The link between the kernel method and the method of adjoints for the generalized index ${}_2F_1$−transform

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Abstract. In this note we show that the two definitions of generalized index ${}_2F_1$-transform given in the previous works [1] and [2] agree for distributions of compact support.

Keywords: index transform, hypergeometric function, distributions of compact support

Classification: 44A15, 46F12

The generalized index ${}_2F_1$-transform of $f \in \mathcal{E}'(I)$, $I = (0, \infty)$, was defined by the kernel method in [1] as:

$$F(\tau) = \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x)\rangle, \quad \tau \in I, \tag{1}$$

where

$$\mathbf{F}(\mu, \alpha, \tau, x) = {}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x)x^{\alpha}$$

and ${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x)$ is the Gauss hypergeometric function defined for $|x| < 1$ as the sum of the Gauss series, while for $|x| \geq 1$, is defined as its analytic continuation [4, p. 431]. Here μ and α are complex parameters.

In [2] the generalized index ${}_2F_1$-transform was analyzed by the method of adjoints. It was required to introduce two Fréchet spaces V and W satisfying that the operator $\mathcal{L} : W \longrightarrow V$ given by

$$(\mathcal{L}\psi)(x) = \int_0^{\infty} \psi(\tau)\mathbf{F}(\mu, \alpha, \tau, x)\, d\tau, \quad x > 0, \tag{2}$$

was continuous whenever $Re\ \mu > -\frac{1}{2}$. To be more precise, the spaces V and W were defined by

$$V = \{\phi \in \mathcal{C}^{\infty}(I) : \gamma_k(\phi) < \infty, \ \ k \in \mathbb{N} \cup \{0\}\}$$

where

$$\gamma_k(\phi) = \sup_{x \in I} \left| x^{-\alpha}(x+1)^{\mu + \frac{3}{2}} A_x^k \phi(x) \right|,$$

A_x being the differential operator defined by

$$x^{\alpha-\mu}(x+1)^{-\mu}D_x x^{\mu+1}(x+1)^{\mu+1}D_x x^{-\alpha},$$

and

$$W = \{\psi \in \mathcal{C}^{\infty}(\mathbb{R}), \quad \text{even}: \ \varrho_r(\psi) < \infty, \ \ r \in \mathbb{N}\cup\{0\}\},$$

where

$$\varrho_r(\psi) = \sup_{x\in I}\left| x^{\frac{3}{2}} e^x \left(\mathcal{F}\left(\frac{\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]^r \psi(\tau)}{\tau\Gamma\left(\mu+\frac{1}{2}+i\tau\right)\Gamma\left(\mu+\frac{1}{2}-i\tau\right)}\right)\right)(x)\right|,$$

and $\mathcal{F}$ denotes Fourier transform with respect to the τ-variable.

The generalized index ${}_2F_1$-transform was defined as the adjoint operator of $\mathcal{L}$, namely,

$$\langle \mathcal{L}' f, \psi\rangle = \langle f, \mathcal{L}\psi\rangle, \quad f \in V', \quad \psi \in W. \tag{3}$$

In this note we establish the link between these two definitions. Here, it will be proved that both definitions agree for distributions of compact support. The corresponding result for the Kontorovich-Lebedev transform was established in [3, Proposition 2.4].

Theorem. *Let $f \in \mathcal{E}'(I)$ and α, μ being complex parameters with $Re\ \alpha > 0$, $Re\ \mu > 0$, $\frac{1}{8} < Re\ (\mu-\alpha) < \frac{1}{4}$ and $Re\ (\mu - 2\alpha) < -1$. Then, for any $\psi \in W$, one has*

$$\langle \mathcal{L}' f, \psi\rangle = \langle T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}, \psi(\tau)\rangle, \tag{4}$$

where $T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}$ is the member of W' given by

$$\langle T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}, \psi(\tau)\rangle = \int_0^{\infty} \langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle \psi(\tau)\, d\tau. \tag{5}$$

Proof: First, observe that $\mathcal{D}(I) \subset V \subset \mathcal{E}(I)$ and the topology of V is stronger than the one induced on it by $\mathcal{E}(I)$. Thus, $\mathcal{E}'(I)$ is a subspace of V'.

Also, from [1, Theorem 3.2, p. 662] the function $F(\tau)$ given by (1) is an entire function such that

(i) $F(\tau) = O(1)$ as $\tau \to 0^+$;

(ii) there exists a $r \in \mathbb{N}\cup\{0\}$ such that $F(\tau) = O\left(\tau^{2r-Re\ \mu-\frac{1}{2}}\right)$ as $\tau\to\infty$.

From this and having into account that, for any $\psi \in W$ one has

(i) $\psi(\tau) = O(\tau^2)$ as $\tau \to 0^+$;

(ii) for all $p \in \mathbb{N}$, $\psi(\tau) = O\ (\tau^{-p})$, as $\tau\to\infty$;

(cf. [2, Proposition 2.2 and Proposition 2.4]) it follows that the integral of the right hand side of (5) has sense.

Clearly, if $f \in \mathcal{D}(I)$, the result of this Theorem holds. In fact, assume that f has its support contained in the closed interval $[a, b] \subset I$, then, using Fubini theorem, one has

$$\begin{aligned}\langle \mathcal{L}' f, \psi \rangle &= \langle f, \mathcal{L}\psi \rangle \\ &= \int_a^b f(x)(\mathcal{L}\psi)(x) dx = \int_a^b f(x) \int_0^\infty \mathbf{F}(\mu, \alpha, \tau, x)\psi(\tau)\, d\tau\, dx \\ &= \int_0^\infty \psi(\tau) \int_a^b f(x)\mathbf{F}(\mu, \alpha, \tau, x)\, dx\, d\tau = \int_0^\infty F(\tau)\psi(\tau)\, d\tau.\end{aligned}$$

So, the result holds for $f \in \mathcal{D}(I)$.

Now, for $f \in \mathcal{E}'(I)$ and using [5, Theorem 28.2 (i), p. 301], there exists a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ belonging to $\mathcal{D}(I)$ which converges to f in $\mathcal{E}'(I)$ and so, in V'. Furthermore, from [2, Proposition 2.3] it follows that $\mathcal{L}'$ is a continuous mapping from V' into W'. Thus $\left\{\mathcal{L}' f_n\right\}_{n \in \mathbb{N}}$ converges to $\mathcal{L}' f$ in W' as $n \to \infty$.

Since, for all $\psi \in W$,

$$\langle \mathcal{L}' f_n, \psi \rangle = \int_0^\infty F_n(\tau)\psi(\tau)\, d\tau$$

where

$$F_n(\tau) = \langle f_n(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle,$$

it follows that

$$\langle \mathcal{L}' f, \psi \rangle = \lim_{n \to \infty} \int_0^\infty F_n(\tau)\psi(\tau)\, d\tau. \tag{6}$$

Clearly, from the asymptotic growth of F_n and ψ, the limit in (6) goes into the integral. Also, noting that

$$\begin{aligned}\lim_{n \to \infty} F_n(\tau) &= \lim_{n \to \infty} \langle f_n(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle \\ &= \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x) \rangle = F(\tau),\end{aligned}$$

the result holds. □

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(*Received* January 16, 1996)