Cleavability and divisibility over developable spaces

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Abstract. Some results on cleavability theory are presented. We also show some new [16]'s results.

Keywords: cleavable space, developable space, subdevelopable space, D-regular space, D-completely regular space, D-compact space, D-normal space, divisible space

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0. Introduction and preliminaries

In 1985 Arhangel'skii in [1], [2], introduced various types of cleavability (originally called splittability) of topological spaces as follows.

Let $\mathcal{P}$ be a class of topological spaces and $\mathcal{M}$ a class of continuous mappings (containing all homeomorphisms). Let A be a subset of a space X. X is said to be $\mathcal{M}$-*cleavable over* $\mathcal{P}$ *along* A if there exist a space $Y \in \mathcal{P}$ and a mapping $f \in \mathcal{M}$, $f : X \to Y$, such that $Y = f(X)$ and $A = f^{-1}f(A)$.

If $\mathcal{A}$ is a family of subsets of X, then we shall say that X is $\mathcal{M}$-*cleavable over* $\mathcal{P}$ *along* $\mathcal{A}$ if it is $\mathcal{M}$-cleavable over $\mathcal{P}$ along each $A \in \mathcal{A}$. X is $\mathcal{M}$-cleavable over $\mathcal{P}$ if it is $\mathcal{M}$-cleavable over $\mathcal{P}$ along each $A \subset X$. When $\mathcal{P}$ is the family of all subsets of a given space Y we speak about $\mathcal{M}$-*cleavability of* X *over* Y instead of $\mathcal{M}$-cleavability over $\mathcal{P}$.

If X is $\mathcal{M}$-cleavable over $\mathcal{P}$ along all singletons $\{x\}$, $x \in X$, one speaks about *pointwise* $\mathcal{M}$-*cleavability* (*of* X) *over* $\mathcal{P}$.

When $\mathcal{M}$ is the class of all continuous (open, closed, perfect, ...) mappings, we use the term *cleavable* (*open cleavable, closed cleavable, perfectly cleavable,* ...) *over* $\mathcal{P}$ instead of $\mathcal{M}$-cleavable over $\mathcal{P}$.

In particular, a cleavable space is a space which is cleavable over the class of all separable metrizable spaces (or equivalently over $\mathbb{R}^\omega$, because every separable metrizable space can be embedded into $\mathbb{R}^\omega$). This case is of particular interest. The paper [7] studied cleavability in details and contains many interested results in this connection.

The following two questions concerning cleavability are quite natural:

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General question A. *Which spaces X are $\mathcal{M}$-cleavable over a class $\mathcal{P}$ (along subset of X or along a collection of subsets of X)?*

General question B. *If a space X is $\mathcal{M}$-cleavable over $\mathcal{P}$, which properties X has? Does X belong to $\mathcal{P}$?*

Let us denote that if there exists a continuous bijection from X onto a space $Y \in \mathcal{P}$, then, obviously, X is cleavable over $\mathcal{P}$. In this case one can say that X is *absolutely cleavable over* $\mathcal{P}$. So, cleavability (over $\mathcal{P}$) may be viewed as a generalization of continuous bijection (onto some $Y \in \mathcal{P}$).

A natural question in this connection is: "When cleavability over $\mathcal{P}$ implies the existence of a continuous bijection onto some $Y \in \mathcal{P}$?" Here is the lemma (which is often used for the proofs of many theorems concerning cleavability) about this:

Lemma 0.1 ([2]). *Let τ be a cardinal, $\mathcal{P}$ a class of spaces. Let a space X be cleavable over $\mathcal{P}$. If $\{A_\alpha : \alpha \in 2^\tau\}$ is a collection of pairwise disjoint subsets of X, then there is a family $\{Y_\beta : \beta \in \tau\} \subset \mathcal{P}$ and a continuous mapping $f : X \to \prod\{Y_\beta : \beta \in \tau\}$ such that $A_\alpha = f^{-1}f(A_\alpha)$ for each $\alpha \in 2^\tau$. In particular, if $\mathcal{P}$ is hereditary and τ-multiplicative class, then if a space X of cardinality $\leq 2^\tau$ is cleavable over $\mathcal{P}$, then it is absolutely cleavable over $\mathcal{P}$.*

One of the most important and useful generalization of metrizable spaces are *developable spaces*. Recall that a space X is developable if there exists a countable collection $\{\mathcal{U}_i : i \in \omega\}$ of open covers of X such that for every $x \in X$ the family $\{St(x,\mathcal{U}_i) : i \in \omega\}$ is a local base for X at x. (Here $St(x,\mathcal{U}_i)$ is the union of all members of $\mathcal{U}_i$ containing x.) A space X is *subdevelopable* if it admits a continuous bijection onto a developable T_1-space.

