 11 it follows that the space Y is metrizable and hence, normal. Now we can fix disjoint open subsets U and V in Y such that $A \subset U$ and $B \subset V$. Then, by Proposition 10 o8), $env(U)$ and $env(V)$ are disjoint open sets in X, containing A and B, respectively. Thus, Y is strongly normal in X. □

Note 3. The discrete bottom line Y of Niemytzky plane X is not properly metrizable in X, though X is first countable, and Y is metrizable in X. Indeed, Y is not normal in X; it remains to apply Theorem 5. The next lemma is an obvious corollary of Proposition 10 o8).

Lemma 3. *If ϱ is a metric on (Y, X) properly defining Y in X, and γ is a locally finite in Y (discrete in Y) family of open subsets of Y, then the family $\{env(U) : U \in \gamma\}$ is locally finite (discrete) at each point of Y.*

Now we can easily prove one of our main results:

Theorem 6. *If Y is properly metrizable in X, then Y is 2-paracompact in X.*

In fact, we shall prove a slightly stronger assertion. Let us recall that Y *is 2-paracompact* (*Aull-paracompact*) *in* X, if for every open covering γ of X (for every family γ of open subsets of X such that $Y \subset \cup\gamma$) there is a family μ of open subsets of X locally finite in X at all points of Y and also satisfying the following conditions: $Y \subset \cup\mu$, and μ refines γ, that is, for each $V \in \mu$ there is $U \in \gamma$ such that $V \subset U$. We shall say that Y is *strictly Aull-paracompact in* X, if in the above definition of Aull-paracompactness of Y in X the family μ can be chosen to satisfy one more condition: μ is σ-discrete in Y, that is, μ is the union of a countable family of its subfamilies, each of which is discrete at all points of Y. Clearly, *Aull-paracompactness of Y in X implies 2-paracompactness of Y in X and paracompactness of Y in itself.*

Theorem 7. *If Y is properly metrizable in X, then Y is strictly Aull-paracompact in X.*

Proof: First, we fix a metric ϱ on (Y, X), that properly metrizes Y in X. Let γ be a family of open subsets of X such that $Y \subset \cup\gamma$. Put $\xi = \{U \cap Y : U \in \gamma\}$. Then ξ is an open covering of Y. By Proposition 11, the space Y is metrizable, and therefore, paracompact. By A.H. Stone's Theorem, there is a σ-discrete locally finite open covering η of the space Y, refining ξ. For each $V \in \eta$ we fix $U_V \in \gamma$ such that $V \subset U_V$. Then the family $\mu = \{env(V) \cap U_V : V \in \eta\}$ is a family of open subsets of X, refining γ, covering Y, and, by Lemma 3, locally finite and σ-discrete at all points of Y. Therefore, Y is strictly Aull-paracompact in X. □

A family $\mathcal{B}$ of open subsets of X is called (*an outer*) *base of Y in X* ([2]), if for each $y \in Y$, it contains a base of X at y. For the next theorem, which naturally generalizes in one direction the well known Bing's metrization condition, we need a slightly stronger notion of relative metric. A symmetric ϱ on (Y, X) will be called *a 2-metric, or an Aull-metric, on* (Y, X), if whenever any two out of the three points in the triangle inequality are in Y, the inequality holds. We shall say that Y is *2-metrizable, or Aull-metrizable, in* X, if there is an Aull-metric ϱ on (Y, X), which satisfies the conditions o1), o2), and o3). Similarly we define when Y is properly Aull-metrizable in X, strictly Aull-metrizable in X, and so on.

Theorem 8. *If Y is properly Aull-metrizable in X, then there is an* (*outer*) *base of Y in X, which is σ-discrete at all points of Y.*

Proof: Again, we fix a 2-metric ϱ on (Y, X) which metrizes Y in X. Since, by Proposition 11, the space Y is metrizable, there is a σ-discrete base $\mathcal{P}$ of Y, that is, $\mathcal{P} = \cup\{\gamma_i : i \in \omega\}$, where each γ_i is a discrete family of open subsets of Y. Then, by Lemma 3, $\mu_i = \{env(V) : V \in \gamma_i\}$ is a family of open subsets of X discrete at all points of Y. It remains to show that the family $\mathcal{B} = \cup\{\mu_i : i \in \omega\}$ is an (outer) base of Y in X.

