Existence of mild solutions for semilinear equation of evolution

Anna Karczewska, Stanisław Wędrychowicz

Abstract. The aim of this paper is to give an existence theorem for a semilinear equation of evolution in the case when the generator of semigroup of operators depends on time parameter. The paper is a generalization of [2]. Basing on the notion of a measure of noncompactness in Banach space, we prove the existence of mild solutions of the equation considered. Additionally, the applicability of the results obtained to control theory is also shown. The main theorem of the paper allows to characterize the set of controls providing solutions of the system considered. Moreover, the application of the main theorem for elliptic equations is given.

Keywords: semilinear equation of evolution, mild solutions, measure of noncompactness, sublinear measure

Classification: 34A10, 49E30

1. Introduction

We consider the following semilinear equation of evolution

$$\begin{cases} x'(t) = A(t)x + f(t,x), & t \in [0,T] \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $A(t) : D(A(t)) \subset \mathbf{X} \to \mathbf{X}$ is a linear operator in Banach space $\mathbf{X}$ for every $t \in [0,T]$ and $f : [0,T] \times \mathbf{X} \to \mathbf{X}$ is a continuous function.

We assume that f satisfies the comparison condition of the type

$$\mu(f(t,X)) \leq \omega(t,\mu(X)),\ X \subset \mathbf{X},\ \text{and } X \text{ is bounded}, \tag{1.2}$$

where μ is the so-called sublinear measure of noncompactness and $\omega(t,\mu)$ is a Kamke comparison function of Coddington and Levinson type [3].

In the paper we prove a theorem on the existence of mild solutions for the semilinear equation of evolution (1.1). The results which we are going to prove generalize those of Pazy [7]–[8], Kato [5], Friedman [4] and others, see e.g. [9]. The considerations of this paper base on the notion of a measure of noncompactness in Banach space. The main theorem of the paper gives a characterization of the set of solutions of the system controlled. Precisely, solution of the system exists when a set of controls is relatively compact.

2. Preliminaries and notation

Let $\mathbf{X}$ be a given Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $\bar{X}$ and $\operatorname{Conv} X$ the closure and the convex closure of the set X, respectively. By $\lambda_1 X + \lambda_2 Y$, $\lambda_1, \lambda_2 \in \mathbb{R}$, we denote the linear combination of sets $X, Y \in \mathbf{X}$. Further, let $\mathbf{M}_X$ denote the family of all nonempty and bounded subsets of $\mathbf{X}$ and $\mathbf{N}_X$ the family of all nonempty and relatively compact sets in $\mathbf{X}$.

By Z^c we denote the family of all closed sets belonging to a nonempty family Z of subsets of the space $\mathbf{X}$.

Definition 2.1. A function $\mu : \mathbf{M}_X \to [0, +\infty)$ is called a *measure of noncompactness* if it satisfies the following conditions:

(1) the family $\mathbf{P} = \{X \in \mathbf{M}_X : \mu(X) = 0\}$ is nonempty and $\mathbf{P} \subset \mathbf{N}_X$,
(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
(3) $\mu(\bar{X}) = \mu(X)$,
(4) $\mu(\operatorname{Conv} X) = \mu(X)$,
(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,
(6) if $X_n \in \mathbf{M}_X$, $\bar{X}_n = X_n$, $X_{n+1} \subset X_n$, $n = 1, 2, \ldots$, and if

$$\lim_{n\to\infty} \mu(X_n') = 0$$

then

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varnothing.$$

The family $\mathbf{P}$ described in (1) is *the kernel of the measure* μ and it is denoted by $\ker\mu$. It may be shown that $(\ker\mu)^c$ forms a closed subspace of the space $\mathbf{M}_X^c$ with respect to the Hausdorff distance (see, e.g. [1]).

Definition 2.1′. The measure of noncompactness μ is called *sublinear* if it satisfies additionally the following two conditions:

(7) $\mu(X + Y) \leq \mu(X) + \mu(Y)$,
(8) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

For a given measure μ in the space $\mathbf{X}$ let us denote:

$$\mathbf{X}_\mu = \{x \in \mathbf{M} : x \in \ker\mu\}.$$

Proposition 2.2. *If μ is a sublinear measure then $\mathbf{X}_\mu$ forms a closed linear subspace of the space $\mathbf{X}$. Additionally, $\mu(x + X) = \mu(X)$ for any sublinear measure μ and $x \in X$.*

Let $(\mathbf{X}, \varrho)$ be a metric space and $A \in \mathbf{X}$. By $\varrho(x, A)$ denote the distance between point x and set A:

$$\varrho(x, A) = \inf_{a\in A} \varrho(x, a).$$

Definition 2.3. Let $A, B \subset \mathbf{X}$ be nonempty bounded sets. The number

$$\varrho_H(A,B) = \max\left\{\sup_{a\in A}\varrho(a,B),\ \sup_{b\in B}\varrho(A,b)\right\},$$

is called *the Hausdorff distance* between A and B. (For its properties we refer, e.g., to [6].)

