Sequential continuity on dyadic compacta and topological groups

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Abstract. We study conditions under which sequentially continuous functions on topological spaces and sequentially continuous homomorphisms of topological groups are continuous.

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§1. Introduction

It is well known that the sequential continuity of a real-valued function defined on a topological space is in general far too weak to imply its continuity. The problem whether sequentially continuous functions on a product of separable metric spaces are continuous is much more delicate.

An investigation originated by Mazur [M] and continued by Noble [N] and Antonovskiĭ and Chudnovskĭ [AC] brought out several interesting results. For instance, the question whether the product of κ separable metric spaces is a space in which sequential continuity suffices for continuity appears to be equivalent to the question whether the Cantor cube D^τ has that property. Moreover, unless certain large cardinals exist, D^τ does have the property for every cardinal number τ. On the other hand, if τ is a real-valued measurable cardinal, then a universal measure on τ provides an example of a sequentially continuous function on D^τ which is not continuous. This suggests that one should not expect the above problem to be decidable within the usual axioms of set theory (see also [P1] and [P2]).

In Section 1 of this article we establish certain conditions under which a sequentially continuous mapping is continuous. We also construct an example destroying some overly optimistic expectations in this respect: It turns out that being a homomorphism of topological groups is not among such conditions, even if the domain is compact. This example sheds new light on a classical result of Varopoulos [Vs], and also leads to a partial solution of a problem posed by Comfort and Remus [CR].

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Section 2 contains a discussion of the role of large cardinals and of compactness-like properties of the sequential leader of D^τ.

§2. Preliminaries

We follow notation and terminology in [E] and [A1]. A *space* means a topological space. Symbols X, Y, Z always stand for topological spaces. Topological groups are assumed to be T_1-spaces, which implies that they are Tychonoff spaces (see [Pn], [RD]).

A space of *point-countable type* is a Hausdorff space X such that for each point $x \in X$ there is a compact subspace F of X satisfying the conditions: $x \in F$, and $\chi(F, X) \leq \omega$, where the last condition means that there is a countable base of neighbourhoods for F in X.

A *Baire subset* of a space X is any member of the σ-algebra generated by the zero-subsets of X.

A space of *countable pseudocharacter* is a space in which all points are G_δ's, and the *tightness* of X is countable if every point in the closure of any subset A of X is in the closure of some countable subset of A.

A cardinal is identified with the smallest ordinal corresponding to it, and an ordinal is considered as the set of all smaller ordinals; $\mathcal{P}(\tau)$ stands for the set of all subsets of τ.

A topological group G is said to be *ω-bounded* (see [A3]), if for each nonempty open subset U of G, there is a countable subset $A \subset G$ such that the set $U \circ A = \{b * a : b \in U, a \in A\}$ coincides with G.

A *dyadic compactum* is a compact Hausdorff space which can be represented as a continuous image of some Cantor cube D^τ. Every compact topological group is a dyadic compactum, according to a famous theorem proved independently by Ivanovskiĭ [Iv] and Kuz'minov [Ku].

We call a (not necessarily Tychonoff) space X *pseudocompact*, if every discrete in X family of nonempty open subsets of X is finite. For Tychonoff spaces this is equivalent to the requirement that every continuous real-valued function on X is bounded.

Recall that a subset A of a topological space X is called *sequentially closed* if it contains a limit of every converging sequence $(x_n)_{n\in\omega}$ of its elements. For every $A \subseteq X$ we denote by A^{seq} the set of all points that are limits of converging sequences from A. (Note that A^{seq} need not be sequentially closed!).

A real-valued function g defined on X is said to be *sequentially continuous* if $\lim_{n\to\infty} g(t_n) = g(t)$ for every (ordinary) sequence $(t_n)_{n\in\omega} \subseteq X$ that converges to t. If g is such a function, then $g^{-1}(F)$ is sequentially closed whenever F is a closed set in the real line.

The space D^κ will sometimes be identified with the power set of κ *via* the mapping $\chi_a \longleftrightarrow a$, $a \subseteq \kappa$. Thus subsets of κ are elements of D^κ, and are denoted by lower case letters.

The subset $A \subseteq D^\kappa$ is said to *depend on a set* $i \subseteq \kappa$ (of coordinates) provided that $a \in A$ and $b \cap i = a \cap i$ imply $b \in A$. It is well known that every zero-set in D^κ depends on a countable set (see [M]). Every subset of D^κ of the form $\{a:\ i \subseteq a \subseteq j^c\}$, where $i, j \subseteq \kappa$ are finite, is called an *elementary set* (here and below Y^c stands for the complement of a set Y). Elementary subsets of D^κ form a standard base for the usual topology of D^κ.

Note that $A^{seq} = \overline{A}$ for every subset of D^κ depending on a countable set $i \subseteq \kappa$. Indeed, let $a \in \overline{A}$. Write i as an increasing union of finite sets i_n and choose, for every n,

$$a_n \in A \cap \{x:\ i_n \cap a \subseteq x \subseteq (i_n \setminus a)^c\}.$$

Put $s_n = (a_n \cap i) \cup (a \setminus i)$; it follows that s_n's form a sequence in A that converges to a.

A measure μ defined on the Borel σ-algebra of a topological space X is said to be *strictly positive* provided $\mu(V) > 0$ for every nonempty open subset V of X. A measure μ is *completion-regular* if for every Borel set B there are Baire sets A_1, A_2 with $A_1 \subseteq B \subseteq A_2$ and $\mu(A_1) = \mu(A_2)$. Since every finite measure is inner-regular on the σ-algebra of Baire sets, the completion-regularity of μ means that for every Borel set B with $\mu(B) > 0$ there exists a zero-set $Z \subseteq B$ such that $\mu(Z) > 0$ (see [Wh]).

The usual product measure on D^κ is strictly positive and completion-regular. This is a particular case of a theorem due to Kakutani stating that a product of any family of strictly positive Borel measures defined on metric spaces is completion-regular. Recall also that the Haar measure on a compact topological group has that property (see [Wh] for references, and [Gr] for more recent results on completion-regularity).

