Factorizations of set-valued mappings with separable range

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Abstract. Right factorizations for a class of l.s.c. mappings with separable metrizable range are constructed. Besides in the selection and dimension theories, these l.s.c. factorizations are also successful in solving the problem of factorizing a class of u.s.c. mappings.

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1. Introduction

Throughout this paper, X is a topological space and Y is a metrizable space. Set

$$2^Y = \{S \subset Y : S \neq \emptyset\} \quad \text{and} \quad \mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}.$$

Let $\mathcal{P}$ be a property of set-valued mappings, and let $\Phi : X \to \mathcal{F}(Y)$ have the property $\mathcal{P}$ (briefly, $\Phi \in \mathcal{P}$). It is called that the triple (Z, h, φ) is a $\mathcal{P}$ *factorization* for Φ (see [7]), if

(a) Z is a metrizable space with $w(Z) \leq w(Y)$,
(b) $h : X \to Z$ is a continuous map, and
(c) $\varphi : Z \to \mathcal{F}(Y)$ is a mapping such that $\varphi \in \mathcal{P}$ and $\Phi = \varphi \circ h$.

Finally, let us recall that a mapping $\Phi : X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set $\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$.

In the present paper, we are concerned with conditions under which an l.s.c. Φ admits an l.s.c. factorization (Z, h, φ). An obvious *necessary* condition for Φ is that $\Phi^{-1}(U) = h^{-1}(\varphi^{-1}(U))$ is a cozero-set in X for every open $U \subset Y$. A purpose of the paper is to show that, for a separable Y, this condition is also sufficient.

In what follows, let us agree to say that $\Phi : X \to \mathcal{F}(Y)$ is *strongly* l.s.c. if $\Phi^{-1}(U)$ is a cozero-set in X for every open $U \subset Y$. Note that, for a perfectly normal X, $\Phi : X \to \mathcal{F}(Y)$ is strongly l.s.c. if and only if it is l.s.c.

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Theorem 1.1. *Let X be a topological space, and let Y be a separable metrizable space. For a mapping $\Phi : X \to \mathcal{F}(Y)$ the following two conditions are equivalent*:

(a) Φ *is strongly l.s.c.,*

(b) Φ *admits an l.s.c. factorization* (Z, h, φ).

The proof of Theorem 1.1 is based on a standard technique (see Section 2). Despite of that, this theorem has some interesting applications. So, it reduces the proofs of several selection and extension theorems for strongly l.s.c. mappings to those for l.s.c. mappings with metrizable domain. By way of example, this is illustrated in Section 4. In the same section, we also obtain a characterization (via selections) for the covering dimension of arbitrary topological spaces (see Theorem 4.2). Finally, and perhaps most interesting, Theorem 1.1 applies successfully in solving a problem of constructing $\mathcal{P}$ factorizations in case $\mathcal{P}$ is the property of *upper semi-continuity*. Let us recall that a set-valued mapping $\Theta : X \to 2^Y$ is *upper semi-continuous*, or u.s.c., if the set $\Theta^\#(U) = \{x \in X : \Theta(x) \subset U\}$ is open in X for every open $U \subset Y$. As before, we shall say that Θ is *strongly* u.s.c. if $\Theta^\#(U)$ is a cozero-set in X for every open $U \subset Y$. In Section 3, we use Theorem 1.1 to obtain the following factorization theorem for u.s.c. mappings.

Theorem 3.1. *Let X be a topological space, and let Y be a separable metrizable space. For a compact-valued mapping $\Theta : X \to \mathcal{F}(Y)$ the following two conditions are equivalent*:

(a) Θ *is strongly u.s.c.,*

(b) Θ *admits a u.s.c. factorization* (Z, h, θ).

2. Proof of Theorem 1.1

Let X, Y and Φ be as in Theorem 1.1. Since (b) $\Rightarrow$ (a) is obvious, we have only to prove (a) $\Rightarrow$ (b). Towards this end, let $\mathcal{B}$ be a countable base for the topology of Y. By (a), for every $U \in \mathcal{B}$ there exists a continuous function $h_U : X \to I_U = [0, 1]$ such that $\Phi^{-1}(U) = h_U^{-1}((0, 1])$. Then let $h : X \to \prod\{I_U : U \in \mathcal{B}\}$ be the diagonal map $h = \triangle\{h_U : U \in \mathcal{B}\}$, and let us show that, for every $x_1, x_2 \in X$,

$$(*) \qquad h(x_1) = h(x_2) \ \text{ implies } \ \Phi(x_1) = \Phi(x_2).$$

Indeed, $h(x_1) = h(x_2)$ implies $h_U(x_1) = h_U(x_2)$, $U \in \mathcal{B}$, and therefore, for every $U \in \mathcal{B}$, $\Phi(x_1) \cap U \neq \emptyset$ if and only if $\Phi(x_2) \cap U \neq \emptyset$. So, $(*)$ holds.