To that end, let us fix $y \in Y$ and an open neighborhood $O(y)$ of y in X. Then the set $F = X \setminus O(y)$ is closed in X, and by 01), $\delta = \varrho(y, F) > 0$. Since the restriction of ϱ to Y metrizes the space Y, there is $V \in \mathcal{P}$ such that $y \in V$ and the ϱ-diameter of V is less than $\delta/2$. Then $env(V)$ is contained in $B(y, \delta)$, by the triangle inequality, and therefore, $env(V) \subset O(y)$. Since $y \in env(V) \in \mathcal{B}$, it follows that $\mathcal{B}$ is an outer base of Y in X. □

After Theorems 5 and 8, keeping in mind Bing's metrization criterion, it is natural to expect that if Y is strongly normal in X, and there is a σ-discrete in Y outer base of Y in X, then Y is properly metrizable in X. Example 1 shows that this is not the case, even if Y is dense in X and X is Tychonoff. In fact, it shows more: that under circumstances, Y need not be metrizable in X. Other examples of this kind can be obtained on the basis of the next two results which also seem to be interesting in itself.

Proposition 12. *If Y is countably compact in X, and ϱ is a metric on (Y, X), defining Y in X, then Y is completely bounded in itself with respect to ϱ, that is, for each $\varepsilon > 0$, there is a finite subset K of Y such that $\varrho(y, K) < \varepsilon$, for each $y \in Y$.*

Proof: Assume the contrary. Then for some $\varepsilon > 0$, we can find an infinite set $A \subset Y$ such that the ϱ-distance between any two different points of A is greater than ε. Since Y is countably compact in X, there is an infinite subset B of A which is not closed in X. Then by o2), there is a point $x \in X$ such that $\varrho(x, B) = 0$. Now we can choose y_1 and y_2 in Y such that $\varrho(y_1, x) < \varepsilon/3$, $\varrho(x, y_2) < \varepsilon/3$, and $y_1 \neq y_2$. Then, by the triangle inequality, $\varrho(y_1, y_2) < \varepsilon/3 + \varepsilon/3 < \varepsilon$, — a contradiction with the choice of A. □

Theorem 9. *If the extent of Y in X is countable, and Y is metrizable in X by a metric ϱ on (Y, X), then the topology $\mathcal{T}(\varrho, Y)$ generated on Y by the restriction of ϱ is separable metrizable, and the space Y (with the original topology) has a countable network.*

Proof: Repeating, with obvious modifications, the argument in the proof of Proposition 12, we see that $(Y, \mathcal{T}(\varrho, Y))$ is separable. By Proposition 4, $\mathcal{T}(\varrho, Y)$ contains the original topology on Y. Therefore, the space Y has a countable network. $\square$

Corollary 5. *If Y is a metrizable non-separable space, and $X = b(Y)$ is a compactification of Y, then Y is not metrizable in X.*

Let us say that Y *satisfies the DCCC in X*, if every family γ of disjoint open sets such that γ is discrete at each point of Y, and $U \cap Y \neq \emptyset$ in X is countable, for every $U \in \gamma$, is countable.

§5. Relative 1-metrics

In this section we impose much stronger restrictions on a metric ϱ on (Y, X), and introduce some other new notions. In what follows we assume that ϱ is a symmetric on X: for the first time in this paper it will matter that the distance $\varrho(x, y)$ is defined even if none of the points x, y is in Y.

A subset A of X is said to be *concentrated on Y*, if $A \subset \overline{A \cap Y}$. A symmetric d on X *strictly defines Y in X*, or *strictly symmetrizes Y in X*, if conditions o1) and o3) are satisfied, as well as the next condition $o^\sharp 2)$, which strengthens condition o2):

$o^\sharp 2)$ if A is a subset of X concentrated on Y such that $d(x, A) > 0$, for each $x \in X \setminus A$, then A is closed in X.