§3. Sufficient and insufficient conditions for continuity of sequentially continuous maps

1. Theorem. *Let X be a dyadic compactum, and let f be a sequentially continuous mapping of X onto a space Y of countable pseudocharacter. Let us also assume that, for each y in Y, the preimage of y under f is a space of countable pseudocharacter. Then the mapping f is continuous, and the spaces X and Y are metrizable.*

Proof: We only have to show that X is metrizable. This already implies that f is continuous, and that Y is a metrizable compact space. Let us assume that X is not metrizable. Then X contains a topological copy of the space D^{ω_1} (see, e.g. Theorem 1 (iv) of [H]). It follows that there exists a nonempty countably compact sequential subspace Z of X such that none of the points in Z is a G_δ-point in Z. Indeed, take a copy of any Σ-product in D^{ω_1}. Let g be the restriction of f to Z. Since the space Z is sequential, g is a continuous mapping of Z onto a subspace M of Y. Take any point y of M. Since $\{y\}$ is a G_δ-subset of M, and g is continuous, $g^{-1}\{y\}$ is a G_δ-set in Z. Clearly, $g^{-1}\{y\}$ is contained in $f^{-1}\{y\}$. Therefore, each point of the space $g^{-1}\{y\}$ is a G_δ-subset of it. It follows that every point of the nonempty set $g^{-1}\{y\}$ is a G_δ-subset of Z — a contradiction with the choice of Z. □

2. Corollary. *Every sequentially continuous one-to-one mapping of a dyadic compactum X onto a first countable T_1-space Y is continuous, X is metrizable, and Y is either not Hausdorff, or also metrizable.*

3. Theorem. *Let f be a sequentially continuous homomorphism of a topological group G of point-countable type into a countably tight topological group F such that the kernel $A = f^{-1}\{e\}$ is closed. Then f is continuous.*

PROOF: Let $H = G/A$ be the topological quotient group of G with respect to A, and let g be the quotient homomorphism of G onto H. Then g is continuous and there is a unique isomorphism h of H onto F such that $f = h \circ g$.

Let $\{y_n : n \in \omega\}$ be any sequence in H, converging to a point $y \in H$. Then there is a sequence $\{x_n : n \in \omega\}$ in G converging to a point x in G, such that $g(x_n) = y_n$ for each $n \in \omega$ (see Theorem 3 of [Pas]). It follows that the mapping h of H into F is sequentially continuous. Clearly, H is a topological group of point-countable type. Therefore, there is a compact subgroup P of H such that the set P has a countable base of neighbourhoods in H. The restriction of h to P is a continuous mapping by the following lemma:

4. Lemma. *Let f be a sequentially continuous mapping of a dyadic compactum X onto a Hausdorff space Y such that the tightness of every compact subspace of Y is countable. Furthermore, assume that $t(f^{-1}\{y\}) = \omega$ for each y in Y. Then f is continuous, and the spaces X and Y are metrizable.*

PROOF: It is enough to prove that X is metrizable. Let us assume the contrary. Then the space X contains a topological copy Z of the space D^{ω_1}. By a theorem of Mazur [M], the restriction of the mapping f to Z is continuous. Then $F = f(Z)$ is compact; therefore $t(F) = \omega$. Also, $t(g^{-1}\{y\}) = \omega$ for every $y \in F$. Thus by Proposition 4.5 in [A1], $t(Z) = \omega$. Since every dyadic compactum of countable tightness is metrizable (see [E]), we get a contradiction. □

By Lemma 4, P is metrizable. Since P has a countable base of neighbourhoods in H, it follows that H is first countable. Thus f is continuous. □

5. Theorem. *If f is a sequentially continuous homomorphism of a topological group G of point-countable type into a countably tight group F such that the kernel $K = f^{-1}(e)$ contains a nonempty Baire subset of G, then f is continuous.*

PROOF: Fix a compact subspace T of G which contains e and has a countable base of neighbourhoods in G. Since K contains a nonempty Baire subset G, there is a nonempty G_δ-subset P of G contained in K. Clearly, we can assume that P is also contained in T. Moreover, we can assume that P is closed and is a subgroup of G (see [Pn], [RD]). Take the quotient space G/P (which is not necessarily a topological group). It is first countable, since P has a countable base of neighbourhoods in G (by transitivity of character in Tychonoff spaces, see [A3]). The natural mapping of G/P into F is well-defined (since $P \subset K$), is sequentially continuous by Theorem 3 of [Pas], and thus is continuous, since G/P

is first countable. Therefore f is a composition of two continuous mappings and is continuous. $\square$

The following example shows that some assumptions on the kernel are necessary in Theorems 3 and 5.

6. Theorem. *Let $\kappa \leq 2^{\aleph_0}$ be a real-valued measurable cardinal. Then there exists a sequentially continuous homomorphism g of D^κ onto a metrizable topological group G that is not pseudocompact.*

Theorem 6 can be proved quite easily using seminorms and the result of Varopoulos [Vs]. The reader interested in this approach and in related results is referred to [U] and [Hu]. Here we present a direct proof that does not depend on Varopoulos's result. The construction of topological groups used in this argument will also be needed in the proof of Theorem 16.

Let κ be an uncountable cardinal, and let λ be an ordinal. Consider a collection $\mathcal{M} = \{\mu_\xi : \xi < \lambda\}$ of probability measures on $\mathcal{P}(\kappa)$. For $\xi < \lambda$, $\varepsilon > 0$ and $X \subseteq \kappa$, let $U(X,\xi,\varepsilon) = \{Y \subseteq \kappa : \mu_\xi(X\Delta Y) < \varepsilon\}$. Let $\mathcal{T}_{\mathcal{M}}$ (or just $\mathcal{T}$ if no confusion can arise) be the topology on $\mathcal{P}(\kappa)$ generated by the subbase $\{U(X,\xi,\varepsilon) : X \subseteq \kappa, \xi < \lambda, \varepsilon > 0\}$, and let $\mathcal{S}$ be the usual topology on $\mathcal{P}(\kappa)$ (i.e. $\mathcal{S}$ is obtained by identifying $\mathcal{P}(\kappa)$ and D^κ with the product topology).