Now, setting $Z = h(X)$ we get a separable metrizable space Z because $\mathcal{B}$ is countable. Define $\varphi : Z \to \mathcal{F}(Y)$ by

$$\varphi(z) = \Phi(x), \quad z \in Z \ \text{ and } \ x \in h^{-1}(z).$$

It follows from $(*)$ that φ is an well defined set-valued mapping such that $\Phi = \varphi \circ h$. So, to finish the proof it only remains to show that φ is l.s.c. That this is so, it follows immediately from the fact that, for every $V \in \mathcal{B}$,

$$\varphi^{-1}(V) = \{(z_U) \in Z : z_V > 0\}.$$

Thus, the proof of Theorem 1.1 completes. □

We conclude this section showing that, for a Polish Y (i.e. a completely metrizable separable space Y), the following refinement of Theorem 1.1 holds true.

Theorem 2.1. *Let X be a topological space, Y be a Polish space, and let $\Phi : X \to \mathcal{F}(Y)$ be strongly l.s.c. Then Φ admits an l.s.c. factorization (Z, h, φ) with Z a Polish space.*

PROOF: Let (Z_0, h, φ_0) be an l.s.c. factorization for Φ, which exists by virtue of Theorem 1.1. Also, let $\tilde{Z}$ be a Polish space containing Z_0. Then, by [2, Theorem 4.7], there exists a G_δ-subset Z of $\tilde{Z}$ containing Z_0, and an l.s.c. $\varphi : Z \to \mathcal{F}(Y)$ such that $\varphi|Z_0 = \varphi_0$. This (Z, h, φ) is as required. □

3. U.s.c. factorizations

In this section we use Theorem 1.1 to obtain the following factorization theorem for u.s.c. mappings.

Theorem 3.1. *Let X be a topological space, and let Y be a separable metrizable space. For a compact-valued mapping $\Theta : X \to \mathcal{F}(Y)$, the following two conditions are equivalent*:

(a) Θ *is strongly u.s.c.,*
(b) Θ *admits a u.s.c. factorization* (Z, h, θ).

Let (Y, d) be a metric space. For $S \in 2^Y$ and $\varepsilon > O$, we use $B^d_\varepsilon(S)$ to denote $\{y \in Y : d(y, S) < \varepsilon\}$. A mapping $\theta : Z \to \mathcal{F}(Y)$ is d-u.s.c. if $\theta^\#(B^d_\varepsilon(\theta(x)))$ is a neighborhood of x for every $x \in X$ and $\varepsilon > 0$.

The most difficult part of the proof of Theorem 3.1 consists in proving the following theorem.

Theorem 3.2. *Let X be a topological space, Y be a separable metrizable space, and let $\Theta : X \to \mathcal{F}(Y)$ be strongly u.s.c. Then there exists a compatible metric d on Y with respect to which Θ admits a d-u.s.c. factorization (Z, h, θ).*

PROOF: Let d be a compatible totally bounded metric on Y. In what follows, we shall consider the set $\mathcal{F}(Y)$ equipped with the topology generated by the *Hausdorff distance*

$$H(d)(S, T) = \inf\{\varepsilon > 0 : S \subset B^d_\varepsilon(T) \text{ and } T \subset B^d_\varepsilon(S)\}, \quad S, T \in \mathcal{F}(Y).$$

Note that $(\mathcal{F}(Y), H(d))$ is totally bounded because so is (Y, d).

Now, define a set-valued mapping $\Phi : X \to 2^{\mathcal{F}(Y)}$ by

$$\Phi(x) = \{K \in \mathcal{F}(Y) : \Theta(x) \subset K\}, \quad x \in X. \tag{1}$$

Clearly, $\Phi : X \to \mathcal{F}(\mathcal{F}(Y))$. Let us check that Φ is strongly l.s.c. Suppose $x_0 \in X$, $K_0 \in \Phi(x_0)$ and $\varepsilon > 0$. Since every countable union of cozero-sets in X is a cozero-set in X and since $\mathcal{F}(Y)$ is separable, it suffices to show that $\Phi^{-1}(B^{H(d)}_\varepsilon(K_0))$ is a cozero-set in $\mathcal{F}(Y)$. Towards this end, let us show that

$$\Phi^{-1}(B^{H(d)}_\varepsilon(K_0)) = \bigcup_{0<\delta<\varepsilon} \Theta^{\#}(B^d_\delta(K_0)). \tag{2}$$