Let $K(x,r)$ denote the closed ball centered at x with radius r.

In the sequel we shall use the following lemma.

Lemma 2.4 ([1]). *If μ is a sublinear measure of noncompactness then*

$$|\mu(X) - \mu(Y)| \le \mu(K(\theta,1))\varrho_H(X,Y)$$

for any $X, Y \subset \mathbf{X}$.

3. Main results

We start with the following definition.

Definition 3.1 (see [8]). A two parameters family of bounded linear operators $U(t,s)$, $0 \le s \le t \le T$, on $\mathbf{X}$ is called *an evolution system* if the following two conditions are satisfied:

(1) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$,
(2) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

Now we can formulate the main lemma.

Lemma 3.2. *If μ is a sublinear measure of noncompactness, $U(t,s)$, $0 \le s \le t \le T$ is an evolution system, and $B \subset \mathbf{X}$ is nonempty and bounded set then*

$$\mu(U(t,s)B) \le \mu(B).$$

Proof: First we shall prove the following equality

$$\mu\Big(\bigcup_{0\le s\le t\le T} U(t,s)B\Big) = \mu(B). \tag{3.1}$$

Let $n \in \mathbb{N}$ and $\delta > 0$ be such that for $T > 0$ we give $[0,T] = [0,n\delta]$. Considering the operators $U(t,s)$ on subintervals we can deduce that

$$\mu\Big(\bigcup_{0\le s\le t\le T} U(t,s)B\Big) = \\ = \max\left\{\mu\Big(\bigcup_{0\le s\le t\le \delta} U(t,s)B\Big), \mu\Big(\bigcup_{\delta\le s\le t\le 2\delta} U(t,s)B\Big), \dots, \mu\Big(\bigcup_{(n-1)\delta\le s\le t\le n\delta} U(t,s)B\Big)\right\}.$$

By Lemma 2.4

$$\begin{aligned}
&\Big|\mu\Big(\bigcup_{0\le s\le t\le\delta} U(t,s)B\Big)-\mu(B)\Big|\\
&\le \mu\Big(K(\theta,1)\varrho_H\Big(\bigcup_{0\le s\le t\le\delta} U(t,s)B,B\Big)\Big)\\
&\le \mu(K(\theta,1)) \sup_{\substack{x\in \operatorname{Conv} F_r(B)\\ 0\le s\le t\le\delta}} \|x-U(t,s)x\|\\
&\le \mu(K(\theta,1)) \sup_{0\le s\le t\le\delta} \|I-U(t,s)\| \sup_{x\in \operatorname{Conv} F_r(B)} \|x\| \le \varepsilon,
\end{aligned}$$

for any $\varepsilon > 0$, where $F_r(B)$ denotes the boundary of B.

Analogously, we obtain

$$\begin{aligned}
&\Big|\mu\Big(\bigcup_{\delta\le s\le t\le 2\delta} U(t,s)B\Big)-\mu(B)\Big|\\
&\le \mu(K(\theta,1))\varrho_H\Big(\bigcup_{\delta\le s\le t\le 2\delta} U(t,s)B,B\Big)\\
&\le \mu(K(\theta,1)) \sup_{\substack{x\in \operatorname{Conv} F_r(B)\\ \delta\le s\le t\le 2\delta}} \|U(t,s)x-x\|\\
&\le \mu(K(\theta,1))\Big(\sup_{\substack{x\in \operatorname{Conv} F_r(B)\\ 0\le s_1\le t_1\le\delta\\ \delta\le s\le t\le 2\delta}} \|U(t,s)-U(t_1,s_1)\|\|x\|\\
&+\sup_{\substack{x\in \operatorname{Conv} F_r(B)\\ 0\le s_1\le t_1\le\delta}} \|U(t_1,s_1)-I\|\|x\|\Big)\le\varepsilon_1,
\end{aligned}$$

for any $\varepsilon_1 > 0$.

By induction and from the above estimation we have

$$\Big|\mu\Big(\bigcup_{\delta(k-1)\le s\le t\le k\delta} U(t,s)B\Big)-\mu(B)\Big|\le\varepsilon_k, \tag{3.2}$$

for any $\varepsilon_k > 0$ and $1 \le k \le n$.