Define an equivalence relation $\sim$ on $\mathcal{P}(\kappa)$ by: $X \sim Y$ iff $\forall\, \xi < \lambda\, \mu_\xi(X\Delta Y) = 0$. Denote $G = \mathcal{P}(\kappa)_{/\sim}$, and let $h : \mathcal{P}(\kappa) \to G$ be the quotient mapping. We shall use the same symbol Δ for operations on $\mathcal{P}(\kappa)$ and G; if $X_{/\sim}, Y_{/\sim} \in G$, then $X_{/\sim}\Delta Y_{/\sim} = (X\Delta Y)_{/\sim}$. Clearly, Δ is a well-defined operation on G. We also shall use the same symbol $\mathcal{T}$ for the quotient topology on G and for the topology on $\mathcal{P}(\kappa)$.

7. Lemma. (a) *The map $id : (\mathcal{P}(\kappa),\mathcal{S}) \to (\mathcal{P}(\kappa),\mathcal{T})$ is sequentially continuous.*

(b) *$((\mathcal{P}(\kappa),\Delta),\mathcal{T})$ and $((G,\Delta),\mathcal{T})$ are topological groups.*

(c) *$h \circ id$ is a sequentially continuous group homomorphism from $((\mathcal{P}(\kappa),\Delta),\mathcal{S})$ onto $((G,\Delta),\mathcal{T})$.*

(d) *If $\lambda < \aleph_1$, then G is metrizable.*

Proof: Note that if $X = \lim_{n\to\infty} X_n$ (in the sense of $\mathcal{S}$), and μ is a probability measure on $\mathcal{P}(\kappa)$, then $\lim_{n\to\infty} \mu(X_n) = \mu(X)$ [Hal]. Using this fact, the verification of (a) is straightforward.

In order to prove that $((\mathcal{P}(\kappa),\Delta),\mathcal{T})$ is a topological group, first note that $X^{-1} = \kappa\Delta X$ for every $X \subseteq \kappa$. Therefore, it suffices to show that the function $\Delta : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ is continuous with respect to $\mathcal{T}$. Now it suffices to note that if μ is a measure on κ, then the function $\varrho_\mu : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ defined by $\varrho_\mu(X,Y) = \mu(X\Delta Y)$ is a pseudometric on $\mathcal{P}(\kappa)$ (see [Hal, §40]), $\mathcal{T}$ is the topology generated by the collection of pseudometrics $\{\varrho_{\mu_\xi} : \xi < \lambda\}$, and the function $\Delta : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ is continuous with respect to every pseudometric ϱ_{μ_ξ}. For a similar reason, $((G,\Delta),\mathcal{T})$ is a topological group.

The verification of (c) is left to the reader.

The result of (d) will only be used if $\lambda = 1$. In this case it suffices to note that ϱ_{μ_0} is a metric on G, and $\mathcal{T}$ is induced by this metric. $\square$

Proof of Theorem 6: Let κ be as in the assumption. For convenience of notation, we shall identify D^κ with $\mathcal{P}(\kappa)$.

Let $\mu : \mathcal{P}(\kappa) \to [0,1]$ be a countably additive probability measure such that $\mu(\{\alpha\}) = 0$ for each $\alpha < \kappa$. Let $\mathcal{T} = \mathcal{T}_{\{\mu\}}$ be the topology on $\mathcal{P}(\kappa)$ defined as in the discussion preceding Lemma 7, let G be the corresponding topological group, and h the quotient mapping. Let $g = h \circ id$.

Then g is a sequentially continuous group homomorphism by Lemma 7(c), and G is metrizable by Lemma 7(d). So we will be done if we prove the following:

8. Claim. *$(G, \mathcal{T})$ is not pseudocompact.*

Proof: Since $(G, \mathcal{T})$ is metrizable, it is enough to show that it is not countably compact. There exists a sequence $(X_n)_{n\in\omega}$ of subsets of κ such that $\mu(X_n) = 0.5$ and

$$(+) \qquad \mu(X_n \Delta X_m) = 0.5 \quad \text{for all} \quad n < m < \omega.$$

This is true in general, and can be very easily observed if we assume that μ extends Lebesgue measure (this does not lead to loss of generality; see [J, p. 302]): In this case, let $(X_n)_{n\in\omega}$ be a standard Rademacher sequence.

But note that the X_n's form an infinite collection A of points whose distance from each other is 0.5 in the metric ϱ_μ. Since $\mathcal{T}$ is induced by this metric, A is a closed discrete subspace of $(G, \mathcal{T})$. Therefore, the space $(G, \mathcal{T})$ is not countably compact. $\square$

The group G constructed in our proof of Theorem 6 is not separable. The density of the space $(G, \mathcal{T})$ is equal to the Maharam type of the measure μ (see [F1, 2.20]). By a theorem of Gitik and Shelah, the Maharam type of μ is at least as big as $\min\{\kappa^{+\omega}, 2^\kappa\}$ (see [F2] for an elementary proof). Hence the following question arises:

Problem 1: What is the smallest possible density of a group G as in Theorem 6? In particular, can one construct an example as in Theorem 6 such that G is separable (but still not compact)?

9. Remark. The map g in the above example is open. Indeed, since

$$\forall\, x, y \in \mathcal{P}(\kappa)\ (|x \Delta y| < \aleph_0 \Rightarrow g(x) = g(y)),$$

the preimage of any point of G under g is dense in $\mathcal{P}(\kappa)$. This implies that g maps every nonempty open subset of $\mathcal{P}(\kappa)$ onto G. Thus g is open.

10. Theorem. *Let f be a sequentially continuous mapping of a dyadic compactum X onto a space Y of countable pseudocharacter such that the preimages of all points are closed. Then f is continuous.*

Proof: The space X is a quotient space of some Cantor cube D^τ under a (continuous) mapping h, and it suffices to show that the composition mapping $g = f \circ h$

is continuous. Clearly, the images of points under g are closed subsets of D^τ, and g is sequentially continuous. Therefore, we may assume that f maps D^τ onto Y, where Y has countable pseudocharacter. If f is sequentially continuous, then f is continuous on the sigma-product $\Sigma = \{f \in D^\tau : |f^{-1}\{1\}| \leq \aleph_0\}$, and hence $f|\Sigma$ depends only on a countable set of coordinates A (see [E]). Now suppose that the fibres of f are closed, and let $y \in Y$. Let $f(x) = y$, and let $p = x|A$. Then $x \in cl(Z)$, where $Z = \{z \in \Sigma : z|A = p\}$. But $cl(Z)$ is all contained in one fibre, by the choice of A. Thus, the whole function f depends only on the coordinates from the countable set A; hence f is continuous. $\square$

We conclude this section with a fairly general measurability condition that implies continuity of a sequentially continuous mapping.