Note 4. In Example 3, symmetric ϱ defines Y in X, while ϱ does not strictly define Y in X. Indeed, the set $P = B \cup C$ is concentrated on Y, and is not closed in X, since $a \in \bar{P} \setminus P$. On the other hand, a is the only point in $\bar{P} \setminus P$, and $\varrho(a, P) = 1$, so that P would have been closed in X, if ϱ strictly defined Y in X. Of course, *if Y is closed in X, and ϱ is a symmetric on (Y, X) which defines Y in X, then ϱ strictly defines Y in X.* In particular, every symmetric d on X always strictly defines X in X with respect to the topology on X generated by d. Therefore, the statement that d strictly defines Y in X does not imply in general that d properly defines Y in X (see Proposition 9). See also Note 3.

We say that ϱ is *a* 1*-metric on* (Y, X), if ϱ is a symmetric on X, and whenever x, y, z are three points in X, at least one of which belongs to Y, then the triangle inequality holds: $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

Proposition 13. *If ϱ is a* 1*-metric on (Y, X), and ϱ strictly defines Y in X, then $\bar{A} = [A]_\varrho$, for each $A \subset Y$.*

Proof: By Proposition 2, $[A]_\varrho \subset \bar{A}$. On the other hand, the set $B = [A]_\varrho$ is closed in X, by condition $o^\sharp 2)$. Indeed, otherwise there is a point $z \in X \setminus B$, such that $d(z, B) = 0$, which obviously contradicts the triangle inequality in the definition of a 1-metric on (Y, X), since $d(z, A) > 0$. $\square$

The next result directly follows from Proposition 13.

Proposition 14. *If ϱ is a* 1*-metric on* (Y, X) *which strictly defines* Y *in* X, *then the restriction of* ϱ *to* Y *generates the topology of* Y.

Proposition 15. *If* ϱ *is a* 1*-metric on* (Y, X), *and* ϱ *strictly defines* Y *in* X, *then:*

a) $\bar{Y} = [Y]_\varrho$, *and*

b) *if at least one of the points* x, y, z *is in* $\bar{Y}$, *then the triangle inequality holds.*

Proof: a) follows from Proposition 13, and b) easily follows from a) and the definition of a 1-metric. □

Proposition 16. *If* ϱ *is a* 1*-metric on* (Y, X), *then* $[[A]_\varrho]_\varrho = [A]_\varrho$, *for each* $A \subset Y$. *In particular, if* ϱ *strictly defines* Y *in* X, *then* $\varrho(x, \bar{Y}) > 0$, *for every* $x \in X \setminus \bar{Y}$.

Proof: The first assertion is obvious, the second follows from Proposition 15. □

The next result is a direct corollary of Proposition 16.

Proposition 17. *If* ϱ *is a* 1*-metric on* (Y, X), *strictly defining* Y *in* X, *then the restriction of* ϱ *to* $\bar{Y}$ *is a* 1*-metric on* $(Y, \bar{Y})$, *strictly defining* Y *in* $\bar{Y}$.

Proposition 18. *If* ϱ *is a* 1*-metric on* (Y, X), *strictly defining* Y *in* X, *then the topology* $\mathcal{T}_\varrho$ *generated on* $\bar{Y}$ *by the restriction of* ϱ *to* $\bar{Y}$, *is contained in the original topology of the subspace* $\bar{Y}$ *of* X.

Proof: In view of Proposition 17, we may assume that $\bar{Y} = X$. Let $A \subset X$ and $x \in X$ be such that $\varrho(x, A) = \varepsilon > 0$. We have to show that then x is not in the closure of A.

Put $M = Y \cap B(A, \varepsilon/3)$. By Proposition 15 a), $A \subset [Y]_\varrho$. Since obviously $A \cap [Y \setminus M]_\varrho = \emptyset$, $A \subset [M]_\varrho$. On the other hand, $\varrho(x, M) \geq \varepsilon/3$, by the triangle inequality. Therefore, x is not in $[M]_\varrho$. Since $[M]_\varrho$ is closed by Proposition 13, it follows that x is not in the closure of A. □

Note that in the course of the proof of Proposition 18, we have also proved the following assertion:

Proposition 19. *If* ϱ *is a* 1*-metric on* (Y, X) *which strictly defines* Y *in* X, *then for each* $A \subset \bar{Y}$ *and each* $\varepsilon > 0$, $A \subset [Y \cap B(A, \varepsilon)]_\varrho$.

