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Varadhan's theorem for capacities

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Abstract. Varadhan's integration theorem, one of the corner stones of large-deviation theory, is generalized to the context of capacities. The theorem appears valid for any integral that obeys four linearity properties. We introduce a collection of integrals that have these properties. Of one of them, known as the Choquet integral, some continuity properties are established as well.

Keywords: capacities, large deviations, Choquet integral, Varadhan's integration theorem

Classification: 28A12, 28A25, 60F10

1. Introduction

A large-deviation principle for a sequence of probability measures gives the exponential rate at which the probability of various sets (or events) tends to 0. A formal statement expresses convergence of vanishing powers of the measures to a lower limit on open and an upper limit on closed sets, quite similar to weak convergence of probability measures. This resemblance was elaborated upon by O'Brien and Vervaat (1991). They noticed that the set C of capacities contains the positive powers of tight probability measures, as well as their limits in large-deviation principles. Here, capacities are monotone, inner and outer regular set functions on a topological space. Capacities have been studied by Norberg (1986), Vervaat (1988) and Norberg and Vervaat (1989) from the viewpoint of random closed sets and extremal processes. Two topologies have been defined on C. The narrow topology extends weak convergence of measures as in Billingsley (1968). The vague topology extends vague convergence of Radon measures as in Berg, Christensen and Ressel (1984). A large-deviation principle can be seen as an instance of $c_n^{\alpha_n} \to c$ in C, where $\alpha_n \downarrow 0$. Thus the space of capacities is the natural topological setting for large-deviation theory.

This view on large deviations has been developed further. In O'Brien and Vervaat (1993) a topological proof is given for some well-known large-deviation results. The new proofs show the advantages of a topological approach. In the same vein we shall generalize Varadhan's theorem in this article. The main ideas of the capacity approach have strongly influenced researchers from fields as diverse as mathematical physics (John Lewis and Charles Pfister) and statistics (Paul Deheuvels).

Varadhan (1966) showed that from a large-deviation principle for probability measures (μ_n) the exponential asymptotic behaviour of integral transforms of μ_n

can be deduced. He proved that, under certain conditions, it follows from $\mu_n^{\alpha_n} \to c$ and $f_n^{\alpha_n} \to f$ that

$$\left(\int_{\cdot} f_n \, d\mu_n\right)^{\alpha_n} \to \sup_{e \in \cdot} f(e)c(\{e\}), \tag{1}$$

where the conclusion may be interpreted as convergence of the indefinite integrals. In this article Varadhan's theorem is extended to the context of capacities. It is presented as a combination of one-sided results, which we shall use in later work to prove a general mixing theorem for large-deviation principles. In turn, this will be used to prove large-deviation results for bootstrap quantities and to obtain a generalization of the Freidlin-Wentzell theorem (cf. Gerritse (1995c)).

In order to prove Varadhan's theorem for capacities we need a theory of integration with respect to capacities. However, Varadhan's theorem is not demanding as far as the type of integral is concerned. In the traditional case only four properties of the Lebesgue integral are needed, and these will also suffice for proving the generalization. Therefore we shall prove Varadhan's theorem assuming solely these properties, without fixing a definition for the integral. This is done in Section 4. In Section 3 possible definitions for the integral will be given, indexed by $p \in [1, \infty]$. The Choquet integral ($p = 1$) is the most natural because it extends the Lebesgue integral and is commonly used in the theory of random closed sets and in mathematical economics. Under mild conditions the indefinite Choquet integral $\int_{\cdot} f \, dc$ is again a capacity and jointly continuous in f and c. This will be proved in Section 5.

Holwerda and Vervaat (1993) have shown that C is a complete lattice with the natural set-wise ordering. In this context O'Brien (1994) observed that for any sequence (c_n) of capacities $\liminf c_n$ and $\limsup c_n$ are well-defined elements of C. A large-deviation principle holds if and only if $\liminf c_n^{\alpha_n}$ and $\limsup c_n^{\alpha_n}$ are equal. We say that a lower (upper) one-sided large-deviation principle holds for (c_n) with limit c if $\liminf c_n^{\alpha_n} \geq c$ ($\limsup c_n^{\alpha_n} \leq c$). The results in Section 4 will be presented in a form that makes them applicable to one-sided large-deviation principles. This approach ultimately yields the mixing theorem that we already mentioned.

The results will be derived in the largest possible topological generality. The reason for this is that we want them to be applicable to large-deviation principles for random capacities, in which case the underlying space is a space of capacities. In general the topological properties of C are not very tractable, which is why we try to avoid a Hausdorff condition on the underlying space. We shall, however, not be able to avoid this all the way. A first use of random capacities can be found in Deheuvels (1994).

2. Preliminaries

As we showed in the introduction, it is natural to formulate large-deviation principles in terms of capacities. Unfortunately, the type of capacity that is needed is not well known, and its basic properties are not easily accessible for reference. In this section we state the properties that we shall need. Most of them can be found in O'Brien and Vervaat (1991). For the others we do give another reference or a proof.

Our desire to prove the main results with minimal topological conditions creates the need for some rather exotic, but weak, topological notions. These will be introduced first, together with some notation concerning the underlying topological space.

Let E be a topological space. Let $\mathcal{G}$ denote the collection of open subsets. We shall write $\mathcal{G}(E)$ instead of $\mathcal{G}$ if it is not clear from the context what the underlying space is. A set $K \subset E$ is called **compact** if every open cover of K has a finite subcover (also if E is not Hausdorff). We shall denote the collections of closed and compact sets by $\mathcal{F}$ and $\mathcal{K}$ respectively. Generic open, closed and compact sets will be denoted by G, F and K.

We call E **locally compact** if for any instance of $K \subset G$ there are K' compact and G' open such that $K \subset G' \subset K' \subset G$. If E is Hausdorff, this simplifies to the usual definition (each point has a compact neighborhood). We call E **Wilker** if for every occurrence of $K \subset G_1 \cup G_2$ there are $K_i \subset G_i$ such that $K \subset K_1 \cup K_2$ (cf. Wilker (1970)), and **inheritary Wilker** if each subset $A \subset E$ with the relative topology is Wilker. We call E **first countable for** $\mathcal{K}$ if each compact set has a countable neighborhood base (cf. O'Brien (1992)).

A function $f : E \to [0, \infty]$ is called **upper semicontinuous** (USC) if for all $x \in (0, \infty)$ the set $\{e \in E : f(e) < x\}$ is open. If f is USC, then f attains its supremum on compact sets. Similarly, f is **lower semicontinuous** (LSC) if $\{e \in E : f(e) > x\}$ is open for all x. If f is LSC, then f attains its infimum on compact sets.

We proceed by giving the definition and elementary properties of capacities. A **capacity** on E is a $[0, \infty]$-valued function c on the subsets of E such that:

(2a) $\quad c(\varnothing) = 0,$

(2b) $\quad c(A) = \sup\{c(K) : K \subset A\}$ for all $A \subset E$ ("inner regularity"),

(2c) $\quad c(K) = \inf\{c(G) : G \supset K\}$ for all $K \in \mathcal{K}$ ("outer regularity").

Choquet (1954) introduced capacities with different regularity conditions. The two notions are in general incomparable, but there is a one-one correspondence between the subcollections of the strongly subadditive capacities (cf. Dellacherie and Meyer (1978) and Norberg and Vervaat (1989)).

All capacities have the following properties. These follow immediately from the definition.

(2d) $\quad c(A) \leq c(B)$ whenever $A \subset B$,

(2e) $\quad c(G_n) \uparrow c(G)$ whenever $G_n \uparrow G$,

(2f) $\quad c(K \cap F_n) \downarrow c(K \cap F)$ whenever $F_n \downarrow F$.

A capacity c is **subadditive** if $c(K_1 \cup K_2) \leq c(K_1) + c(K_2)$ for all $K_1, K_2 \in \mathcal{K}$. It is **additive** if $c(K_1 \cup K_2) + c(K_1 \cap K_2) = c(K_1) + c(K_2)$. It is a **sup measure** if $c(K_1 \cup K_2) = c(K_1) \vee c(K_2)$. A capacity c is **Radon** if $c(K) < \infty$ for all $K \in \mathcal{K}$. The set of all capacities on E is denoted by C or $C(E)$. The spaces of all subadditive capacities, all additive capacities and all sup measures are denoted by SA, AD and SM respectively. These subspaces are of special interest. Therefore we shall give some more information about them, mainly concerning the extension of the defining relation to other than compact sets.

For a capacity c the function $e \mapsto c(\{e\})$ is USC by (2c, d). If c is a sup measure, then $c(A) = \sup\{c(\{e\}) : e \in A\}$ for all $A \subset E$. Conversely, any $[0, \infty]$-valued USC function f determines a sup measure c via $c(A) = \sup\{f(e) : e \in A\}$. Thus there is a one-one correspondence between the set US of all $[0, \infty]$-valued USC functions on E and SM. Closed sets F have USC indicator functions $\mathbf{1}_F$, which embed $\mathcal{F}$ into SM (cf. Vervaat (1988)).

Subadditivity extends to a much larger class of subsets than the compact sets. If E is a Wilker space, subadditivity holds for instance for arbitrary collections of open sets (i.e., $c(\cup_\alpha G_\alpha) \leq \sum_\alpha c(G_\alpha)$). O'Brien and Vervaat (1994) explore how far subadditivity can be extended. The following proposition is a corollary of their Lemma 4.3. However, O'Brien and Vervaat (1994) assume E to be Hausdorff. In their Sections 3 and 4 this is needed only to prove the fact that a subadditive capacity on $A \cup B$ is arbitrarily subadditive on $\mathcal{G}(A \cup B)$. The Wilker property of $A \cup B$ guarantees this.