Let X be a compact (Hausdorff) topological space. We denote by $\mathcal{Z}(X)$ the family of all zero subsets of X; we also put $\mathcal{Z}^+(X) = \mathcal{Z}(X)\backslash\{\emptyset\}$.

We define two families of subsets of X in the following way:

$$\mathcal{S}_0(X) = \{A \subseteq X : \forall V \ (V \cap A \neq \emptyset \Rightarrow \exists Z \in \mathcal{Z}^+(X) \ Z \subseteq V \cap A)\},$$
$$\mathcal{S}(X) = \{A \subseteq X : \forall V \, \exists Z \in \mathcal{Z}^+(X) \ (Z \subseteq V \cap A \vee Z \subseteq V \cap A^c)\},$$

where V denotes an open subset of X.

We shall list some basic properties of the families $\mathcal{S}(X)$ and $\mathcal{S}_0(X)$ thus defined.

(i) $A \in \mathcal{S}_0(X)$ if and only if there exists a family $\mathcal{Z} \subseteq \mathcal{Z}(X)$ such that $\bigcup \mathcal{Z} \subseteq A \subseteq \overline{\bigcup \mathcal{Z}}$;

(ii) $\mathcal{S}_0(X)$ contains all zero-sets and all open sets in X, and is closed under arbitrary unions;

(iii) $\mathcal{S}(X)$ is closed under complements;

(iv) every subset of X having the Baire property belongs to $\mathcal{S}(X)$;

(v) $\mathcal{S}(X)$ contains μ-measurable sets for every completion-regular and strictly positive measure μ on X.

Indeed, it is immediate from the definition of $\mathcal{S}_0(X)$ that if $A \in \mathcal{S}_0(X)$, then the union of all zero-sets contained in A is dense in A. On the other hand, if $\bigcup \mathcal{Z} \subseteq A \subseteq \overline{\bigcup \mathcal{Z}}$, then every open set V with $A \cap V \neq \emptyset$ meets some $Z \in \mathcal{Z}$; since $Z \cap V$ contains a non-empty zero-set, so does $V \cap A$.

Properties (ii) and (iii) are clear. To check (iv) it suffices to notice that if V is open and non-empty, and N is a set of the first category in X, then $V \setminus N$ contains some $Z \in \mathcal{Z}^+(X)$.

Let μ be a measure as in (v) and let A be measurable. If V is open and nonempty, then either $\mu(V \cap A) > 0$ or $\mu(V \setminus A) > 0$, so by completion-regularity we can find a non-empty zero-set either in $V \cap A$ or in $V \setminus A$.

Let us remark that $\mathcal{S}_0(X)$ is the whole power set of X whenever X is first-countable (since then singletons are zero-sets). It is also worth noticing that $\mathcal{S}(X)$ may be fairly large in other cases.

For instance, *every* subset A of $D^{\mathbf{c}}$ (where $\mathbf{c}$ is the cardinality of the real line) can be written as $A = B \cup C$ with $B, C \in \mathcal{S}(D^{\mathbf{c}})$. Indeed, there is a decomposition $D^{\mathbf{c}} = M \cup N$, where N is of first category and M is a null-set with respect to the usual product measure λ (which is strictly positive and completion-regular). It follows from the remarks above that $A = (A \cap M) \cup (A \cap N)$ is the decomposition as required. On the other hand, if P is the family of all countable subsets of $\mathbf{c}$, then $P \notin \mathcal{S}(D^{\mathbf{c}})$. This shows that $\mathcal{S}(X)$ need not be closed under finite unions.

In the sequel, we shall say that a real-valued function g defined on a space X is $\mathcal{S}(X)$-*measurable* provided $g^{-1}(H) \in \mathcal{S}(X)$ for very open subset H of the reals (equivalently: for every closed H). It is clear from the above observations that every function having the Baire property is $\mathcal{S}(X)$-measurable, and so is every function which is measurable with respect to some strictly positive and completion-regular measure.

11. Proposition. *If X is a dyadic compactum and $A \in \mathcal{S}_0(X)$, then the sequential closure and the closure of A coincide.*

Proof: The substantial part of the argument is contained in the following well-known fact (see [E1], [EK]):

12. Lemma. *Let $\mathcal{D}$ be a family of subsets of D^κ such that every $D \in \mathcal{D}$ depends on a countable set of coordinates. Then there exists a countable subfamily $\mathcal{D}_0 \subseteq \mathcal{D}$ such that $\overline{\bigcup \mathcal{D}_0} = \overline{\bigcup \mathcal{D}}$.*

In order to prove Proposition 11, suppose that we have a non-empty set $A \in \mathcal{S}_0(X)$, and let us fix a continuous surjection θ from D^κ onto X. We let $\mathcal{Z}$ to be the family of all zero-sets in X that are contained in A, and put $\mathcal{D} = \{\theta^{-1}(Z) : Z \in \mathcal{Z}\}$. Since every set of the form $\theta^{-1}(Z)$, where $Z \in \mathcal{Z}(X)$, is a zero subset of D^κ, it depends on a countable subset of κ. Now it follows from Lemma 12 and the remark from Section 2 that $\overline{\bigcup \mathcal{D}} = (\bigcup \mathcal{D})^{seq}$. Since the union of $\mathcal{Z}$ is dense in A (see (i) above), we get

$$\overline{A} = \overline{\bigcup \mathcal{Z}} = \overline{\theta\left(\bigcup \mathcal{D}\right)} = \theta\left(\overline{\bigcup \mathcal{D}}\right) = \theta\left(\left(\bigcup \mathcal{D}\right)^{seq}\right) \subseteq \left(\theta\left(\bigcup \mathcal{D}\right)\right)^{seq} \subseteq A^{seq},$$

and we are done. $\square$

13. Theorem. *Let g be a sequentially continuous real-valued function defined on a dyadic compactum X. If g is $\mathcal{S}(X)$-measurable, then g is continuous.*

Proof: We shall check first that

(1) $\quad g^{-1}(H) \in \mathcal{S}_0(X)$ whenever $H \subseteq \mathbf{R}$ is open.