Theorem 10. *If* X *is regular, and* ϱ *is a* 1*-metric on* (Y, X) *strictly defining* Y *in* X, *then* ϱ *metrizes the subspace* $\bar{Y}$, *that is, the restriction of* ϱ *to* $\bar{Y}$ *generates the original topology of* $\bar{Y}$.

Proof: In view of Proposition 18, it would suffice to check that if P is a closed subset of $\bar{Y}$, then $\varrho(x, P) > 0$, for each $x \in X \setminus P$. Let us fix such an x. Since X is regular, we can choose an open set U in X such that $P \subset U$ and $x \notin \bar{U}$. Put $B = \bar{U}$. Then B is closed in X, and x is not in B. Therefore, by condition o3), $\varrho(x, B \cap Y) = \varepsilon > 0$. On the other hand, P is obviously contained in the closure of $B \cap Y$. Now it follows from Proposition 13, that $P \subset [B \cap Y]_\varrho$. By the triangle inequality, $\varrho(x, [B \cap Y]_\varrho) = \varrho(x, B \cap Y) = \varepsilon > 0$. Hence, $\varrho(x, P) > 0$. $\square$

Note 5. Let X, Y, and ϱ be the same as in Example 3. Then ϱ is a 1-metric on (Y, X), and ϱ defines Y in X. On the other hand, for the set $C = \{(1/n, 1/m) : n, m \in N\}$ we obviously have: $[C]_\varrho = C \cup B \neq \bar{C} = X$, since a is not in $[C]_\varrho$. This implies that Propositions 13–19 and Theorem 11 stop to be true if we drop the word "strictly" in their formulations.

Proposition 20. *If ϱ is a 1-metric on (Y, X), and ϱ generates the topology of X, then ϱ properly defines Y in X.*

Proof: Let $A \subset X$ and $y \in Y$. Because of the triangle inequality, we have: $d(y, A) = d(y, [A]_\varrho)$. Since iterations of the operator $[\ ]_\varrho$ along all countable ordinal numbers result in the closure of A in X (see [3], [11]), we conclude that $d(y, \bar{A}) = d(y, A) > 0$. Therefore, y is not in $\bar{A}$. $\square$

If a symmetric d strictly and properly defines Y in X, we say that d *perfectly defines* Y *in* X. Accordingly, Y is *perfectly metrizable in* X, if there is a metric ϱ on (Y, X) perfectly defining Y in X. Similarly it is defined when Y is *strictly,* (*properly, perfectly*) 1-*metrizable in* X.

From Proposition 20 we easily get the next

Proposition 21. *If ϱ is a 1-metric on (Y, X), and ϱ generates the topology of X, then ϱ perfectly defines Y in X.*

The next result directly follows from Propositions 15 and 21.

Theorem 11. *If ϱ is a 1-metric on (Y, X), and ϱ generates the topology of X, then ϱ is a metric on $(\bar{Y}, X)$ and ϱ perfectly defines $\bar{Y}$ in X. Therefore, $\bar{Y}$ is perfectly 1-metrizable in X.*

From Theorem 6 and Proposition 20, we get the following corollary:

Corollary 6. *If ϱ is a 1-metric on (Y, X) which generates the topology of X, then Y is 2-paracompact in X.*

It is good to compare the last results and Proposition 9 with the following general assertion, which is basically known and can be proved by a standard argument (see [11]).

Proposition 22. *If X is Hausdorff and first countable at all points of Y, and d is a symmetric on X which generates the topology of X and defines Y in X, then d properly defines Y in X.*

It follows from Proposition 22 and Theorem 5, that if d is a symmetric on the Niemytzky plane X which generates the topology of X, then d cannot be a 2-metric on (Y, X), where Y is the discrete bottom line. Note, that X is symmetrizable by a rather nice symmetric, and Y is metrizable (see [11]).