Take a point $x \in \bigcup_{0<\delta<\varepsilon} \Theta^{\#}(B^d_\delta(K_0))$. Then $\Theta(x) \subset B^d_\delta(K_0)$ for some $0 < \delta < \varepsilon$, and therefore $K = \Theta(x) \cup K_0 \in \Phi(x)$ has the property that $K \subset B^d_\delta(K_0)$ and $K_0 \subset B^d_\delta(K)$. That is, $H(d)(K, K_0) \le \delta < \varepsilon$, which implies $x \in \Phi^{-1}(B^{H(d)}_\varepsilon(K_0))$. Let now $x \in \Phi^{-1}(B^{H(d)}_\varepsilon(K_0))$. Then there is $K \in \Phi(x)$ such that $H(d)(K, K_0) < \delta < \varepsilon$. By (1), $\Theta(x) \subset K \subset B^d_\delta(K_0)$, which completes the verification of (2).

It now follows from Theorem 1.1 that Φ admits an l.s.c. factorization $(\tilde{Z}, h, \varphi)$. Set $Z = h(X)$, and then define a mapping $\theta : Z \to \mathcal{F}(Y) \cup \{\emptyset\}$ by

$$\theta(z) = \bigcap \varphi(z), \quad z \in Z. \tag{3}$$

Note that, for every $x \in X$, (1) implies

$$\Theta(x) = \bigcap \Phi(x) = \bigcap \varphi(h(x)) = \theta(h(x)).$$

So, $\theta : Z \to \mathcal{F}(Y)$ and $\Theta = \theta \circ h$. Thus, to finish the proof, it only remains to show that θ is d-u.s.c. Let $z \in Z$ and let $\varepsilon > 0$. Also, let $x \in h^{-1}(z)$. Then, by (1), (2) and (3),

$$\begin{aligned}
\theta^{\#}(B^d_\varepsilon(\theta(z))) &= h(\Theta^{\#}(B^d_\varepsilon(\Theta(x)))) \\
&\supset h(\Phi^{-1}(B^{H(d)}_\varepsilon(\Theta(x)))) \\
&= \varphi^{-1}(B^{H(d)}_\varepsilon(\Theta(x))) \;=\; \varphi^{-1}(B^{H(d)}_\varepsilon(\theta(z))),
\end{aligned}$$

which completes the proof because φ is l.s.c. □

Proof of Theorem 3.1: (b) ⇒ (a) is obvious. As for (a) ⇒ (b), it follows immediately from Theorem 3.2 because, for compact-valued mappings, the concept "d-u.s.c." is actually equivalent to "u.s.c.". □

4. Selections and extensions

As it was mentioned in the introduction, Theorem 1.1 reduces the proofs of several selection and extension theorems for strongly l.s.c. mappings to those for l.s.c. mappings with metrizable domain. In this section, we shall restrict our attention only to a few of these reductions that may have some general interest.

Let Y be a Banach space. We write

$$\mathcal{K}(Y) = \{S \in 2^Y : S \text{ is convex}\} \text{ and } \mathcal{F}_c(Y) = \{S \in \mathcal{F}(Y) : S \in \mathcal{K}(Y)\}.$$

Also, we recall the following terminology: If $K \in \mathcal{F}_c(Y)$, then a *supporting set* of K is a closed convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K will be denoted by $\mathrm{I}(K)$. Finally, as in [5], we define

$$\mathcal{D}(Y) = \{B \in \mathcal{K}(Y) : B \supset \mathrm{I}(\overline{B})\}.$$

The following are well-known (see [5]):

(1) $\mathcal{F}_c(Y) \subset \mathcal{D}(Y)$;
(2) every $B \in \mathcal{K}(Y)$ with $\mathrm{Int}(B) \neq \emptyset$ belongs to $\mathcal{D}(Y)$;
(3) every finite-dimensional $B \in \mathcal{K}(Y)$ belongs to $\mathcal{D}(Y)$.