Let V be open subset of X; suppose that $g^{-1}(H) \cap V$ contains no elements of $\mathcal{Z}^+(X)$. It follows that $g^{-1}(H^c) \cap V \in \mathcal{S}_0(X)$, and $g^{-1}(H^c) \cap V$ is dense in V. Thus, applying Proposition 11 and the fact that $g^{-1}(H^c)$ is sequentially closed, we have

$$V \subseteq \overline{g^{-1}(H^c) \cap V} = (g^{-1}(H^c) \cap V)^{seq} \subseteq g^{-1}(H^c),$$

so $V \cap g^{-1}(H) = \emptyset$.

Now we shall check that

(2) $\overline{g^{-1}(H)} \subseteq g^{-1}(\overline{H})$ for every open H.

Indeed, using (1) and Proposition 11 once again, we can write

$$\overline{g^{-1}(H)} = (g^{-1}(H))^{seq} \subseteq (g^{-1}(\overline{H}))^{seq} = g^{-1}(\overline{H}).$$

Property (2) is equivalent to continuity of g.

Indeed, for a closed set $F \subseteq \mathbf{R}$ we can find open sets H_n such that $F = \bigcap_{n\in\omega} H_n = \bigcap_{n\in\omega} \overline{H_n}$. Thus

$$g^{-1}(F) = \bigcap_{n\in\omega} g^{-1}(\overline{H_n}) \supseteq \bigcap_{n\in\omega} \overline{g^{-1}(H_n)} \supseteq \overline{\bigcap_{n\in\omega} g^{-1}(H_n)} \supseteq \overline{g^{-1}(F)},$$

so the set $g^{-1}(F)$ is closed. □

14. Corollary. *If g is a sequentially continuous function on a compact space X, then g is continuous provided any of the following is satisfied:*

(i) *X is a dyadic compactum and g has the Baire property;*
(ii) *X is a compact group and g is measurable with respect to its Haar measure.*

PROOF: If (i) holds, the assertion follows from Theorem 13, since functions having the Baire property are measurable with respect to $\mathcal{S}(X)$.

Recall again that every compact topological group is a dyadic compactum ([Iv], [Ku]). As we already noticed, functions that are measurable with respect to a strictly positive and completion-regular measure on X are $\mathcal{S}(X)$-measurable. Moreover, the Haar measure is such a one. Thus, by Theorem 13 again, if (ii) holds, then g is continuous. □

§4. The role of large cardinals and compactness-like properties of the sequential leader

Some large cardinal assumption is needed in Theorem 6. This was first observed by Mazur in [M], where it is shown that unless κ is at least as big as the first weakly inaccessible cardinal, all sequentially continuous functions from D^κ into metric (and thus, into Tychonoff) spaces are continuous. The lower bound for κ was later improved by Antonovskiĭ and Chudnovskiĭ [AC] and Ciesielski [C]. It is still unknown whether a real-valued measurable cardinal is needed.

The proof of Theorem 6 can be modified to yield further interesting results. The following theorem answers Question 8 in [A2].

15. Theorem. *If $\kappa \leq 2^{\aleph_0}$ is a real-valued measurable cardinal, then there exists a sequentially continuous isomorphism of the topological group D^κ onto a topological group G that is not ω-bounded.*

Proof: Let f be a sequentially continuous mapping of a (Tychonoff) space X onto a (Tychonoff) space Y. Let $\mathcal{T}_1$ be the smallest (Tychonoff) topology on the set X, containing the topology $\mathcal{T}$ of the space X and all inverse images of open subsets of Y. The identity mapping $i : (X, \mathcal{T}) \to (X, \mathcal{T}_1)$, defined by the rule: $i(x) = x$ for each x in X, is sequentially continuous and one-to-one. Obviously, i is continuous if and only if f is continuous. If X and Y are topological groups, and f is a homomorphism, then $(X, \mathcal{T}_1)$ is also a topological group, and i is an algebraic isomorphism. One can prove Theorem 15 by applying this technique to the example of Theorem 6. Of course, the resulting isomorphism is not continuous. □

W. Comfort and D. Remus asked whether every compact group H of measurable cardinality admits a strictly stronger countably compact group topology of weight $2^{|H|}$ (see Question 5.4(b) of [CR]). The following theorem gives a positive answer for the group $(\mathcal{P}(\kappa), \Delta)$ if κ is strongly compact (note that strongly compact cardinals are measurable, but not vice versa).

16. Theorem. *Let κ be a strongly compact cardinal, and let $\mathcal{S}$ be the usual (product) topology on $\mathcal{P}(\kappa)$. Then there exists an extension $\Theta \supset \mathcal{S}$ such that*

(1) $((\mathcal{P}(\kappa), \Delta), \Theta)$ *is a topological group,*
(2) $(\mathcal{P}(\kappa), \Theta)$ *is countably compact,*
(3) $w((\mathcal{P}(\kappa), \Theta)) = 2^{2^\kappa}$.

Proof: Let κ be a cardinal, and let $\mathcal{M} = \{\mu_\xi : \xi < \lambda\}$ be a family of two-valued κ-additive probability measures on κ such that $\lambda > \kappa$ and for $\alpha < \kappa$ we have $\mu_\alpha(X) = 1$ iff $\alpha \in X$. Let Θ be the topology on $\mathcal{P}(\kappa)$ induced by the subbasis

$$\{\{x \subseteq \kappa : \mu_\xi(x) = 0\}, \{x \subseteq \kappa : \mu_\xi(x) = 1\} : \xi < \lambda\}.$$

It is not hard to see that $\Theta = \mathcal{T}_\mathcal{M}$. Thus, by Lemma 7, $((\mathcal{P}(\kappa), \Delta), \Theta)$ is a topological group. □