Example 7. Let X be Mrowka-Isbell space Ψ (see [10], [11]). Then $X = N \cup \Omega$, where N is the set of positive integers and Ω is an uncountable almost disjoint family of infinite subsets of N. We define a symmetric d on (N, X) in the following way. If n, m are in N and $n \neq m$, then $d(n,m) = 1/n+1/m$. Let $a \in \Omega$ and $n \in N$. If $n \in a$, then $d(n,a) = 1/n$; if n is not in a, we put $d(n,a) = 1$. Finally, if a and b are two different points of Ω, then $d(a,b) = 1$. We endow X with the topology generated on X by symmetric d; the space obtained is the classical space Ψ. From this definition it follows (see Proposition 1 and Note 4) that d strictly (in fact, perfectly) defines N in X. It is also easy to see that d is a metric on (N, X). Note that d is not a 2-metric on (N, X). Indeed, take $a \in \Omega$, $n \in a$, such that $n > 3$, and $m \in N \setminus a$ such that $m > 3$. Then $d(m,a) = 1 > 1/3+1/3 > d(m,n)+d(n,a)$. The space X is locally compact and Hausdorff (therefore, it is Tychonoff). Let $Z = X \cup \{b\}$ be the one-point compactification of X, and $Y = N \cup \{b\}$. Then Z is a compact Hausdorff sequential scattered space, and Y is a countable non-sequential subspace of X with only one non-isolated point.

Let us extend our symmetric d from X to Z in the following trivial way: define the distance from b to any other point of Z to be 1. The symmetric ϱ on Z so obtained does not generate the topology of Z, since X is not closed in Z, while the distance from b to X is 1, and b is the only point of Z not in X. Nevertheless, it is easily seen that ϱ properly defines N in Z and defines Y in Z. On the other hand, ϱ does not strictly define N in Z, and does not strictly define Y in Z. It is also clear that ϱ is a metric on (N, Z) and on (Y, Z). Thus, we see, that *there is a non-metrizable Hausdorff compactification of the countable discrete space N, in which N is metrizable.*

§6. Relative 1-metrics and relative star normality

An important property of metrizable spaces is that all of them are star-normal. It is natural to ask, which relative properties of star normality type are generated by relative metrics.

If y is a point and γ is a family of sets, we put $St_\gamma(y) = \cup\{U : y \in U \in \gamma\}$. The set $St_\gamma(y)$ is called *the star of y with respect to γ*. Let γ and μ be two families of sets, and let y be a point. We say that μ *star-refines* γ *at* y, if there is $U \in \gamma$ such that $St_\mu(y) \subset U$.

Let us say that Y *is strongly star-normal in* X, if for each family γ of open sets in X covering Y there is a family μ of open subsets of X which covers Y and star refines γ at each point of $\cup\mu$.

Theorem 12. *If Y is properly 1-metrizable in X, then Y is strongly star-normal in X.*

Proof: Let ϱ be a 1-metric on (Y, X) properly defining Y in X, and let γ be a family of open subsets of X covering Y. For any positive $n \in \omega$ we denote by $\mathcal{E}_n$ the family of all open subsets U of X satisfying the next two conditions:

1) $\varrho(x_1, x_2) < 1/n$, whenever x_1 and x_2 are in U, and

2) there exists $G \in \gamma$ such that $B(U, 1/n) \subset G$.

Let us show that the family $\mu = \cup\{\mathcal{E}_n : n \in \omega \setminus \{0\}\}$ covers Y.

Take any $y \in Y$. There is $W \in \gamma$ such that $y \in W$. Since W is open in X and $y \in Y$, condition o1) implies that $\varepsilon = \varrho(y, X \setminus U) > 0$. There is $n \in \omega$ such that $1/n < \varepsilon/2$. Then

$$V = O_{1/2n}(y) \in \mathcal{E}_n \subset \mu.$$

Indeed, V is the interior in X of the set $B(y, 1/2n)$, and by the triangle inequality, we have: $B(V, 1/n) \subset B(B(y, 1/2n), 1/n) \subset B(y, 2/n) \subset W$. Besides, V is an open subset of X. Since ϱ properly metrizes Y in X, y belongs to V. Since y is also in Y, the triangle inequality implies that as soon as x_1 and x_2 are in V, $\varrho(x_1, x_2) < 1/n$. Thus, $y \in \cup\mu$.

Now take any $z \in \cup\mu$; let us prove that μ star refines γ at z. For each $U \in \mu$ we fix a positive number $n(U) \in \omega$ such that $U \in \mathcal{E}_{n(U)}$ (this is possible by the definition of μ). Then $N_z = \{n(U) : z \in U \in \mu\}$ is a non-empty subset of $\omega \setminus \{0\}$, since $z \in \cup\mu$.