Proposition 1. *If E is inheritary Wilker, c is a subadditive capacity and A and B are subsets of E such that A is either open or closed in the relative topology on $A \cup B$, then $c(A \cup B) \leq c(A) + c(B)$.*

On Hausdorff E an additive capacity c is countably additive on the Borel sets and is therefore an extension of a measure. A finite measure on a Hausdorff space E that is first countable for $\mathcal{K}$ is extendable to a capacity iff it is tight (cf. O'Brien (1992)).

We turn to the two topologies on C. The **vague topology** on C is the coarsest topology that makes the evaluations $c \mapsto c(A)$ LSC for open A and USC for compact A. This means that the collection of all sets of the form $\{c \in C : c(G) > x\}$ or $\{c \in C : c(K) < x\}$ is a subbase ($x \in (0, \infty)$), and that a sequence (c_n) in C converges vaguely to c (this will be denoted by $c_n \xrightarrow{v} c$) iff

(3a) $\qquad \liminf c_n(G) \geq c(G) \quad \text{for all} \quad G \in \mathcal{G},$

(3b) $\qquad \limsup c_n(K) \leq c(K) \quad \text{for all} \quad K \in \mathcal{K}.$

The subbase for the vague topology consists of two types of sets. Two smaller topologies will be needed as well. If C is endowed with the topology generated by the sets $\{c \in C : c(G) > x\}$, it is denoted $C\uparrow$. In this topology convergence is characterized by (3a). Dually, $C\downarrow$ denotes C endowed with the topology generated by the sets $\{c \in C : c(K) < x\}$. Convergence is characterized by (3b). We shall call convergence results in these two topologies **one-sided**, for obvious reasons.

With the vague topology, C is compact. If E is locally compact, then C is Hausdorff and the subspaces SA, AD, SM and $\mathcal{F}$ are closed and compact. If E is first countable for $\mathcal{K}$, then sequences in C have at most one limit (we shall say that C is sequentially Hausdorff) and the subspaces mentioned are sequentially closed (cf. O'Brien (1992)).

Replacing 'compact' by 'closed' gives the **narrow topology** on C. It is the coarsest topology that makes $c \mapsto c(A)$ LSC for open A and USC for closed A. A subbase is given by all sets of the form $\{c \in C : c(G) > x\}$ or $\{c \in C : c(F) < x\}$. A sequence (c_n) converges narrowly to c (denoted by $c_n \xrightarrow{n} c$) iff (3a) holds and

$$\text{(3c)} \qquad \limsup c_n(F) \leq c(F) \quad \text{for all} \quad F \in \mathcal{F}.$$

When C is provided with the topology generated by the sets $\{c \in C : c(F) < x\}$, it is denoted $C\Downarrow$. If E is regular and Hausdorff, then C is narrowly Hausdorff and its subspaces SA, AD, SM and $\mathcal{F}$ are narrowly closed.

A set Π of capacities is **equitight** if there is a net (K_m) in $\mathcal{K}$ (a **tightening net**) such that $c(K \cap K_m) \to c(K)$ as $m \to \infty$ uniformly for $K \in \mathcal{K}$ and $c \in \Pi$. A capacity c is **tight** if $\{c\}$ is equitight. Let C_t denote the set of all tight capacities and let SA_t $(\mathrm{AD}_t, \mathrm{SM}_t, \mathcal{F}_t)$ denote the intersection of C_t and SA $(\mathrm{AD}, \mathrm{SM}, \mathcal{F})$. If E is Hausdorff and c is a tight capacity, then (2b, c, f) hold with 'K' replaced by 'F'. Prohorov's theorem extends to capacities: an equitight set of capacities is narrowly relatively compact and the converse holds if E is Polish.

A **vague large-deviation principle** (VLDP) is any instance of $c_n^{\alpha_n} \xrightarrow{v} c$ where $c_n \in \mathrm{SA}$, $c \in \mathrm{SM}$ and $\alpha_n \in (0,1)$ such that $\alpha_n \downarrow 0$. The assumption on c follows from the other assumptions if E is Hausdorff and first countable for $\mathcal{K}$ (Theorem 2.8 in O'Brien (1992)). A **narrow large-deviation principle** (NLDP) is any instance of $c_n^{\alpha_n} \xrightarrow{n} c$, where c_n, c and α_n are as above. The assumption on c follows from the other assumptions if E is Hausdorff and regular. A VLDP (NLDP) holds if inequalities (3a) and (3b) ((3c)) hold for $(c_n^{\alpha_n})$ rather than (c_n). We say that a **one-sided large-deviation principle** holds if one of these inequalities holds (this is a bit less general than the definition from O'Brien (1995) cited in the introduction, but sufficient for our purposes). We want to emphasize (again) that for any sequence $(c_n^{\alpha_n})$ two one-sided LDPs hold, possibly with different limits.

On C we define an order and an addition by setting $c_1 \leq c_2$ iff $c_1(A) \leq c_2(A)$ for all $A \subset E$, and $(c_1 + c_2)(A) = c_1(A) + c_2(A)$. With this order C is a complete lattice, and many properties can be given a lattice-theoretical proof (cf. Holwerda and Vervaat (1993)). The addition is continuous in both topologies.

We end this section with some notational conventions and an elementary proposition. Let x, y be real numbers, $f : E \to \overline{\mathbb{R}}$ and $A \subset E$. By $x \wedge y$ we denote the minimum of x and y, by $x \vee y$ the maximum, by $f^{\wedge}(A)$ the infimum of f on A and by $f^\vee(A)$ the supremum. In several proofs the following property of upper limits of sequences of real numbers will be needed. It is proved by the observation that $x_n \vee y_n \le x_n + y_n \le 2\,(x_n \vee y_n)$ and $\lim 2^{\alpha_n} = 1$.

Proposition 2. *Let* (x_n) *and* (y_n) *be sequences in* $[0,\infty]$ *and* (α_n) *a sequence in* $(0,\infty)$ *that tends to* 0. *Then*

$$\limsup_{n\to\infty} (x_n + y_n)^{\alpha_n} = \limsup_{n\to\infty} (x_n)^{\alpha_n} \vee \limsup_{n\to\infty} (y_n)^{\alpha_n}. \tag{4}$$

3. Integration with respect to capacities

In this section we present possible definitions for the integral $\int f\,dc$ of a $[0,\infty]$-valued function f with respect to a capacity c. All integrals in this section will have the following properties:

$$\int_A x\,dc = xc(A) \quad \text{for all} \quad x \in [0,\infty] \quad \text{and all} \quad A \subset E, \tag{5a}$$

$$\int f\,dc \le \int g\,dc \quad \text{whenever} \quad f \le g, \tag{5b}$$

$$\int (f \vee g)\,dc \le \int f\,dc + \int g\,dc \quad \text{for semicontinuous } f, \text{ any } g \text{ and } c \in \mathrm{SA}, \tag{5c}$$

$$\int f\,d(c_1 + c_2) \le \int f\,dc_1 + \int f\,dc_2 \quad \text{for all } f \text{ and all } c_1, c_2. \tag{5d}$$

(Here and in the sequel we use the notation $\int_A f\,dc := \int f\mathbf{1}_A\,dc$.) In Section 4 we shall prove Varadhan's theorem without referring to a specific definition for the integral; we only use properties (5a-d), where (5a) is actually needed only for $A \in \mathcal{K}$, $\mathcal{F}$ or $\mathcal{G}$, depending on the context.

Choquet (1954) has defined the integral of a non-negative function with respect to Choquet capacities. His definition can be copied for capacities in our sense. For $c \in C$ and $f : E \to [0,\infty]$ the function $t \mapsto c(\{e \in E : f(e) > t\})$ on $(0,\infty)$ is non-increasing, hence measurable with at most countably many discontinuities. Only at these points the function $t \mapsto c(\{e \in E : f(e) \ge t\})$ may be different. The **Choquet integral** of f with respect to c is

$$\int^1 f\,dc = \int_0^\infty c(\{e \in E : f(e) \ge t\})\,dt = \int_0^\infty c(\{e \in E : f(e) > t\})\,dt \tag{6}$$

(the '1' on top labels the type of integral; it is not a boundary of integration whatsoever). If c, restricted to the Borel sets, is a σ-finite measure and f is measurable, then the Choquet integral of f with respect to c is equal to the Lebesgue integral, as can be seen easily from Fubini's theorem. Therefore the Choquet integral is a natural candidate for an integral with respect to capacities. In Section 5 it is studied more closely. For now we restrict our attention to the properties above.

Theorem 1. *If E is inheritary Wilker, then the Choquet integral has properties* (5a-d).

PROOF: As $\{e \in E : x\mathbf{1}_A(e) > t\}$ equals A for $t < x$ and $\varnothing$ for $t \geq x$, equation (6) gives (5a). If $f \leq g$ we have $\{e \in E : f(e) \geq t\} \subset \{e \in E : g(e) \geq t\}$ for every t and (5b) follows by (2d). For all functions f and g and all t

$$\{e \in E : (f \vee g)(e) \geq t\} = \{e \in E : f(e) \geq t\} \cup \{e \in E : g(e) \geq t\}. \tag{7}$$

If f is USC the first set in the right-hand side (RHS) is closed, so (5c) follows by subadditivity of c and Proposition 1. If f is LSC, we can replace '$\geq$' by '$>$' and 'closed' by 'open' to arrive at the same conclusion. Finally, (5d) is trivial and holds with equality. □

Our second definition is inspired by the RHS in (1), which can be seen as an integral of a function with respect to a sup measure. For $c \in C$ and $f : E \to [0, \infty]$ the **sup integral** of f with respect to c is

$$\int^{\infty} f\, dc = \sup\left\{f^{\wedge}(K)\, c(K) : K \in \mathcal{K}\right\}. \tag{8}$$

If c is a sup measure this reduces to the RHS in (1), but in general it will give a larger and therefore better lower bound in Varadhan's integration theorem, cf. Remark 3. The following proposition shows that the sup integral can be seen as the supremum equivalent of the Choquet integral, compare (6). Norberg (1986) proves the special case $c \in \mathrm{SM}$.