17. Lemma. $(\mathcal{P}(\kappa), \Theta)$ *is countably compact.*

Proof: Let $\{x_n : n \in \omega\}$ be a countable infinite subspace of $(\mathcal{P}(\kappa), \Theta)$. We show that the set $\{x_n : n \in \omega\}$ has a cluster point. Let $\mathcal{F}$ be a nonprincipal ultrafilter on ω, and let $e = \lim_\mathcal{F} x_n$ (i.e. $\alpha \in e$ if and only if $\{n \in \omega : \alpha \in x_n\} \in \mathcal{F}$). Consider an arbitrary basic open neighborhood U of e. We may assume that $U = U_0 \cap \ldots \cap U_k$ for some $k \in \omega$, and for each $i \leq k$ there exist $\xi_i < \lambda$ and $\varepsilon_i \in \{0, 1\}$ such that $U_i = \{X \subseteq \kappa : \mu_{\xi_i} = \varepsilon_i\}$. For $A \in \mathcal{F}$, let $e_A = \{\alpha \in \kappa : \forall n \in A\, (\alpha \in e \leftrightarrow \alpha \in x_n)\}$. Clearly, $\bigcup_{A \in \mathcal{F}} e_A = \kappa$. Since $2^{\aleph_0} < \kappa$, by κ-completeness of the measures μ_ξ there exists an $A \in \mathcal{F}$ such that $\mu_{\xi_i}(e_A) = 1$ for all $i \leq k$. It is not hard to see that the latter implies that $x_n \in U$ for all

$n \in \omega$. Since U was arbitrary, we have shown that e is a cluster point of the set $\{x_n : n \in \omega\}$. $\square$

Our work so far did not depend on the size or other properties of the family $\mathcal{M}$ of measures on κ. Now we show that if κ is strongly compact, then we can find a family $\mathcal{M}$ such that the weight of Θ is as large as postulated in Theorem 16(3). We need a technical lemma. In its formulation, the concept of a κ-independent family is used. We call a subfamily $\mathcal{A}$ of $\mathcal{P}(\kappa)$ *κ-independent* if for every subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} = \{B_\eta : \eta < \sigma < \kappa\}$ and every function $h : \sigma \to \{0,1\}$ the intersection $\bigcap_{\eta<\sigma} B_\eta^{h(\eta)}$ has cardinality κ, where $B^0 = B$ and $B^1 = \kappa \backslash B$.

18. Lemma. *Let κ be strongly inaccessible. Then there exists a κ-independent family $\mathcal{A}$ of size 2^κ in $\mathcal{P}(\kappa)$.*

This lemma is a slight variation on the theme of a well-known result of Fichtenholz, Kantorovich and Hausdorff. Before we prove it, we show how it can be used to prove Theorem 16. Let $\{g_\alpha : \alpha < 2^\kappa\}$ be a set of functions from 2^{2^κ} into $\{0,1\}$ that is dense in ${}^{2^{2^\kappa}}D$ with the product topology. Denote $\lambda = 2^{2^\kappa}$. Let $\{h_\xi : \xi < \lambda\}$ be the set of functions from 2^κ into $\{0,1\}$ defined as follows:

$$h_\xi(\alpha) = g_\alpha(\xi).$$

Note that the functions h_ξ are pairwise different: If $\eta \neq \xi$, then there is some α such that $g_\alpha(\eta) = 1 - g_\alpha(\xi)$.

Let $\mathcal{A} = \{A_\alpha : \alpha < 2^\kappa\}$ be a family as in Lemma 18. For each $\xi < \lambda$, let F_ξ be the filter on κ generated by the family $\{A_\alpha : h_\xi(\alpha) = 1\} \cup \{\kappa \backslash A_\alpha : h_\xi(\alpha) = 0\}$. The family $\mathcal{A}$ was chosen in such a way that F_ξ can be extended to a κ-complete filter G_ξ on κ that is uniform (i.e. all elements of G_ξ have cardinality κ). Since κ was assumed to be a strongly compact cardinal, each G_ξ can be extended to a κ-complete nonprincipal ultrafilter H_ξ on κ. For each $\xi < \lambda$, let μ_ξ be the two-valued probability measure associated with H_ξ, let $\mathcal{M} = \{\mu_\xi : \xi < \lambda\}$, and let $\Theta = \mathcal{T}_{\mathcal{M}}$.

We show that $w(\mathcal{P}(\kappa), \Theta) = \lambda$. The inequality $w(\mathcal{P}(\kappa), \Theta) \leq \lambda$ follows immediately from the way the topology was introduced; so we need only prove the inequality $w(\mathcal{P}(\kappa), \Theta) \geq \lambda$. Suppose toward a contradiction that $\mathcal{B}$ is a base for this space of cardinality less than λ. Since the space $\mathcal{P}(\kappa)$ has cardinality less than λ, we can without loss of generality assume that each $B \in \mathcal{B}$ is of the form $B = \{x \subseteq \kappa : \{\alpha_0^B, \ldots, \alpha_i^B\} \subset x \,\&\, \{\alpha_{i+1}^B, \ldots, \alpha_j^B\} \cap x = \emptyset \,\&\, \mu_{\xi_0^B}(x) = \ldots = \mu_{\xi_k^B}(x) = 0 \,\&\, \mu_{\xi_{k+1}^B}(x) = \ldots = \mu_{\xi_\ell^B}(x) = 1\}$ for some $\alpha_0^B, \ldots, \alpha_j^B \in \kappa$ and $\xi_0^B, \ldots, \xi_\ell^B \in \lambda$. Now let $\xi \in \lambda$ be such that $\xi \neq \xi_n^B$ for any $B \in \mathcal{B}$ and $n \in \omega$. Let $V = \{x \subseteq \kappa : \mu_\xi(x) = 1\}$. There must be some $B \in \mathcal{B}$ with $B \subseteq V$. Assume $B = \{x \subseteq \kappa : \{\alpha_0^B, \ldots, \alpha_i^B\} \subset x \,\&\, \{\alpha_{i+1}^B, \ldots, \alpha_j^B\} \cap x = \emptyset \,\&\, \mu_{\xi_0^B}(x) = \ldots = \mu_{\xi_k^B}(x) = 0 \,\&\, \mu_{\xi_{k+1}^B}(x) = \ldots = \mu_{\xi_\ell^B}(x) = 1\}$. By the choice of the functions g_α, there exists some $\alpha \neq \alpha_0^B, \ldots, \alpha_j^B$ such that $g_\alpha(\xi) = 0$, $g_\alpha(\xi_0^B) = \cdots g_\alpha(\xi_k^B) = 0$

and $g_\alpha(\xi^B_{k+1}) = \cdots = g_\alpha(\xi^B_\ell) = 1$. Let $a = (A_\alpha \cup \{\alpha^B_0, \dots, \alpha^B_i\}) \setminus \{\alpha^B_{i+1}, \dots, \alpha^B_j\}$. A straightforward verification of the relevant definitions shows that $a \in B$. On the other hand, $a \notin H_\xi$, and therefore, $a \notin V$. This contradicts the assumption that $B \subseteq V$. □

It remains to prove Lemma 18.