Let $m = min(N_z)$. There is $U^* \in \mu$ such that $m = n(U^*)$. Then $U^* \in \mathcal{E}_m$ and therefore, there is $G^* \in \gamma$ such that $B(U^*, 1/m) \subset G^*$.

Take any $U \in \mu$ such that $y \in U$. Then, from the choice of m, $m \leq n(U)$. Now condition 1) implies that

$$U \subset B(z, 1/n(U)) \subset B(z, 1/m) \subset B(U^*, 1/m) \subset G^*.$$

Therefore, $St_\mu(z) \subset G^*$, that is, μ star refines γ at z. □

From Theorems 11 and 12, and Proposition 20 we get the next corollary:

Corollary 7. *If ϱ is a 1-metric on (Y, X) generating the topology of X, then both Y and $\bar{Y}$ are strongly star-normal in X.*

§7. Some remarks and open questions

A routine argument shows *that if Y is strongly star-normal in X, then Y is strongly normal in X.* It is also clear that if γ is a family of open sets in X which covers Y, then star refining γ two or more times, we obtain a covering ν of Y by open sets in X such that the stars of elements of ν with respect to ν are contained in some elements of the family γ.

The notion of strong star-normality of Y in X, introduced above, is much stronger than the following very natural concept of relative star-normality. If for each open covering γ of X there is a family μ of open subsets of X covering Y and star refining γ at every point of Y, we say that Y *is star-normal in* X. If the above condition holds for every family γ of open subsets of X covering Y, we say that Y is *α-star-normal in* X. Finally, we call Y *weakly star-normal in* X, if for every open covering γ of the space X there is a covering μ of Y by open subsets of Y which star refines γ at all points of Y.

Example 8. The discrete bottom line Y of Niemytzky plane is evidently α-star-normal in X. On the other hand, Y is not strongly normal in X ([4]). Therefore, Y is not strongly star-normal in X.

Question 1. Find a first countable compact Hausdorff space X with a countable dense subspace Y such that Y is not metrizable in X (is not properly metrizable in X).

Question 2. Let Y be metrizable in X. Is then Y paracompact (in itself)? Normal (in itself)?

Question 3. Let X be Tychonoff pseudocompact, Y dense in X, and Y metrizable in X. Is Y separable? Is Y metrizable? Is the Souslin number of X countable?

Question 4. Let Y be properly metrizable (properly Aull-metrizable) in X. Is then Y strongly star-normal in X? Star-normal in X?

Question 5. Let Y be 1-metrizable in X. Is then Y star-normal (in itself)? Normal (in itself)?

Question 6. Let Y be properly metrizable in X. Is then true that there exists an (outer) base of Y in X which is σ-discrete at all points of Y?

Question 7. Let Y be properly metrizable in X. Is then Y properly Aull-metrizable in X?

Question 8. Find an example, where X is Tychonoff, Y is weakly star-normal in X, and Y is not star-normal in X.

Question 9. Is it true that if Y is strongly star-normal in X, then Y is 2-paracompact in X?

There are some interesting questions, involving very natural notions of potential metrizability and potential symmetrizability of a given space in a class of spaces, which we are going to define now.

Let $\mathcal{P}$ be a class of topological spaces. We shall say that a space Y *is potentially metrizable in the class* $\mathcal{P}$, or $\mathcal{P}$*-potentially metrizable*, if there is a space X in $\mathcal{P}$ which contains Y as a subspace in such a way that Y is metrizable in X. Similarly, *potential symmetrizability in a class of spaces* is defined, as well as other potential properties. We write that Y *is* T_2*-potentially metrizable*, if Y is metrizable in the class of Hausdorff spaces. Similarly, for T_3-potentially metrizable, and so on.

General Question 10. Characterize in terms of the topology of Y, when Y is potentially metrizable in a class $\mathcal{P}$ of spaces. What if $\mathcal{P}$ is the class of all Tychonoff spaces? Of all regular spaces? Of all Hausdorff spaces? Of all compact Hausdorff spaces? If $\mathcal{P}$ is the class of all topological spaces, we drop $\mathcal{P}$ in the above notation.

Observe, that a non-metrizable countable Tychonoff space can be metrizable in a larger Tychonoff space (see Example 5). Note also the next two corollaries of Theorem 9.