Theorem 4.1. *Let X be a topological space, Y a separable Banach space, and let $\Phi : X \to \mathcal{D}(Y)$ be strongly l.s.c. Then Φ admits a single-valued continuous selection.*

Proof: Define $\tilde{\Phi} : X \to \mathcal{F}_c(Y)$ by $\tilde{\Phi}(x) = \overline{\Phi(x)}$, $x \in X$. Since $\tilde{\Phi}^{-1}(U) = \Phi^{-1}(U)$ for every open $U \subset Y$, it follows that $\tilde{\Phi}$ is strongly l.s.c. too. Then, by Theorem 1.1, $\tilde{\Phi}$ admits an l.s.c. factorization $(\tilde{Z}, h, \varphi)$. Set $Z = h(X)$. Being a metrizable, Z is perfectly normal. Hence, by [5, Theorem 3.1$'''$], there exists a continuous $g : Z \to Y$ such that $g(z) \in \mathrm{I}(\varphi(z))$ for all $z \in Z$. Finally, $f = g \circ h$ is as required because

$$f(x) \in \mathrm{I}(\varphi(h(x))) = \mathrm{I}(\tilde{\Phi}(x)) = \mathrm{I}(\overline{\Phi(x)}) \subset \Phi(x) \quad \text{for every} \quad x \in X. \qquad \square$$

In our next result, $\dim(X) \leq n$ means that every finite cozero-set cover of X admits a finite cozero-set refinement of order $\leq n+1$, i.e. the covering dimension of a space X in the sense of Morita [6].

Theorem 4.2. *For a topological space X and $n \geq 0$ the following two conditions are equivalent*:

(a) $\dim(X) \leq n$;
(b) *if Y is a Polish space, then every strongly l.s.c. $\Phi : X \to \mathcal{F}(Y)$ admits a strongly u.s.c. selection $\theta : X \to 2^Y$ such that $|\theta(x)| \leq n+1$ for all $x \in X$.*

Proof: (a) $\Rightarrow$ (b). Suppose Y is a Polish space and $\Phi : X \to \mathcal{F}(Y)$ is strongly l.s.c. By Theorem 1.1, Φ admits an l.s.c. factorization (Z, h, φ). Then, by [6, Lemma 2.2], there is a metrizable space M with $\dim(M) \leq n$, and continuous maps $f : X \to M$ and $l : M \to Z$ such that $h = l \circ f$. Finally, by [1, Theorem 11.1], there exists a u.s.c. selection $\theta_0 : M \to 2^Y$ for $\varphi \circ l$ such that $|\theta_0(t)| \leq n+1$ for every $t \in M$. The mapping $\theta = \theta_0 \circ f$ is as required in (b).

(b) $\Rightarrow$ (a). Let $\mathcal{U} = \{U_y : y \in Y\}$ be a cozero-set cover of X, where Y is a finite set. It suffices to show that $\mathcal{U}$ admits a finite zero-set refinement of order $\leq n+1$. To this end, we consider Y as a Polish space endowed with the discrete topology. Next, we define a strongly l.s.c. $\Phi : X \to \mathcal{F}(Y)$ by

$\Phi(x) = \{y \in Y : x \in U_y\}$, $x \in X$. By (b), Φ admits a strongly u.s.c. selection $\theta : X \to 2^Y$ such that $|\theta(x)| \leq n+1$ for all $x \in X$. Then $\{\theta^{-1}(\{y\}) : y \in Y\}$ satisfies all our requirements. $\square$

We conclude this paper with two extension results.

Theorem 4.3. *Let X be a topological space, $A \subset X$, Y a Polish space, and let $\Phi : A \to \mathcal{F}(Y)$ be strongly l.s.c. Then there exists a G_δ subset $\tilde{X}$ of X containing A, and a strongly l.s.c. $\tilde{\Phi} : \tilde{X} \to \mathcal{F}(Y)$ such that $\tilde{\Phi}|A = \Phi$.*

Proof: By Theorem 2.1, Φ admits an l.s.c. factorization (Z, h, φ) with Z a Polish space. Then, by a result of Lavrentieff [4], h can be extended to a continuous $\tilde{h} : \tilde{X} \to Z$ for some G_δ subset $\tilde{X}$ of X containing A. The mapping $\tilde{\Phi} = \tilde{h} \circ \varphi$ is as required. $\square$

Theorem 4.4. *Let X be a completely regular space, Y be a compact metric space, and let $\Phi : X \to \mathcal{F}(Y)$ be strongly l.s.c. Then there exists an l.s.c. mapping $\beta\Phi : \beta X \to \mathcal{F}(Y)$, where βX is the Čech-Stone compactification of X, such that $\beta\Phi|X = \Phi$.*

Proof: Let (Z, h, φ) be an l.s.c. factorization of Φ. By [3, Theorem 5.1], there exists an l.s.c. mapping $\beta\varphi : \beta Z \to \mathcal{F}(Y)$ such that $\beta\varphi|Z = \varphi$. Then $\beta\Phi = \beta\varphi \circ \beta h$ is as required, where $\beta h : \beta X \to \beta Z$ is the extension of h. $\square$

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