Proof of Lemma 18: It will be convenient to enumerate the elements of the family $\mathcal{A}$ that we are going to construct by functions $g : \kappa \to \{0,1\}$. Also since κ is strongly inaccessible (in fact, it suffices to assume that κ is strongly limit), we can enumerate the family $[\kappa]^{<\kappa} = \{d_\alpha : \alpha < \kappa\}$ of all subsets of κ of cardinality less than κ in such a way that each $d \in [\kappa]^{<\kappa}$ appears cofinally often in the enumeration. Also, we can partition $\kappa = \{b_\alpha : \alpha < \kappa\}$ into pairwise disjoint sets such that $|b_\alpha| = 2^{2^{|d_\alpha|}}$. We enumerate: $b_\alpha = \{\beta^\alpha_H : H : \{0,1\}^{d_\alpha} \to \{0,1\}\}$.

After all these preliminaries we are ready to construct $\mathcal{A} = \{A_g : g \in \{0,1\}^\kappa\}$. We set:

$$\beta^\alpha_H \in A_g \;\Leftrightarrow\; H(g|d_\alpha) = 1.$$

We show that this family is κ-independent. Let $h : C \to \{0,1\}$ be a function defined on a subset C of $\{0,1\}^\kappa$ of cardinality $< \kappa$. Let $d \subset \kappa$ be such that for each pair of different $g_0, g_1 \in C$ there is some $\delta \in d$ with $g_0(\delta) \neq g_1(\delta)$. Let $H : \{0,1\}^d \to \{0,1\}$ be any function such that $H(g|d) = h(g)$ for each $g \in C$. Let α be such that $d = d_\alpha$. Then $\beta^\alpha_H \in \bigcap_{g\in D} A^{h(g)}_g$. Since there are κ many such α's, we are done. □

Let τ be a cardinal. By D^τ_{seq} we denote D^τ with the strongest topology extending the Tychonoff product topology and still preserving convergence of sequences in D^τ. We shall distinguish the two different closure operations in D^τ by writing $cl(X)$ for the closure in the product topology, and $cl_{seq}(X)$ for the closure operation in D^τ_{seq}. The space D^τ_{seq} is called the *sequential leader* of D^τ. This space is Hausdorff, but it need not be a Tychonoff space. In this section we show that unless some large cardinals exist, the space D^τ_{seq} is pseudocompact. Before proving it, let us compare this result with some other compactness-like properties of D^τ.

19. Theorem. *Let τ be a cardinal. The following are equivalent:*

(a) *D^τ_{seq} is countably compact;*

(b) *D^τ_{seq} is sequentially compact;*

(c) *τ is less than the splitting number* $\mathbf{s}$.

Theorem 19 is a minor variation on the theme of Theorem 1 in [B], Theorem 5.12 in [Vn], respectively Theorem 6.1 in [vD]. The definition of the splitting number $\mathbf{s}$ can be found in [vD] or [Vn]. For our purposes it suffices to know that $\mathbf{s}$ is an uncountable cardinal not exceeding $2^{\aleph_0}$. In particular, it follows that $D^{2^{\aleph_0}}_{seq}$ is never countably compact, even if $2^{\aleph_0}$ is as small as ω_1. For pseudocompactness, we get a very different picture.

20. Theorem. (a) *If τ is at least as big as the first real-valued measurable cardinal, then D^τ_{seq} is not pseudocompact.*

(b) *If τ is smaller than the first weakly inaccessible cardinal, then D^τ_{seq} is pseudocompact.*

Proof: Note that (a) follows from Theorem 6: If we consider the function g of this theorem as a function with domain D^τ_{seq}, then g is continuous.

We prove (b) by an argument closely resembling the reasoning in [M]. As in the proof of Theorem 6, it will be convenient to identify elements of D^τ with subsets of τ. Call a subset U of D^τ_{seq} *special* if all finite subsets of τ are in U, and $\tau \notin cl_{seq}U$. The following lemma was first proved in a different terminology by Mazur [M].

21. Lemma. *Suppose κ is such that there exists a sequentially continuous map $f : D^\kappa \to \mathbf{R}$ that is not continuous. Then, for some $\tau \le \kappa$, there exists a special subset U of D^τ_{seq} that is both closed and G_δ in D^τ_{seq}.*

We shall not prove Lemma 21 here. For our purposes, a very similar lemma will be relevant.

22. Lemma. *Suppose κ is such that D^κ_{seq} is not pseudocompact. Then, for some $\tau \le \kappa$, there exists a special open subset U of D^τ_{seq}.*

Proof: Let κ be as in the assumption, and let $\{U_n : \ n \in \omega\}$ be a countable discrete family of nonempty open subsets of D^κ_{seq}. By compactness of D^κ (with the usual product topology), there must be an n such that $cl(U_n)$ has empty interior in the product topology on D^κ. To simplify notation, assume $n = 0$ is such.

Let $x \in U_0$. There is a finite set $F \subset \kappa$ such that if G is a finite subset of $\kappa \backslash F$ and $y \in D^\kappa$ is such that $x\Delta y \subseteq G$, then $y \in U_0$: If not, then we could inductively construct a sequence $(y_k)_{k\in\omega}$ of elements of $D^\kappa \backslash U_0$ such that the sequence $(x\Delta y_k)_{k\in\omega}$ consists of pairwise disjoint finite subsets of κ, and thus $\lim_{k\to\omega} y_k = x$. The latter is impossible though, since U_0 is a neighborhood of x and no $y_k \in U_0$.