Corollary 8. *Every Lindelöf space, which is potentially metrizable, has a countable network.*

Corollary 9. *If a countably compact space is potentially metrizable, then it has a countable network* (*and is, therefore, compact*).

Question 11. Is every Tychonoff pseudocompact potentially metrizable (potentially Tychonoff metrizable) space compact and metrizable?

Some information in the direction of Question 10 is provided by the next result, which follows from Corollary 1.

Theorem 13. *A sequential space X is T_2-potentially metrizable, if and only if X is metrizable.*

Example 9. Let N be the discrete space of positive integers, and $\beta(N)$ the Stone-Čech compactification of N. Take any $\xi \in \beta(N) \setminus N$, and put $N_\xi = N \cup \{\xi\}$. Let $\mathcal{E}$ be the family of all infinite subsets of N, which are not in ξ. Now take any maximal (with respect to $\mathcal{E}$) almost disjoint subfamily Ω of $\mathcal{E}$. We introduce a topology $\mathcal{T}$ on the set X in the following, rather standard, way. All subsets of N are open in X. A basic open neighborhood of a point $a \in \Omega$ consists of the point a itself and of all but finitely many points of the subset a of N. A basic open neighborhood of the point ξ in X has the form: $\xi \cup P$, where P is any element of ξ. Clearly, N_ξ as a subspace of X has the same topology as N_ξ as a subspace of βN. The space X is, of course, Hausdorff and not regular. Now we introduce a metric ϱ on X as follows. For any x in X which is different from ξ, $\varrho(\xi, x) = 1$. If a and b are any two different points of Ω, then again $\varrho(a, b) = 1$. If m, n are two different points of N, then $\varrho(m, n) = 1/m + 1/n$. Finally, let $n \in N$, and $a \in \Omega$. Then $\varrho(n, a) = 1/n$, if $n \in a$, and $\varrho(n, a) = 1$ otherwise. It is clear that ϱ is a metric on (N_ξ, X), though ϱ is not a metric on X. A routine verification shows that ϱ defines N_ξ in X. Therefore, *N_ξ is T_2-potentially metrizable.* It follows that N_ξ is sequential in X (see Proposition 5). Thus, the space *N_ξ is T_2-potentially sequential.*

Question 12. Is the space N_ξ T_3-potentially metrizable? Is it T_3-potentially sequential? Is N_ξ potentially metrizable (potentially sequential) in the class of Tychonoff spaces?

Another notion worthy of further investigation. Let d and ϱ be symmetrics on (Y, X). We shall say that d and ϱ *are equivalent* (on (Y, X)), if whenever $y \in Y$ and $x \in X$, $d(x, y) = \varrho(x, y)$.

Note 6. If symmetrics d and ϱ on (Y, X) are equivalent, and one of them defines (properly defines) Y in X, then the other one also defines (properly defines) Y in X. This is obvious from the definitions.

Question 13. Let d be a 2-metric on (Y, X). When is there a 1-metric ϱ on (Y, X), which is equivalent to d?

Question 14. Let d be a symmetric (a metric) on (Y, X), which defines Y in X. When is there a symmetric (a metric) ϱ on (Y, X), which is equivalent to d and generates the topology of the space X?

A very weak form of relative metrizability can be defined as follows. Let us say that ϱ is *a weak metric on* (Y, X), if ϱ is a symmetric on (Y, X) and the restriction of ϱ to Y is a metric on the set Y. We say that Y is *weakly metrizable in* X, if there is a weak metric ϱ on (Y, X), which defines Y in X. Note, that in the definitions above, we do not require that the restriction of ϱ to Y generates the topology given on Y — this need not be the case.

There are a few interesting open questions on the relationship between cardinal invariants in relatively metrizable spaces.

Question 15. Let Y be metrizable in X. Is then true that the following implications hold:

a) If $c(Y) \leq \omega$, then Y is separable?
b) If Y is separable, then Y is Lindelöf?
c) If Y is Lindelöf then Y has a countable network?

Question 16. What are the answers to questions 15, if we assume only that Y is weakly metrizable in X?

Clearly, many of the questions formulated above should be considered under various separation restrictions on X and Y. This is especially true for Questions 15 and 16.

One other direction for investigation could be to combine relative metrizability with relative dimension invariants.

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