Proposition 3. *For all f and all c*

$$\int^{\infty} f\, dc \;=\; \sup_{t>0} t\, c(\{e \in E : f(e) \geq t\}) \;=\; \sup_{t>0} t\, c(\{e \in E : f(e) > t\}). \tag{9}$$

PROOF: We have

$$\begin{aligned} &\sup_{t>0} t\, c(\{e \in E : f(e) \geq t\}) \\ &\quad = \sup\{t\, c(K) : t > 0, K \subset \{e \in E : f(e) \geq t\}\} \\ &\quad = \sup\{f^{\wedge}(K)\, c(K) : K \in \mathcal{K}\}, \end{aligned} \tag{10}$$

where the first equality holds because c is inner regular and the second because $K \subset \{e \in E : f(e) \geq t\}$ iff $f^{\wedge}(K) \geq t$. By (2d) we have

$$\sup_{t>0} t\, c(\{e \in E : f(e) \geq t\}) \;\geq\; \sup_{t>0} t\, c(\{e \in E : f(e) > t\}). \tag{11}$$

In order to prove the reverse inequality take $x > 0$ strictly smaller than the first supremum. There exist a $t_0 > 0$ such that $x < t_0\, c(\{e \in E : f(e) \geq t_0\})$ and a $t_1 < t_0$ such that $x < t_1\, c(\{e \in E : f(e) \geq t_0\})$. By (2d) we have

$$x < t_1\, c(\{e \in E : f(e) > t_1\}) \;\leq\; \sup_{t>0} t\, c(\{e \in E : f(e) > t\}). \tag{12}$$

Take the supremum over all possible x. □

Theorem 2. *If E is inheritary Wilker, then the sup integral has properties* (5a-d).

Proof: As $x\mathbf{1}_A^\wedge(K)$ equals x if $K \subset A$ and 0 otherwise, (5a) holds. When $f \leq g$ we have $f^\wedge(K) \leq g^\wedge(K)$ for all $K \in \mathcal{K}$, so (5b) holds. Now copy the corresponding part of the proof of Theorem 1, and (9) gives (5c). For all capacities c_1 and c_2 we have

$$\begin{aligned}\int^\infty f\, d\,(c_1 + c_2) &= \sup_{K\in\mathcal{K}} f^\wedge(K)(c_1(K) + c_2(K)) \\ &\leq \sup_{K\in\mathcal{K}} f^\wedge(K)c_1(K) + \sup_{K\in\mathcal{K}} f^\wedge(K)c_2(K)\end{aligned} \tag{13}$$

and (5d) follows. □

By adding superscripts '1' and '∞' to the integral symbols we already introduced the two extreme elements of a collection of integrals. For $p \in (1,\infty)$ let the **p-integral** of f with respect to c be

$$\int^p f\, dc = \left(\int^1 f^p\, dc^p\right)^{1/p}. \tag{14}$$

Theorem 3. *If E is inheritary Wilker, then the p-integral has properties* (5a-d).

Proof: The properties of the Choquet integral imply (5a, b). If $\int^p f \vee g\, dc$ is finite it is equal to the $\mathcal{L}^p$-norm of $\varphi : t \mapsto c(\{e \in E : (f \vee g)^p(e) \geq t\})$. For f usc and c subadditive it follows, as in the proof of Theorem 1, that φ is less than or equal to $t \mapsto c(\{e \in E : f^p(e) \geq t\}) + c(\{e \in E : g^p(e) \geq t\})$. Now (5c) follows by (5b) and the triangle inequality for the $\mathcal{L}^p$-norm. For f lsc the previous holds with all '$\geq$' replaced by '$>$'. Let c_1 and c_2 be capacities and define $\varphi_i(t) = c_i(\{e \in E : f^p(e) \geq t\})$. If the rhs of (5d) is finite, then φ_1 and φ_2 are both in $\mathcal{L}^p$, so (5d) holds by the triangle inequality for the $\mathcal{L}^p$-norm. □

Remark 1. The function $p \mapsto \int^p f\, dc$ is continuous on $[1,\infty)$ if f and c are sufficiently nice. This follows from Theorem 14. Varadhan's theorem as presented in Section 4 shows that $\int^p f\, dc \to \int^\infty f\, dc$ as $p \to \infty$, if c is a sup measure.

4. Varadhan's integration theorem

Throughout this section, c_n ($n \in \mathbb{N}$) and c are subadditive capacities on E, f_n and f are functions $E \to [0,\infty]$, and (α_n) is a sequence in $(0,1)$ tending to 0. Furthermore, we assume that integration of $[0,\infty]$-valued functions with respect to capacities is defined in such a way that (5a-d) hold.

We shall prove results of the form: if $c_n^{\alpha_n}$ and $f_n^{\alpha_n}$ converge, then $\left(\int f_n\, dc_n\right)^{\alpha_n}$ converges. These convergences will be interpreted in several ways, both one-sided (e.g., $c_n^{\alpha_n}$ converges in $C\!\uparrow$) and two-sided (e.g., $c_n^{\alpha_n}$ converges narrowly in C). The one-sided results will play important roles in future research.

There are two situations in which results will be obtained. To allow a combined proof we introduce a class $\mathcal{B}$ of subsets of E which is either $\mathcal{K}$ or $\mathcal{F}$. The two situations will be referred to as 'the case $\mathcal{B} = \mathcal{K}$' and 'the case $\mathcal{B} = \mathcal{F}$'. E is assumed to be first countable for $\mathcal{K}$ in case $\mathcal{B} = \mathcal{K}$, and regular and Hausdorff in case $\mathcal{B} = \mathcal{F}$.

The hypotheses on the convergence of the capacities and the functions in our results will be taken from the following collection:

$$\liminf c_n^{\alpha_n}(G) \geq c(G) \quad \text{for all} \quad G \in \mathcal{G}, \tag{15a}$$

$$\limsup c_n^{\alpha_n}(B) \leq c(B) \quad \text{for all} \quad B \in \mathcal{B}, \tag{15b}$$

$$\liminf \left(f_n^{\wedge}(B)\right)^{\alpha_n} \geq f^{\wedge}(B) \quad \text{for all} \quad B \in \mathcal{B}, \tag{16a}$$

$$\limsup \left(f_n^{\vee}(B)\right)^{\alpha_n} \leq f^{\vee}(B) \quad \text{for all} \quad B \in \mathcal{B}. \tag{16b}$$

Thus (15a) and (15b) together state that a VLDP (in case $\mathcal{B} = \mathcal{K}$) or an NLDP (in case $\mathcal{B} = \mathcal{F}$) holds. If f is continuous, then (16a) and (16b) together are equivalent with uniform convergence of $f_n^{\alpha_n}$ to f on sets in $\mathcal{B}$ (cf. Gerritse (1995b)).

Splitting the convergences allows us to split Varadhan's theorem accordingly. The advantage is that these results apply to one-sided LDPs, as announced in the introduction. Theorem 4 applies to lower LDPs, Theorems 5 and 7 to upper. The first two theorems will each be preceded by a proposition that presents the partial result for which the topological assumptions are needed. This is done in order to allow easy adaptation to other contexts (e.g., (c_n) being a net rather than a sequence (cf. Remark 5)).

Proposition 4. *Assume* (15a). *Let $G \in \mathcal{G}$ and $x \in (0, \infty)$ such that $c(G) > x$. Then there is a $B \in \mathcal{B}$ such that $B \subset G$ and $\liminf c_n^{\alpha_n}(B) > x$.*

PROOF: By (2b) there is a compact $K \subset G$ such that $c(K) > x$. In case $\mathcal{B} = \mathcal{K}$ Corollary 2.3 from O'Brien (1992) gives the desired result. In case $\mathcal{B} = \mathcal{F}$ there are an open set U and a closed set F such that $K \subset U \subset F \subset G$. Now $\liminf c_n^{\alpha_n}(F) \geq \liminf c_n^{\alpha_n}(U) \geq c(U) \geq c(K) > x$. ☐

Theorem 4. *Assume* (15a) *and* (16a), *and that f is* LSC. *Then for all $G \in \mathcal{G}$*

$$\liminf \left(\int_G f_n \, dc_n \right)^{\alpha_n} \geq \sup \left\{ f^{\wedge}(K)\, c(K) : K \in \mathcal{K}, K \subset G \right\}. \tag{17}$$

PROOF: Let K be a compact subset of G and let x and y be real numbers such that $c(K) > x$ and $f^{\wedge}(K) > y$. We need consider only the case $x, y > 0$. As f is LSC there exists an open set $U \supset K$ such that $f^{\wedge}(U) > y$. Now $c(G \cap U) > x$, and by Proposition 4 there is a $B \in \mathcal{B}$ such that $B \subset G \cap U$ and $\liminf c_n^{\alpha_n}(B) > x$.

Hence:

$$
\begin{aligned}
\liminf \left(\int_G f_n\, dc_n\right)^{\alpha_n} &\geq \liminf \left(\int_B f_n\, dc_n\right)^{\alpha_n} \\
&\geq \liminf \left(f_n^\wedge(B)\, c_n(B)\right)^{\alpha_n} \\
&\geq \liminf \left(f_n^\wedge(B)\right)^{\alpha_n} \liminf c_n^{\alpha_n}(B) \\
&> f^\wedge(B)\, x \quad > \quad yx.
\end{aligned} \tag{18}
$$

□

Remark 2. In case $\mathcal{B} = \mathcal{F}$, if f is continuous, the conclusion of Theorem 4 holds without any assumption on E. This has a similar proof, using the fact that by continuity of f (16a) extends to open sets.