Next observe that there is some $y \in D^\kappa \backslash cl(U_0)$ with $y \cap F = x \cap F$ (otherwise U_0 would contain the basic open neighborhood of x in D^κ determined by F). Fix such y. Then $x\Delta y$ is an infinite set. To simplify notation we may assume that $x\Delta y = \tau$ for some infinite cardinal τ. Consider the set

$$V = \{x\Delta z : \ z \in U_0 \ \& \ x\Delta z \subseteq \tau\}.$$

This set is open in D^τ_{seq}, since if $w_n \notin V$ for every $n \in \omega$ and $\lim_{n\to\infty} w_n = w_\omega$ (in D^τ_{seq}), then for each $n \le \omega$ there exists exactly one $z_n \in D^\kappa$ such that $x\Delta z_n = w_n$. Moreover, $\lim_{n\to\infty} z_n = z_\omega$, and $z_\omega \notin U_0$ since U_0 is open in D^κ_{seq}. It follows that $w_\omega \notin V$. A similar reasoning shows that $\tau = x\Delta y \notin cl_{seq}(V)$. Moreover, if G is a finite subset of τ, and $z\Delta x = G$, then $z \in U_0$ (by the choice of F), and thus $G \in V$. We conclude that V is a special open subset of D^τ_{seq}. $\square$

Now let τ_o be the smallest cardinal τ such that there exists a special open subset of D^{τ}_{seq}. In view of Lemma 22, point (b) of Theorem 20 is a consequence of the following.

23. Lemma. *τ_o is a weakly inaccessible cardinal.*

PROOF: The proof is based on the following observations.

24. Claim. *Suppose U is a special subset of some D^{τ}_{seq}, and $X \subset \tau$ is such that $X \notin cl_{seq}U$. Then the family $U \cap \mathcal{P}(X)$ is a special subset of D^{X}_{seq}. Moreover, $U \cap \mathcal{P}(X)$ is open in D^{X}_{seq} whenever U is open in D^{τ}_{seq}.*

PROOF: Follows from the definition of a special subset and the fact that $\mathcal{P}(X)$ can be treated as a subspace of D^{τ}_{seq}. □

25. Claim. *If U is a special open subset of $D^{\tau_o}_{seq}$, and X is a subset of τ_o of cardinality smaller than τ_o, then $X \in U$.*

PROOF: Immediate from Claim 24 and the choice of τ_o. □

26. Claim. *If U is a special open subset of τ_o, and $\mathcal{X}$ is a family of pairwise disjoint nonempty subsets of τ_o such that $|\mathcal{X}| < \tau_o$ and $\bigcup \mathcal{X} = \tau_o$, then there is a finite subfamily $\mathcal{Y} \subseteq \mathcal{X}$ such that $\bigcup \mathcal{Y} \notin cl_{seq}U$.*

PROOF: Suppose $\mathcal{X} = \{X_\alpha : \alpha \in \lambda < \tau_o\}$ is a counterexample. Then the function $h : \mathcal{P}(\lambda) \to \mathcal{P}(\tau_o)$ defined by: $h(Y) = \bigcup\{X_\alpha : \alpha \in Y\}$ is a homeomorphism between $\mathcal{P}(\lambda)_{seq}$ and a closed subspace of $\mathcal{P}(\tau_o)_{seq}$. Let $V = h^{-1}U$. By the assumptions on $\mathcal{X}$, $\lambda \notin cl_{seq}V$, each finite subset of λ is in V, and V is open in $\mathcal{P}(\lambda)_{seq}$. Thus λ contradicts the choice of τ. □

27. Corollary. *τ_o is a regular uncountable cardinal.*

PROOF: Note that every subset of ω is the limit of a sequence of finite subsets of ω. Therefore, $\tau_o > \omega$. Now suppose τ_o is singular, and consider a partition $\mathcal{X}$ of τ_o into pairwise disjoint subsets of smaller cardinality. Let $\mathcal{Y}$ be as in Claim 26. Then $\bigcup \mathcal{Y}$ has cardinality less than τ_o which contradicts Claim 25. □

Since $D^{\aleph_0}_{seq}$ is the same as $D^{\aleph_0}$, there are no special subsets whatsoever of $D^{\aleph_0}_{seq}$. In other words, τ_o is an uncountable cardinal. Thus, the following is the last brick needed to conclude the proof of Lemma 23.

28. Claim. *τ_o is a limit cardinal.*

PROOF: Suppose $\tau_o = \lambda^+$ for some infinite cardinal λ. Let $\{A_{\alpha,\zeta} : \alpha < \lambda, \zeta < \tau_o\}$ be an Ulam matrix of subsets of τ_o. That is,

(A) $A_{\alpha,\zeta} \cap A_{\beta,\zeta} = \emptyset$ for all $\alpha < \beta < \lambda$ and $\zeta < \tau_o$;
(B) $A_{\alpha,\xi} \cap A_{\alpha,\zeta} = \emptyset$ for all $\alpha < \lambda$ and $\xi < \zeta < \tau_o$;
(C) $\bigcup_{\alpha<\lambda} A_{\alpha,\zeta} = \tau_o \backslash (\zeta + 1)$ for all $\zeta < \lambda$.

By Claims 25 and 26, for each $\zeta < \tau$, there exists a subfamily $\mathcal{A}_\zeta \subset \{A_\alpha, \zeta : \alpha < \lambda\} \cup \{\zeta + 1\}$ of size $k(\zeta) < \omega$ such that $\bigcup \mathcal{A}_\zeta \notin cl_{seq}(U)$. (For this proof, we only need that $\bigcup \mathcal{A}_\zeta \notin U$.) Let $(\zeta_n : n \in \omega)$ be an increasing sequence of ordinals less than τ_o such that $k(\zeta_n) = k$ for a fixed k and all $n \in \omega$. Let $\xi < \tau_o$. By (B), ξ belongs to at most k among the sets $\bigcup \mathcal{A}_{\zeta_n}$, and hence there exists a subsequence $(\zeta_{n_k})_{k\in\omega}$ such that either $\lim_{k\to\infty} \bigcup \mathcal{A}_{\zeta_{n_k}} = \emptyset$, or $\lim_{k\to\infty} \bigcup \mathcal{A}_{\zeta_{n_k}} = \bigcup_{k\in\omega} \zeta_{n_k}$. This is impossible, since both $\emptyset \in U$ and $\bigcup_{k\in\omega} \zeta_{n_k} \in U$. Since U is open in D^τ_{seq}, an element of U cannot be the limit of a convergent sequence whose terms are all outside U. □

Problem 2: Determine exactly how large the cardinal τ_o has to be. Is it equal to the first cardinal τ such that there exists a sequentially continuous, discontinuous map from D^τ into some Tychonoff space?

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