In 1978, H. Brandenburg began the systematic investigation of topological spaces generated by developable spaces (instead of metrizable spaces) and obtained some new classes of spaces, as *D-completely regular, D-regular, D-compact* and so on (for details see Brandenburg's nice survey [10]). Besides, among developable spaces there is an analogue of the real line, in fact a space, denoted by $\mathbb{D}_1$, of cardinality 2^ω whose countable power D_1^ω is universal for the class $\mathcal{D}_c$ of all second countable developable T_1-space (i.e. every second countable developable T_1-space can be embedded into D_1^ω) [10].

In this paper we continue the previous two lines of investigation and study cleavability over the class of developable T_1-spaces (that generalize metrizable spaces) and over the class of second countable developable T_1-space (which generalize separable metrizable spaces); these classes of spaces we shall denote by $\mathcal{D}$, $\mathcal{D}_c$ respectively. We clarify which results concerning cleavability over $\mathbb{R}^\omega$ can be or cannot be generalized to the case of cleavability over $\mathcal{D}$ and over $\mathcal{D}_c$.

Give now some definitions:

Definition 0.2 ([9]–[10]). *A space X is called:*

(1) *D-regular if each point $x \in X$ has a local base consisting of F_σ-sets (not necessarily open);*

(2) *weakly-D-completely regular if it has a base consisting of open F_σ-sets;*

(3) *D-completely regular if it can be embedded into a product of developable T_1-spaces;*

(4) *D-normal (weakly-D-normal) if for every two disjoint closed subsets A and B of X there exists a continuous mapping f from X into some developable T_1-space such that $\overline{f(A)} \cap \overline{f(B)} = \varnothing$ ($f(A) \cap f(B) = \varnothing$);*

(5) *D-compact if every open cover of X has a finite refinement consisting of open F_σ-sets;*

(6) *perfect if every open set is an F_σ-set;*

(7) *R_0 when every open set is an union of closed sets.*

Definition 0.3 ([9]–[10])**.** *A subset A of a topological space X is said to be D-closed iff there exist a continuous mapping (onto) $f : X \to Y$, Y is some developable space, and a closed subset B of Y such that $A = f^{-1}(B)$.*

Remark 1. Every D-closed subset A of X is a G_δ-set of X.

1. Separation axioms and cleavability

It is known that if a space X admits a continuous bijection onto a regular (D-regular) space, then X need be regular (D-regular). In this connection we have the following result.

Proposition 1.1. *A space X is cleavable over the class $\mathcal{P}$ of D-regular (resp. D-completely regular, weakly D-regular) spaces if and only if X admits a continuous bijection onto some space in $\mathcal{P}$ (but X need not be in $\mathcal{P}$).*

It is known that D-complete regularity is not inversely preserved even under open perfect mappings and that weak D-complete regularity is not preserved in the preimage direction by perfect mappings. Perfect preimages of D-normal spaces are not necessarily D-normal (see [10]). However we have the following result:

Proposition 1.2. *If a space X is closed pointwise cleavable over the class $\mathcal{P}$ of D-regular (resp. weakly-D-completely regular) spaces, then $X \in \mathcal{P}$. If X is closed cleavable over the class of all D-completely regular (D-normal) spaces, then X is also D-completely regular (D-normal).*

For a class of spaces the previous result concerning cleavability over the class of weakly D-completely regular may be improved.

Theorem 1.1. *If a hereditary Lindelöf space X is closed pointwise cleavable over the class of all weakly D-completely regular space, then X is subdevelopable.*

2. Concerning cleavability over $\mathcal{D}$ and over $\mathcal{D}_c$

As was mentioned, cleavability of a space over the class $\mathcal{D}_c$ of second countable developable T_1-space is equivalent to the cleavability of that space over D_1^ω. However, this cleavability is equivalent to cleavability over each of the following two classes of spaces:

(i) the class of all second countable weakly D-completely regular T_1-spaces,

(ii) the class of all second countable D-regular T_1-spaces.

That follows from the fact that these two classes of spaces coincide with the class $\mathcal{D}_c$.

Now we shall give some results regarding cleavability over D_1, $\mathcal{D}$ and $\mathcal{D}_c$.