Remark 3. The RHS in (17) is greater than or equal to $\sup\{f(e)\, c(\{e\}) : e \in G\}$, the RHS in (1). Inequality can occur only for one-sided LDPs, since in case of a full LDP the limit is a sup measure and then the two expressions are equal. The following example shows that inequality can occur.

Example 1. This is a modification of an example by O'Brien (1995), which exhibits a limit in a lower LDP that is not a sup measure. Let $E = \{-1, 0, 1\}$ with the discrete topology. Let (β_n) be positive real numbers such that $\beta_n \downarrow 0$ and $\beta_n^{1/n} \to 1$. Let c_n be the probability measure on E with $c_n(\{0\}) = 1 - \beta_n$, $c_n(\{1\}) = \beta_n$ if n is even and $c_n(\{-1\}) = \beta_n$ if n is odd. Let c be the capacity that is 0 on the sets $\varnothing$, $\{-1\}$ and $\{1\}$ and 1 on all other subsets of E. Let $f = \mathbf{1}_{\{-1,1\}}$ and $f_n = f^n$. Now all conditions of Theorem 4 are met (with $c = \liminf c_n^{1/n}$ and $c \notin \mathrm{SM}$). In this case the RHS in (17) equals $f^\wedge(\{-1,1\})\, c(\{-1,1\}) = 1$ and the RHS in (1) equals 0.

Proposition 5. *Let d be a sup measure on E and let (d_n) be a sequence of set functions on E satisfying* (2a, b) *and such that* $\limsup d_n(B) \leq d(B)$ *for all $B \in \mathcal{B}$. Then for each $K \in \mathcal{K}$ and $x \in (0, \infty)$ such that $d(K) < x$ there is an open set $G \supset K$ such that* $\limsup d_n(G) < x$.

PROOF: The proof of Lemma 2.2 from O'Brien (1992) does not use the outer regularity (2c) of capacities, so the case $\mathcal{B} = \mathcal{K}$ is covered by O'Brien's Corollary 2.4. In case $\mathcal{B} = \mathcal{F}$ there exists by outer regularity (2c) of d an open set $U \supset K$ with $d(U) < x$. By the topological regularity of E there exist an open set G and a closed set F such that $K \subset G \subset F \subset U$. Now $\limsup d_n(G) \leq \limsup d_n(F) \leq d(F) \leq d(U) < x$. □

Remark 4. Not requiring (2c) for d_n makes this proposition applicable to $(f_n^\vee(\,\cdot\,))^{\alpha_n}$.

Remark 5. The restriction to sequences is necessary for applying Corollaries 2.3 and 2.4 from O'Brien (1992). It is not needed in the case $\mathcal{B} = \mathcal{F}$. The stronger assumption that E be locally compact allows in the case $\mathcal{B} = \mathcal{K}$ proofs for Propositions 4 and 5 similar to the ones given for $\mathcal{B} = \mathcal{F}$ and renders the propositions, and also the Theorems 4, 5 and 7, valid for nets in general.

Before we can give the lim sup counterpart of Theorem 4 we introduce the upper semicontinuous hull of the functions f_n. It will be needed in Theorem 6 to avoid problems arising from the semicontinuity condition on f in (5c). Let $g : E \to [0,\infty]$ be an arbitrary function. The **hypograph** of g is the set

$$\text{hypo}\,(g) = \{(e,x) \in E \times [0,\infty] : g(e) \geq x\}. \tag{19}$$

The hypograph of g is closed in the product topology on $E \times [0,\infty]$ iff g is USC. The **upper semicontinuous hull** of g is the function $\bar{g} : E \to [0,\infty]$ given by

$$\bar{g}(e) = \inf\{h(e) : h \in \mathrm{US}(E), h \geq g\}. \tag{20}$$

It is the smallest USC function that is greater than or equal to g. It is characterized by $\text{hypo}\,(\bar{g}) = \overline{\text{hypo}\,(g)}$, the closure of $\text{hypo}\,(g)$ in $E \times [0,\infty]$.

Proposition 6. *The upper semicontinuous hull has the following properties:*

(i) *if $e \in G$ and $\bar{g}(e) > x$, then $g^\vee(G) > x$,*
(ii) *if $\alpha \in (0,\infty)$, then $\bar{g}^\alpha = \overline{g^\alpha}$.*

PROOF: (i) Set $y = \bar{g}(e)$, so that $(e,y) \in \overline{\text{hypo}\,(g)}$. There exists a net $(e_n, y_n) \to (e,y)$ with $(e_n,y_n) \in \text{hypo}\,(g)$. For large n we have $e_n \in G$ and $y_n > x$, and therefore $g^\vee(G) \geq g(e_n) \geq y_n > x$.

(ii) This easily follows from the fact that a function h is USC iff h^α is USC. □

Corollary 1. $\bar{g}^\vee(G) = g^\vee(G)$.

PROOF: For all $x < \bar{g}^\vee(G)$ there is an $e \in G$ such that $\bar{g}(e) > x$, thus $g^\vee(G) > x$. This proves $g^\vee(G) \geq \bar{g}^\vee(G)$. The reverse inequality follows from $\bar{g} \geq g$. □

Theorem 5. *Assume* (15b) *and* (16b) *and that f is* USC. *Then for all $K \in \mathcal{K}$ for which $f^\vee(K) < \infty$ and $c(K) < \infty$*

$$\limsup \left(\int_K f_n \, dc_n \right)^{\alpha_n} \leq \sup\{f(e)\,c(\{e\}) : e \in K\}. \tag{21}$$

The same holds with $\bar{f}_n$ replacing f_n.

PROOF: Let K be compact (so $K \in \mathcal{B}$ in either case) and such that $f^\vee(K) < \infty$ and $c(K) < \infty$. Let $x > \sup\{f(e)\,c(\{e\}) : e \in K\}$. For each $e \in K$ we can write $x = y_e z_e$ where $y_e > f(e)$ and $z_e > c(\{e\})$, and find by Proposition 5 (applied twice with $K = \{e\}$) an open set $G_e \ni e$ such that $\limsup \big(f_n^\vee(G_e)\big)^{\alpha_n} < y_e$ and $\limsup c_n^{\alpha_n}(G_e) < z_e$. Now K is covered by the collection $\{G_e : e \in K\}$, so there is a finite set $I \subset K$ such that already $K \subset \bigcup_{e\in I} G_e$. By (5a, c) and Corollary 1 we have

$$\begin{aligned} \int_K \bar{f}_n \, dc_n &\le \int_E \max\{\bar{f}_n{}^\vee(G_e)\mathbf{1}_{G_e} : e \in I\}\, dc_n \\ &\le \sum_{e\in I} \bar{f}_n{}^\vee(G_e)\, c_n(G_e) = \sum_{e\in I} f_n^\vee(G_e)\, c_n(G_e). \end{aligned} \tag{22}$$

Consequently, by Proposition 2,

$$\begin{aligned} \limsup \left(\int_K \bar{f}_n \, dc_n\right)^{\alpha_n} &\le \max_{e\in I} \limsup \left(f_n^\vee(G_e)\, c_n(G_e)\right)^{\alpha_n} \\ &< \max_{e\in I} y_e z_e \;=\; x. \end{aligned} \tag{23}$$

Now (21) follows from the fact that $f_n \le \bar{f}_n$ and (5b). □

Remark 6. The conclusion of Theorem 5 trivially holds whenever the RHS in (21) is infinite. If E is Wilker and the RHS is finite, then the conditions $f^\vee(K) < \infty$ and $c(K) < \infty$ are equivalent to the seemingly weaker condition that for all $e \in K$ the pair $(f(e), c(\{e\}))$ does not equal $(0, \infty)$ or $(\infty, 0)$.

We shall improve on Theorem 5 in two directions. We shall prove its conclusion for closed sets, and we shall weaken the finiteness assumption. The former can be achieved by general capacity theory, using the fact that under weak conditions the indefinite integrals are again capacities. The conclusions of Theorems 4 and 5 together state that these satisfy a VLDP. In general, in Hausdorff spaces vague convergence can be upgraded to narrow convergence by proving **controlledness** (cf. O'Brien (1992)). We shall present a more direct method that makes no use of the indefinite integrals being capacities.

Theorem 6. *Let E be Hausdorff. Assume* (15b) *and* (16b) *and that f is* USC. *Let $F \in \mathcal{F}$. If there is a net (K_m) of compact sets such that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left(\int_{FK_m^c} \bar{f}_n \, dc_n\right)^{\alpha_n} = 0, \tag{24}$$

and if $f^\vee(F) < \infty$ and $c(F) < \infty$, then

$$\limsup \left(\int_F f_n \, dc_n\right)^{\alpha_n} \le \; \sup\{f(e)\, c(\{e\}) : e \in F\}. \tag{25}$$

The same holds with $\bar{f}_n$ replacing f_n.

Proof: For each m

$$\begin{aligned}
\limsup\left(\int_F \bar f_n\,dc_n\right)^{\alpha_n} &\le \limsup\left(\int \bar f_n \mathbf{1}_{FK_m} \vee \bar f_n \mathbf{1}_{FK_m^c}\,dc_n\right)^{\alpha_n}\\
&\le \limsup\left(\int_{FK_m} \bar f_n\,dc_n + \int_{FK_m^c} \bar f_n\,dc_n\right)^{\alpha_n} \qquad (26)\\
&\le \limsup\left(\int_{FK_m} \bar f_n\,dc_n\right)^{\alpha_n} \vee \limsup\left(\int_{FK_m^c} \bar f_n\,dc_n\right)^{\alpha_n}.
\end{aligned}$$

The first term is less than or equal to $\sup\{f(e)\,c(\{e\}) : e \in F\}$ by Theorem 5; the second term tends to 0 as $m \to \infty$ by assumption. □

Remark 7. The Hausdorff condition is needed to ensure that $\bar f_n \mathbf{1}_{FK_m}$ is USC and consequently justify application of (5c) in the second inequality. In case the integral is a p-integral (including $p = 1$ and $p = \infty$), the Hausdorff condition can be replaced by an extra subadditivity-like condition on c_n (compare the proofs of Theorems 1, 2 and 3). A sufficient condition is that all c_n are measures.

Remark 8. In case $\mathcal{B} = \mathcal{F}$, condition (24) is satisfied whenever $(c_n^{\alpha_n})$ is equitight (this corresponds with exponential tightness in the literature) and (16b) holds with $\bar f_n$ replacing f_n. Let (K_m) be a tightening net for $(c_n^{\alpha_n})$ and note that

$$\limsup\left(\int_{FK_m^c} \bar f_n\,dc_n\right)^{\alpha_n} \le \limsup\left(\bar f_n{}^{\vee}(F)\right)^{\alpha_n} \limsup c_n^{\alpha_n}(K_m^c). \tag{27}$$

We need consider only F for which $f^{\vee}(F) < \infty$, in which case the first upper limit is bounded. The second upper limit vanishes by equitightness.

The finiteness conditions in Theorem 5 and 6 can be replaced by the weaker, and global, conditions (28) and (29) below. For $n \in \mathbb{N}$ and $L > 1$ let $c_n^{(L)}$ be the truncation of c_n given by $c_n^{(L)}(A) = c_n(A) \wedge L^{1/\alpha_n}$ and let $c^{(L)}$ be the truncation of c given by $c^{(L)}(A) = c(A) \wedge L$. It is easy to verify that $c^{(L)}$, $c_n^{(L)}$ and $c_n - c_n^{(L)}$ are capacities on E and that (15b) holds for $((c_n^{(L)})^{\alpha_n})$ with limit $c^{(L)}$, whenever it holds for $(c_n^{\alpha_n})$ with limit c and also that (16b) holds for $((f_n \mathbf{1}_{[\bar f_n{}^{\alpha_n} < L]})^{\alpha_n})$ with limit $f \wedge L$ whenever it holds for $(f_n^{\alpha_n})$ with limit f (here $[\bar f_n{}^{\alpha_n} < L]$ is an abbreviation for $\{e \in E : \bar f_n{}^{\alpha_n}(e) < L\}$). Now sufficient conditions for (21), resp. (25), to hold for all K, resp. F, are

$$\lim_{L\to\infty} \limsup_{n\to\infty} \left(\int_{[\bar f_n{}^{\alpha_n} \ge L]} \bar f_n\,dc_n^{(L)}\right)^{\alpha_n} = 0, \tag{28}$$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \left(\int_E \bar f_n\,d\Big(c_n - c_n^{(L)}\Big)\right)^{\alpha_n} = 0. \tag{29}$$

Condition (29) is trivially satisfied if $\limsup c_n(E) < \infty$, in particular, if all c_n are probability measures. In this last case condition (28) reduces to condition (3.4) in Varadhan (1966). In case $f_n^{\alpha_n} = f$ it reduces further to condition (2.1.9) in Deuschel and Stroock (1989).

Theorem 7. *Assume* (15b) *and* (16b) *and that* f *is* USC. *If* (28) *and* (29) *hold, then* (21) *holds for all* $K \in \mathcal{K}$.

PROOF: Let $K \in \mathcal{K}$. For all $L > 0$

$$
\begin{aligned}
\int_K \bar{f}_n\, dc_n &\le \int_K \bar{f}_n\, dc_n^{(L)} + \int_K \bar{f}_n\, d(c_n - c_n^{(L)}) \\
&\le \int \bar{f}_n \mathbf{1}_{K\cap[\bar{f}_n{}^{\alpha_n}<L]} \vee \bar{f}_n \mathbf{1}_{[\bar{f}_n{}^{\alpha_n}\ge L]}\, dc_n^{(L)} + \int_E \bar{f}_n\, d(c_n - c_n^{(L)}) \qquad (30)\\
&\le \int_K \overline{f_n \mathbf{1}}_{[\bar{f}_n{}^{\alpha_n}<L]}\, dc_n^{(L)} + \int_{[\bar{f}_n{}^{\alpha_n}\ge L]} \bar{f}_n\, dc_n^{(L)} + \int_E \bar{f}_n\, d(c_n - c_n^{(L)}).
\end{aligned}
$$

We can apply Theorem 5 to the first term for the sequences $(f_n \mathbf{1}_{[\bar{f}_n{}^{\alpha_n}<L]})$, with upper limit $f \wedge L$, and $(c_n^{(L)})$, with upper limit $c^{(L)}$. Combining Proposition 2 and the assumptions we get (21). □

Theorem 8. *Let* E *be Hausdorff. Assume* (15b) *and* (16b), *that* f *is* USC *and that there is a net* (K_m) *of compact sets such that* (24) *holds for* $F = E$ *and consequently for all* $F \in \mathcal{F}$. *If* (28) *and* (29) *hold, then* (25) *holds for all* $F \in \mathcal{F}$.

PROOF: A simple adaption of the proof of Theorem 7. □

Continuity of f allowed us to drop the assumptions on E in Theorem 4 in case $\mathcal{B} = \mathcal{F}$ (cf. Remark 2). It has an even stronger effect in the case of Theorem 7. The following theorem puts no assumptions on E, and the condition around (24) is dropped as well. However, it is valid only in case $\mathcal{B} = \mathcal{F}$ and all f_n USC. Furthermore condition (28) must be strengthened a bit: let (28*) be (28) with $[\bar{f}_n{}^{\alpha_n} \ge L]$ replaced by $[f \ge L]$. The main idea in the proof comes from Varadhan (1984).

Theorem 9. *Assume* (15b) *and* (16b) *with* $\mathcal{B} = \mathcal{F}$, *that* f *is continuous and* f_n *is* USC *for all* n, *and that* $c \in$ SM. *If* (28*) *and* (29) *hold, then* (25) *holds for all* $F \in \mathcal{F}$.

PROOF: First we prove the case $c(E) < \infty$. Let $F \in \mathcal{F}$. Choose $\varepsilon > 0$. Let $L > 0$ be such that

$$
\limsup_{n\to\infty} \left(\int_{[f\ge L]} f_n\, dc_n \right)^{\alpha_n} < \varepsilon. \tag{31}
$$

Let $M \in \mathbb{N}$ be such that $\frac{L}{M} \le \varepsilon$. Define for $k \in \{1, \dots, M\}$ the closed set $F_k := F \cap [(k-1)\frac{L}{M} \le f \le k\frac{L}{M}]$. Now $F \cap [f \le L] = \cup_{k=1}^M F_k$, and therefore

$$
\int_F f_n\, dc_n \le \sum_{k=1}^M \int_{F_k} f_n\, dc_n + \int_{[f\ge L]} f_n\, dc_n. \tag{32}
$$

There is an $N \in \mathbb{N}$ such that for all $n \geq N$ and all k

$$(f_n^\vee(F_k))^{\alpha_n} \leq f^\vee(F_k) + \varepsilon \leq k\tfrac{L}{M} + \varepsilon. \tag{33}$$

For these n

$$\int_{F_k} f_n \, dc_n \leq \left(k\tfrac{L}{M} + \varepsilon\right)^{1/\alpha_n} c_n(F_k) \tag{34}$$

and thus, from (32) and Proposition 2,

$$\begin{aligned} \limsup \left(\int_F f_n \, dc_n\right)^{\alpha_n} &\leq \max_k \limsup \left(k\tfrac{L}{M} + \varepsilon\right) c_n^{\alpha_n}(F_k) \ \vee \ \varepsilon \\ &\leq \max_k \left(k\tfrac{L}{M} + \varepsilon\right) c(F_k) \ \vee \ \varepsilon. \end{aligned} \tag{35}$$

For all $e \in F_k$ we have $k\frac{L}{M} + \varepsilon \leq f(e) + 2\varepsilon$, hence

$$\limsup \left(\int_F f_n \, dc_n\right)^{\alpha_n} \leq \sup_{e \in F} (f(e) + 2\varepsilon)\, c(\{e\}) \ \vee \ \varepsilon. \tag{36}$$

This holds for all $\varepsilon > 0$, and (25) follows because c is bounded.