Proposition 2.1. *If a space X is pointwise cleavable over the class $\mathcal{D}$ (or over D_1), then X is a T_1-space of countable pseudocharacter. If X is closed pointwise cleavable over D_1, then X is a first countable space.*

Proposition 2.2. *If a space X is perfectly cleavable over $\mathcal{D}$ (over $\mathcal{D}_c$ or over D_1), then X belongs to $\mathcal{D}$ ($\mathcal{D}_c$).*

The following three results are related to **General question A**.

Proposition 2.3. *Every space X is cleavable over $\mathcal{D}$ (over D_1) along each D-closed set (and thus along each D-open set).*

Since every closed set in a perfect space is D-closed (in fact, a G_δ-set), we have:

Proposition 2.4. *Every perfect space is cleavable over $\mathcal{D}$ along each closed set (and thus, along each open set).*

Theorem 2.6. *Every perfect weakly D-completely regular Lindelöf space X is cleavable over $\mathcal{D}_c$ along any disjoint family of open subsets of X.*

As a nice application of this theorem we have the following result:

Corollary 2.7. *Let a perfect weakly D-completely regular Lindelöf space X admits a perfect mapping onto a space $\mathcal{D}_c$. Then $c(X) \leq \omega$.*

Now we give some results devoted to **General question B**.

Proposition 2.8. *If a Lindelöf space X is cleavable over $\mathcal{D}_c$, then X is a subdevelopable T_1-space (and thus a G_δ-diagonal).*

Proposition 2.9. *A regular Lindelöf space is cleavable over the class $\mathcal{D}_c$ if and only if it is cleavable over the class of separable metrizable spaces.*

In [7] it was shown that every compact cleavable space is metrizable. Now we give a generalization of that result.

Theorem 2.10. *If a H-closed space X is closed cleavable over the class $\mathcal{D}_c$, then X is subdevelopable.*

Corollary 2.11. *If a minimal Hausdorff space X is closed cleavable over the class of second countable developable T_2-spaces, then X is developable.*

Recall that a subset A is called D-*embedded* if every continuous mapping f from A into D_1 can be extended to a continuous mapping $F : X \to D_1$ such that $F|A = f$. The following two results should be compared with the corresponding result in [7] concerning cleavability over $\mathbb{R}^\omega$.

Theorem 2.12. *Let X be the union of an increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ of D-closed of X. If every X_n is cleavable over $\mathcal{D}_c$, then X is also cleavable over $\mathcal{D}_c$.*

Theorem 2.13. *Let X be a D-completely regular space. If $X = \bigoplus\{X_\alpha : \alpha \in 2^\omega\}$ and every X_α is cleavable over $\mathcal{D}_c$, then X is also cleavable over $\mathcal{D}_c$.*

<u>ONE RESULT ON DIVISIBILITY</u>. We recall the following

Definition 2.11 ([3]). *Let X be a topological space and A be a subset of X. We say that a family $\mathcal{S}_A$ of subsets of X is a divisor* (*or separator*) *for A if for every $x \in A$ and every $y \in X - A$ there exists $S \in \mathcal{S}_A$ such that $x \in S$ and $y \notin S$. If all members of $\mathcal{S}_A$ are open* (*closed*) *in X, then we say that $\mathcal{S}_A$ is an open* (*closed*) *divisor for A. We say also that a space X is divisible if for every $A \subset X$ there is a countable closed divisor for A.*

Now we shall see one relation between divisibility and cleavability over the class $\mathcal{D}_c$. Kočinac remarked that a perfectly normal space is divisible if and only if it is cleavable (over $\mathbb{R}^\omega$). Here we have:

Theorem 2.12. *A perfect space X is divisible if and only if X is cleavable over $\mathcal{D}_c$.*

3. Some open problems

The following questions remain open.

Question 3.1. *Characterize spaces which are cleavable over D_1 or over the class $\mathcal{D}_c$.*

Question 3.2. *If spaces X and Y are cleavable over $\mathcal{D}$ or over $\mathcal{D}_c$, is then the product $X \times Y$ cleavable over the same class?*

Question 3.3. *Characterize D-completely regular spaces X whose D-compactification is cleavable over $\mathcal{D}_c$ or over $\mathbb{R}^\omega$ along X.*

This problem is related to the fact that every D-completely regular space has a T_1-D-compactification (i.e. a D-compact space in which it is dense) [10].

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