Now the case $c(E) = \infty$. Let $F \in \mathcal{F}$. Choose $\delta > 0$. Let $L > 0$ be such that

$$\limsup \left(\int_E f_n \, d\left(c_n - c_n^{(L)}\right)\right)^{\alpha_n} < \delta. \tag{37}$$

The foregoing applies to the sequence $(c_n^{(L)})$ with limit $c^{(L)}$. This gives

$$\begin{aligned} \limsup \left(\int_F f_n \, dc_n\right)^{\alpha_n} &\leq \limsup \left(\int_F f_n \, dc_n^{(L)}\right)^{\alpha_n} \vee \ \delta \\ &\leq \sup_{e \in F} f(e) c^{(L)}(\{e\}) \vee \delta \ \leq \ \sup_{e \in F} f(e) c(\{e\}) \vee \delta \end{aligned} \tag{38}$$

and (25) follows. □

We have proved the following two versions of Varadhan's theorem:

Theorem 10. *Assume that E is first countable for $\mathcal{K}$, $c_n^{\alpha_n} \xrightarrow{v} c$ and $f_n^{\alpha_n} \to f$ uniformly on compact sets and that f is continuous. If* (28) *and* (29) *hold, then* (17) *holds for all $G \in \mathcal{G}$ and* (21) *holds for all $K \in \mathcal{K}$.*

Theorem 11. *Assume that $c_n^{\alpha_n} \xrightarrow{n} c \in \mathrm{SM}$ and $(c_n^{\alpha_n})$ is equitight. Assume also that $f_n^{\alpha_n} \to f$ uniformly and that f is continuous and f_n is* USC *for all n. If* (28*) *and* (29) *hold, then* (17) *holds for all $G \in \mathcal{G}$ and* (21) *holds for all $F \in \mathcal{F}$.*

Remark 9. The conclusions in these theorems amount to new LDPs: if all indefinite integrals are capacities (cf. Section 5) Theorem 10 gives a new VLDP and Theorem 11 gives a new NLDP.

5. The indefinite Choquet integral

For a function $f : E \to [0, \infty]$ and a capacity c the indefinite Choquet integral $\int_\cdot^1 f\,dc$ of f with respect to c is the map $A \mapsto \int_A^1 f\,dc = \int_0^\infty c(\{e \in A : f(e) \geq t\})\,dt$. In this section we try to determine under what conditions it is again a capacity and when the map $(f, c) \mapsto \int_\cdot^1 f\,dc$ is continuous.

To find an answer to the first question we have to verify inner and outer regularity. Following Vervaat (1988) and Norberg and Vervaat (1989) we interpret this as semicontinuity of the restrictions to $\mathcal{G}$ and $\mathcal{K}$. We endow $\mathcal{G}$ with the topology with as a base all sets $\{G \in \mathcal{G} : K \subset G\}$ with $K \in \mathcal{K}$. Similarly, we endow $\mathcal{K}$ with the topology generated by $\{K \in \mathcal{K} : K \subset G\}$ with $G \in \mathcal{G}$. Now $\gamma : \mathcal{G} \cup \mathcal{K} \to [0, \infty]$ can be extended to a capacity by (2b) iff the restriction $\gamma : \mathcal{G} \to [0, \infty]$ is LSC and the restriction $\gamma : \mathcal{K} \to [0, \infty]$ is USC.

Assume for the moment that all indefinite integrals are capacities. To answer the second question we have to verify semicontinuity of evaluations. The map $(f, c) \mapsto \int_\cdot^1 f\,dc$ is vaguely (narrowly) continuous iff $(f, c) \mapsto \int_A^1 f\,dc$ is LSC for all $A \in \mathcal{G}$ and USC for all $A \in \mathcal{K}$ ($A \in \mathcal{F}$).

These observations suggest that we study the joint semicontinuity of the maps $(f, c, A) \mapsto \int_A^1 f\,dc$ for $A \in \mathcal{G}$, $A \in \mathcal{K}$ and $A \in \mathcal{F}$. For now we shall concentrate on $\mathcal{G}$ and $\mathcal{K}$. Both maps will be split into parts, that will be treated in five lemmas. To be able to formulate these properly we have to establish some further notation. Recall that $C{\uparrow}$ and $C{\downarrow}$ denote C with one-sided topologies. With $\mathrm{US}(E){\downarrow}$ (or $\mathrm{US}{\downarrow}$) we denote $\mathrm{US} = \mathrm{SM}$ with the relative topology of $C{\downarrow}$. Analogously, $\mathrm{LS}(E){\uparrow}$ is the collection $\mathrm{LS}(E)$ (or LS) of all LSC functions $f : E \to [0, \infty]$, with the topology generated by the sets of the form $\{f : f^\wedge(K) > x\}$ with $K \in \mathcal{K}$ and $x \in (0, \infty)$. Thus, identifying sets with their indicator functions, $\mathcal{G}$ has the relative topology of $\mathrm{LS}(E){\uparrow}$. If E is Hausdorff, $\mathcal{K}$ has the relative topology of $C(E){\Downarrow}$.

Lemma 1. *If E is Wilker, the map $\Pi : \mathrm{LS}(E){\uparrow} \times \mathrm{LS}(E){\uparrow} \to \mathrm{LS}(E){\uparrow}$ that maps $(f, g) \mapsto fg$ is continuous.*

Proof: Observe that for all $x \in (0, \infty)$

$$\{e \in E : f(e)g(e) > x\} = \bigcup_{y>0} \{e \in E : f(e) > y, g(e) > \frac{x}{y}\} \tag{39}$$

which guarantees $fg \in \mathrm{LS}$.

Let $\Lambda = \{h \in \mathrm{LS} : h^\wedge(K) > x\}$ be an open subbase element in $\mathrm{LS}(E){\uparrow}$ and take $(f, g) \in \Pi^{-1}(\Lambda)$. Now $(fg)^\wedge(K) > x$ and with each $e \in K$ there are $y_e, z_e > 0$ such that $f(e) > y_e$, $g(e) > z_e$ and $y_e z_e = x$. By lower semicontinuity of f and g there is an open $G_e \ni e$ such that $f^\wedge(G_e) > y_e$ and $g^\wedge(G_e) > z_e$. The collection $\{G_e : e \in K\}$ covers K. There is a finite set $I \subset K$ such that K is already covered by $\{G_e : e \in I\}$. Because E is Wilker there exist compact sets $K_e \subset G_e$, for $e \in I$, such that $K \subset \cup_{e \in I} K_e$. Now

$$\bigcap_{e \in I} \{h \in \mathrm{LS} : h^\wedge(K_e) > y_e\} \times \bigcap_{e \in I} \{h \in \mathrm{LS} : h^\wedge(K_e) > z_e\} \tag{40}$$

is an open neighborhood of (f,g). For all its elements (h_1,h_2) we have

$$\begin{aligned}\Pi(h_1,h_2)^\wedge(K) &= \inf\{h_1(e)h_2(e) : e \in K\} \\ &\geq \min_{e\in I} h_1^\wedge(K_e)h_2^\wedge(K_e) \\ &> \min_{e\in I} y_e z_e \;=\; x\end{aligned} \tag{40}$$

hence $\Pi^{-1}(\Lambda)$ is open. □

Corollary 2. *If E is Wilker, then the map $\mathrm{LS}(E){\uparrow}\times \mathcal{G}(E) \to \mathrm{LS}(E){\uparrow}$ that maps $(f,G) \mapsto f\mathbf{1}_G$ is continuous.*

Lemma 2. *If E is locally compact, then the map $\Phi : \mathrm{LS}(E){\uparrow} \times C(E){\uparrow} \to \mathrm{LS}((0,\infty)){\uparrow}$ that maps (f,c) to the function $\varphi : t \mapsto c(\{e \in E : f(e) > t\})$ is continuous.*

Proof: From (2d) it is clear that φ is decreasing. If $t_n \downarrow t$, then $\varphi(t_n) \uparrow \varphi(t)$ by (2e), hence φ is LSC and indeed Φ is a map into $\mathrm{LS}((0,\infty))$.

Let $\Lambda = \{\psi \in \mathrm{LS}((0,\infty)) : \psi^\wedge(I) > x\}$, with $I \subset (0,\infty)$ compact and $x \in (0,\infty)$. Take an element (f,c) in the inverse image of Λ. Let φ be the image of (f,c). Put $t_0 = \sup I$. Because φ is decreasing, $\varphi \in \Lambda$ means that $c(\{e \in E : f(e) > t_0\}) > x$. Hence by (2b) there is a compact set $K \subset \{e \in E : f(e) > t_0\}$ such that $c(K) > x$. By local compactness of E there are open U and compact L such that $K \subset U \subset L \subset \{e \in E : f(e) > t_0\}$. Now $\{g \in \mathrm{LS} : g^\wedge(L) > t_0\}$ is open in $\mathrm{LS}(E){\uparrow}$ and contains f, and $\{d \in C : d(U) > x\}$ is open in $C(E){\uparrow}$ and contains c. For (g,d) in their product one has $\Phi(g,d)^\wedge(I) = d(\{e \in E : g(e) > t_0\}) \geq d(L) \geq d(U) > x$, hence the image of (g,d) is in Λ. □

Lambda 3. *The map from $\mathrm{LS}((0,\infty)){\uparrow}$ to $[0,\infty]$ that maps φ to $\int_0^\infty \varphi(t)\,dt$ is* LSC.

Proof: Fix $x \in (0,\infty)$ and choose $\varphi \in \mathrm{LS}((0,\infty))$ such that $\int_0^\infty \varphi(t)\,dt > x$. There exists a simple function $g < \varphi$ such that also $\int_0^\infty g(t)\,dt > x$. It can be written as $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ with A_i disjoint. Now $\int_0^\infty g(t)\,dt = \sum_{i=1}^n a_i\,\lambda(A_i)$, where λ denotes the Lebesgue measure. There are compact $I_i \subset A_i$ such that even $\sum_{i=1}^n a_i\,\lambda(I_i) > x$. For $t \in I_i$ we have $\varphi(t) > g(t) = a_i$ and therefore $\varphi^\wedge(I_i) > a_i$. Now define

$$\Lambda \;=\; \bigcap_{i=1}^n \{\psi \in \mathrm{LS}((0,\infty)) : \psi^\wedge(I_i) > a_i\}, \tag{42}$$

an open neighborhood of φ. For $\psi \in \Lambda$ we have $\psi > \sum_{i=1}^n a_i \mathbf{1}_{I_i}$ and therefore $\int_0^\infty \psi(t)\,dt > x$. □

Remark 10. Lemma 3 strongly resembles Fatou's lemma. The stronger convergence assumption ($\liminf \varphi_n^\wedge(I) \geq \varphi^\wedge(I)$ for all compact $I \subset (0,\infty)$ instead of $\liminf \varphi_n(x) \geq \varphi(x)$ for all $x \in (0,\infty)$) is essential. One can easily construct an example that shows that Fatou's lemma does not hold for nets in general.

Taking Corollary 2 and Lemmas 2 and 3 together we arrive at

Corollary 3. *If E is locally compact, then the map $(f,c,G)\mapsto \int_G^1 f\,dc$ is* LSC *on* $\mathrm{LS}(E)\!\uparrow\!\times C(E)\!\uparrow\!\times \mathcal{G}(E)$.

The maps $f\mapsto \int_G^1 f\,dc$, $c\mapsto \int_G^1 f\,dc$ and $G\mapsto \int_G^1 f\,dc$ are LSC even without the local compactness assumption. This can be seen by an easy adaptation of the proofs above. Regarding the second question we don't gain more than that. However, later on we shall see that an ad-hoc proof does give a sharper answer to the first question.

We now proceed with upper semicontinuous analogues to the lemmas above.

Lemma 4. *If E is locally compact, the map Ψ from* $\mathrm{US}(E)\!\downarrow\!\times C(E)\!\downarrow\!\times \mathcal{K}(E)$ *to* $\mathrm{US}((0,\infty))\!\downarrow$ *that maps (f,c,K) to the function $\psi : t\mapsto c(\{e\in K : f(e)\geq t\})$ is continuous.*

Proof: From (2d) it is clear that ψ is decreasing. Take $t_n\uparrow t_0$, and define $F_n := \{e\in E : f(e)\geq t_n\}$. Then $F_n\downarrow F_0$ and by (2f) also $c(KF_n)\downarrow c(KF_0)$. This is $\psi(t_n)\downarrow\psi(t_0)$, and it follows that indeed ψ is USC.

Let $\Lambda = \{\xi\in \mathrm{US}((0,\infty)) : \xi^{\vee}(I) < x\}$ be a subbasic open set, and let $(f,c,K)\in\Psi^{-1}(\Lambda)$. Set $t_0 = \inf I$. Then $c(\{e\in K : f(e)\geq t_0\}) < x$ and by (2c) there exists an open $G\supset\{e\in K : f(e)\geq t_0\}$ such that $c(G)<x$. There exist an open U and a compact L such that $\{e\in K : f(e)\geq t_0\}\subset U\subset L\subset G$. We have $K\subset U\cup\{e\in E : f(e)<t_0\}$ and there exist an open V and a compact C such that $K\subset V\subset C\subset U\cup\{e\in E : f(e)<t_0\}$. Now the sets $\{g\in \mathrm{US} : g^{\vee}(C\setminus U) < t_0\}$, $\{d\in C : d(L)<x\}$ and $\{M\in\mathcal{K} : M\subset V\}$ are open neighborhoods of f, c and K respectively. For (g,d,M) in their product, it follows that $\xi := \Psi(g,d,M)\in\Lambda$, since $\xi^{\vee}(I) = d(\{e\in M : g(e)\geq t_0\})\leq d(\{e\in C : g(e)\geq t_0\})\leq d(U)\leq d(L) < x$. □

Lemma 5. *If for a net $\chi_m\to\chi$ in* $\mathrm{US}((0,\infty))\!\downarrow$ *there exists a* $\varphi\in\mathrm{LS}((0,\infty))$ *such that $\varphi\geq\chi_m,\chi$ and $\int_0^\infty \varphi(t)\,dt<\infty$, then* $\limsup\int_0^\infty \chi_m(t)\,dt\leq\int_0^\infty\chi(t)\,dt$.

Proof: Apply Lemma 3 to the net $(\varphi-\chi_m)$. The following propositions guarantee that $\varphi-\chi_m\in\mathrm{LS}((0,\infty))$ and that $\varphi-\chi_m\to\varphi-\chi$ in $\mathrm{LS}((0,\infty))\!\uparrow$. □

Here $\varphi-\chi$ and $\varphi-\chi_m$ are defined pointwise with the convention $\infty-\infty=0$.

Proposition 7. *If $\varphi\in\mathrm{LS}(E)$ and $\chi\in\mathrm{US}(E)$ are such that $\varphi\geq\chi$, then $\varphi-\chi\in\mathrm{LS}$.*

Proof: For each $x\in(0,\infty)$ the set

$$\{e\in E : \varphi(e)-\chi(e) > x\} = \bigcup_{y\in(x,\infty)}\{e\in E : \varphi(e)>y, \chi(e)<y-x\} \tag{43}$$

is open in E. □

Proposition 8. *Suppose that E is Wilker, that $\chi_m \to \chi$ in* $\mathrm{US}(E)\!\downarrow$ *and that $\varphi \in \mathrm{LS}(E)$ is such that $\varphi \geq \chi_m, \chi$. Then $\varphi - \chi_m \to \varphi - \chi$ in* $\mathrm{LS}(E)\!\uparrow$.

Proof: Let $K \in \mathcal{K}$ and $x \in (0,\infty)$ be such that $(\varphi - \chi)^\wedge(K) > x$, corresponding with a subbasic open neighborhood of $\varphi - \chi$. With each $e \in K$ there are $y_e, z_e > 0$ such that $y_e - z_e = x$ and $\varphi(e) > y_e$, $\chi(e) < z_e$, and there is an open $G_e \ni e$ such that $\varphi^\wedge(G_e) > y_e$ and $\chi^\vee(G_e) < z_e$. The collection $\{G_e : e \in K\}$ covers K, so there is a finite $I \subset K$ such that $K \subset \cup_{i\in I} G_i$. Because E is Wilker there exist compact sets $K_i \subset G_i$ such that $K \subset \cup_{i\in I} K_i$. With each $i \in I$ there is an m_i with $\chi_m^\vee(K_i) < z_i$ for all $m \geq m_i$. Now if $m \geq m_i$ for all $i \in I$ we have

$$(\varphi - \chi_m)^\wedge(K) \;\geq\; \min_{i\in I} \inf_{e\in K_i} (\varphi(e) - \chi_m(e)) \;>\; \min_{i\in I} (y_i - z_i) \;=\; x. \tag{44}$$

□

The function φ is an integrable majorant for the functions χ_m. If for given f and c we want to decide if the indefinite Choquet integral is a capacity, i.e., answer our first question, the functions to which we have to apply Lemma 5 are of the form $t \mapsto c(\{e \in K_m : f(e) \geq t\})$ for nets $K_m \to K$ in $\mathcal{K}$. If $c(K) < \infty$ and $f^\vee(K) < \infty$, there is a $G \supset K$ such that $c(G) < \infty$ and $f^\vee(G) < \infty$. Now the LSC function $c(G)\mathbf{1}_{(0,f^\vee(G)+1)}$ is an integrable majorant, since $K_m \subset G$ for large m. Thus we arrive at the following result.

Theorem 12. *If E is locally compact, c is Radon and f is continuous and finite valued, then the indefinite Choquet integral is a capacity.*

However, this result is not optimal. The following example shows that Theorem 12 and the lemmas preceding it are not powerful enough in an essential case.

Example 2. Let F be a closed subset of E and c a capacity on E. Then also $c(F \cap \cdot)$ is a capacity. This capacity is the indefinite Choquet integral of $\mathbf{1}_F$ with respect to c.

This suggests that if we do not insist on joint continuity in three variables in the conclusion in Lemma 2, we can do without some of the hypotheses. Indeed, in the same style one can prove that with c fixed the map $(f, G) \mapsto \varphi$ is continuous even if E is not locally compact and that with f and c fixed the map $G \mapsto \varphi$ is continuous even if f is not LSC. However, in this last case LS is no longer the intermediate space, so Lemma 3 cannot be applied. By a direct proof we shall see that any indefinite Choquet integral is inner regular, hence the map $G \mapsto \int_G^1 f\,dc$ is always LSC.

Lemma 6. *Let E be a topological space. Let $f : E \to [0,\infty]$ and $c \in C(E)$. Let $\gamma = \int_\cdot^1 f\,dc$. Then*

$$\gamma(A) = \sup\{\gamma(K) : K \subset A\} \tag{45}$$

for all $A \subset E$.

Proof: Only the case $\gamma(A) > 0$ needs a proof. Let $x < \gamma(A)$. As $\gamma(A)$ is the Lebesgue integral of the decreasing function $t \mapsto c(\{e \in A : f(e) \geq t\})$ we can approximate $\gamma(A)$ from below by the integral of a simple function: there are $n \in \mathbb{N}$ and real numbers $0 \leq t_0 < t_1 < \ldots < t_n < \infty$ such that

$$x \; < \; \sum_{i=1}^{n} (t_i - t_{i-1})\, c(\{e \in A : f(e) \geq t_i\}) \; \leq \; \gamma(A). \tag{46}$$

There exist $x_i < (t_i - t_{i-1})\, c(\{e \in A : f(e) \geq t_i\})$ such that $x = x_1 + \ldots + x_n$. By (2b) there exist $K_i \subset \{e \in A : f(e) \geq t_i\}$ such that $x_i < (t_i - t_{i-1})\, c(K_i)$. Set $K := K_1 \cup \ldots \cup K_n$. Now $K \in \mathcal{K}$ and $K \subset A$ and

$$\begin{aligned} \gamma(K) &= \int_0^\infty c(\{e \in K : f(e) \geq t\})\, dt \\ &\geq \sum_{i=1}^{n} (t_i - t_{i-1})\, c(\{e \in K : f(e) \geq t_i\}) \\ &\geq \sum_{i=1}^{n} (t_i - t_{i-1})\, c(K_i) \; > \; \sum_{i=1}^{n} x_i = x \end{aligned} \tag{47}$$

where the second inequality holds because $K_i \subset \{e \in K : f(e) \geq t_i\}$. □

We have established inner regularity of $\int_\cdot^1 f\, dc$ in the largest generality possible. We may not hope for a similar statement for outer regularity (2c), as the following example shows. Just as in Theorem 5 (see the remark following it) the points e where the pair $(f(e), c(\{e\}))$ equals $(0, \infty)$ or $(\infty, 0)$ cause trouble.

Example 3. Let $E = \mathbb{R}$ and $f : x \mapsto |x|^{-1}$ and let c be the Lebesgue measure. Then $\gamma(\{0\}) = f(0)c(\{0\}) = \infty \cdot 0 = 0$ and $\gamma(G) = \infty$ for all open $G \ni 0$.

However, by keeping two variables constant, combining Lemmas 4 and 5 and immediately introducing the boundedness of f and c we find that we may drop the local compactness.

Lemma 7. *Let E be a topological space. Let $f : E \to [0, \infty]$ be* usc *and let $c \in C(E)$. Define $\gamma = \int_\cdot^1 f\, dc$. Then*

$$\gamma(K) = \inf\{\gamma(G) : K \subset G\} \tag{48}$$

for all $K \in \mathcal{K}(E)$ for which $f^\vee(K) < \infty$ and $c(K) < \infty$.

Proof: Let $K \in \mathcal{K}$ be such that $f^\vee(K) < \infty$ and $c(K) < \infty$. Obviously, $\gamma(K)$ is finite. Let $x > \gamma(K)$. As $\gamma(K)$ is the Lebesgue integral of the finite decreasing function $t \mapsto c(\{e \in K : f(e) \geq t\})$, which has support in $[0, f^\vee(K)]$, we can approximate $\gamma(K)$ from above by the integral of a simple function: there are $n \in \mathbb{N}$ and real numbers $0 < t_1 < \ldots < t_n < \infty$ such that $t_n > f^\vee(K)$ and, with $t_0 = 0$,

$$\gamma(K) \;\leq\; \sum_{i=0}^{n-1} (t_{i+1} - t_i)\; c(\{e \in K : f(e) \geq t_i\}) \;<\; x. \tag{49}$$

There exist $x_i > (t_{i+1} - t_i)\, c(\{e \in K : f(e) \geq t_i\})$ such that $x = x_0 + \ldots + x_{n-1}$. By (2c), for $i \in \{0, \ldots, n-1\}$ there exist $G_i \supset \{e \in K : f(e) \geq t_i\}$ such that $(t_{i+1} - t_i)\, c(G_i) < x_i$. Furthermore, let $G_n \supset K$ be such that $f^\vee(G_n) < t_n$. Set

$$G = \bigcap_{i=0}^{n-1} (G_i \cup \{e \in E : f(e) < t_i\}) \cap G_n. \tag{50}$$

Now G is open, $K \subset G$ and

$$\begin{aligned} \gamma(G) &= \int_0^\infty c(\{e \in G : f(e) \geq t\})\, dt \\ &\leq \sum_{i=0}^{n-1} (t_{i+1} - t_i)\; c(\{e \in G : f(e) \geq t_i\}) \\ &\leq \sum_{i=0}^{n-1} (t_{i+1} - t_i)\; c(G_i) \;<\; \sum_{i=0}^{n-1} x_i \;=\; x. \end{aligned} \tag{51}$$

The first inequality holds because $c(\{e \in G : f(e) \geq t\}) = 0$ if $t > t_n$ and $c(G) \leq c(G_0) < \infty$, and the second because $\{e \in G : f(e) \geq t_i\} \subset G_i$. □

Lemmas 6 and 7 yield the following improvement of Theorem 12, which covers the case described in Example 2.

Theorem 13. *If $c \in C$ is Radon and $f : E \to [0, \infty]$ is* USC *and finite valued, then the indefinite Choquet integral is a capacity.*

Remark 11. The finiteness conditions in Theorem 13 can be relaxed. Let C be the set of all points e where the pair $(f(e), c(\{e\}))$ equals $(0, \infty)$ or $(\infty, 0)$. It can be proved that if c is subadditive, then γ is outer regular in all K for which $K \cap C = \varnothing$.

Remark 12. If c is subadditive and f is USC, then the indefinite Choquet integral is subadditive. The same holds for additivity. Both statements follow by direct verification. If c is subadditive and tight and f is bounded, then the indefinite Choquet integral is tight, because $\int_{K^c}^1 f\, dc \leq \int_{K^c}^1 f^\vee(E)\, dc = f^\vee(E)\, c(K^c)$.

So far we have concentrated on answering the first question. We now formulate an answer to the second question as it is given by Lemmas 1–5. Again we first have to establish some notation. Let $CF_+(E)$ be the set of all continuous

$[0, \infty)$-valued functions on E, provided with the topology of uniform convergence on compact sets. Let $C_r(E)$ be the set of all Radon capacities, with the vague topology.

Theorem 14. *If E is locally compact, then the map*

$$CF_+(E) \times C_r(E) \ \ni \ (f,c) \mapsto \int_\cdot^1 f\,dc \ \in \ C_r(E)$$

is vaguely continuous.

Inspecting the proof of Lemma 2 shows that even with G fixed we cannot abandon the assumption of local compactness of E. So Theorem 14 is the best we can at present.

If we want to give an analogue of Theorem 14 for the narrow topology, with all capacities tight, we shall have to adapt Lemmas 1–4. Formulating a narrow equivalent of Lemmas 1 and 2 requires a change of topology on LS and $\mathcal{G}$. By $\mathrm{LS}(E)\Uparrow$ we denote LS with the topology generated by the sets $\{f : f^\wedge(F) > x\}$. By $\mathcal{G}(E)\Uparrow$ we denote $\mathcal{G}$ with the relative topology, with base $\{G : F \subset G\}$.

Lemma 8. *If E is regular, then the map $\Phi : \mathrm{LS}(E)\Uparrow \times C_t(E)\uparrow \times \mathcal{G}(E)\Uparrow \to \mathrm{LS}((0,\infty))\uparrow$ that maps (f,c,G) to the function $t \mapsto c(\{e \in G : f(e) > t\})$ is continuous.*

Proof: Take Λ, I, and x as in the proof of Lemma 2. Let $(f,c,G) \in \Phi^{-1}(\Lambda)$ and set $\varphi = \Phi(f,c,G)$ and $t_0 = \sup I$. Then $\varphi(t_0) > x$, and by lower semicontinuity of φ there is an $s > t_0$ such that $\varphi(s) > x$. There is a compact $K \subset \{e \in G : f(e) > s\}$ such that $c(K) > x$, and there are an open U and a closed L such that $K \subset U \subset L \subset \{e \in G : f(e) > s\}$. Now as open neighborhoods for f, c and G with product in the inverse image of Λ we can take $\{g \in \mathrm{LS} : g^\wedge(L) > t_0\}$, $\{d \in C_t : d(U) > x\}$ and $\{V \in \mathcal{G} : L \subset V\}$. □

Let $\mathrm{US}(E)\Downarrow$ (resp. $\mathcal{F}(E)\Downarrow$) be $\mathrm{US}(E)$ ($\mathcal{F}(E)$) with the relative topology from $C\Downarrow$, generated by the sets $\{f \in \mathrm{US}(E) : f^\vee(F) < x\}$ ($\{F \in \mathcal{F}(E) : F \subset G\}$).

Lemma 9. *If E is normal, then the map $\Psi : \mathrm{US}(E)\Downarrow \times C_t(E)\Downarrow \times \mathcal{F}(E)\Downarrow \to \mathrm{US}((0,\infty))\downarrow$ that maps (f,c,F) to the function $\psi : t \mapsto c(\{e \in F : f(e) \geq t\})$ is continuous.*

Proof: As in the proof of Lemma 4 it follows that ψ is usc. Let $\Lambda = \{\xi \in \mathrm{US}((0,\infty)) : \xi^\vee(I) < x\}$ be a subbasic open set, and let $(f,c,F) \in \Psi^{-1}(\Lambda)$. Set $\psi = \Psi(f,c,F)$ and $t_0 = \inf I$. Then $\psi(t_0) < x$, and by upper semicontinuity of ψ there is an $s < t_0$ such that $\psi(s) < x$. By tightness of c there is an open $G \supset \{e \in F : f(e) \geq s\}$ such that $c(G) < x$. There are closed L and open U such that $\{e \in F : f(e) \geq s\} \subset U \subset L \subset G$. Now for each $e \in F \cap U^c$ we have $f(e) < s$, and there is an open $G_e \ni e$ such that $f^\vee(G_e) < s$. Taking the union we get an open $V \supset F \cap U^c$ for which $f^\vee(V) \leq s < t$. As $F \subset V \cup U$, we can find an open W and a closed M with $F \subset W \subset M \subset V \cup U$. Now as open neighborhoods for f, c and F with product in the inverse image of Λ we can take $\{g \in \mathrm{US} : g^\vee(M \cap U^c) < t\}$, $\{d \in C : d(L) < x\}$ and $\{N \in \mathcal{F} : N \subset W\}$. □

Theorem 14 has the following narrow equivalent, where $CF_+(E)$ now has the topology of uniform convergence and $C_{r,t}(E)$ is the space of all tight elements of C_r (cf. also Remark 12).

Theorem 15. *If E is normal, then the map*

$$CF_+(E) \times C_{r,t}(E) \ \ni \ (f, c) \mapsto \int_\cdot^1 f \, dc \ \in \ C_r(E) \tag{53}$$

is narrowly continuous.

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