Using the inequality (3.2) and (3.1) we get

$$\mu(B)=\mu\Big(\bigcup_{0\le s\le t\le T} U(t,s)B\Big). \tag{3.3}$$

Then, by the equality (3.3) and property of the measure of noncompactness we obtain that

$$\mu(U(t,s)B) \le \mu(B)$$

for $s < t$ and $s, t \in [0,T]$. □

Now we can introduce the following definitions.

Definition 3.3. Let $\{A(t)\}_{t\in[0,T]}$ satisfy some regularity conditions and let $U(t,s)$, $0 \le s \le t \le T$ be the evolution system generated by $\{A(t)\}_{t\in[0,T]}$. The continuous function $x = x(t)$ such that

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))\,ds, \qquad t \in [0,T],$$

is called *the mild solution of the initial values problem* (1.1).

Definition 3.4. By the class D we denote the family of functions $\omega(t,u) = \omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, $\omega(t,0) = 0$, which are locally Lebesgue integrable and satisfy the Carathéodory conditions i.e. they are Lebesgue measurable with respect to t for any u and continuous in u for any t. Moreover, for each $t_0 \in (0,T]$ and $u_0 > 0$ there exists a function $h(t)$ Lebesgue integrable on the interval $[t_0,T]$ such that $\omega(t,u) \le h(t)$ for $(t,u) \in [t_0,T] \times [0,u_0]$. Furthermore we assume that the only continuous function on the interval $[0,T]$ which satisfies the inequality

$$u(\bar{t}) - u(t) \le \int_t^{\bar{t}} \omega(s,u(s))\,ds, \qquad 0 \le t \le \bar{t} \le T,$$

and such that $\lim_{t\to 0_+} \frac{u(t)}{t} = u(0) = 0$, is the function $u(t) \equiv 0$.

Let $C = C([0,T],\mathbf{X})$ be the Banach space of all continuous functions acting from the interval $[0,T]$ into $\mathbf{X}$ with the usual maximum norm $\|x\|_c = \max\{\|x(t)\| : t \in [0,T]\}$. For a given set $B \in \mathbf{M}_C$ let us denote

$$B(t) = \{x(t) : x \in B\}$$
$$\int_0^t B(s)\,ds = \left\{ \int_0^t x(s)\,ds : x \in B \right\}.$$

Lemma 3.5 ([1])**.** *If all functions belonging to B are equicontinuous then*

$$\mu\Big(\int_0^t B(s)\,ds\Big) \le \int_0^t \mu(B(s))\,ds.$$

Definition 3.6. Assume that $f : [0,T] \times \mathbf{X} \to \mathbf{X}$ is continuous and bounded: $\|f(t,x)\| \leq E$ and μ is a sublinear measure of noncompactness in $\mathbf{X}$. We say that *the function f satisfies the Kamke comparison condition* with respect to the measure μ if for any set $B \in \mathbf{M}_X$ and almost all $t \in [0,T]$ the following inequality holds

$$\mu(f(t,B)) \leq \omega(t,\mu(B)),$$

where $\omega(t,u)$ is a comparison function from the class of Coddington and Levinson.

From the Definition 3.4, for any point $x \in \mathbf{X}_\mu$ we have

$$\mu(f(t,\{x\})) \leq \omega(t,\mu(\{x\})) = 0$$

for almost all $t \in [0,T]$, so that in view of the continuity of f we obtain that $f : [0,T] \times \mathbf{X}_\mu \to \mathbf{X}_\mu$.

Proposition 3.7. *Let us assume that $A(t)$ is a bounded linear operator on $\mathbf{X}$ for $0 \leq t \leq T$ and $t \to A(t)$ is continuous in the uniform operator topology. Then*

$$\|U(t,s)\| \leq M \quad \text{for} \quad t,s \in [0,T], \text{ where } M \in \mathbb{R}_+. \tag{3.4}$$

Now we are in a position to formulate the main theorem of the paper.

Theorem 3.8. *Assume that the function f is uniformly continuous on $[0,T] \times K(x_0,r)$. Suppose $\|f(x,t)\| \leq E$, $ETM \leq r$ and f satisfies the Kamke comparison condition of the form (1.2) with respect to sublinear measure μ. Let $A(t)$ be a linear operator in Banach space $\mathbf{X}$ for every $t \in [0,T]$ and satisfy the condition of Proposition 3.7. Then the system (1.1) has at least mild solution x such that $x(t) \in \mathbf{X}_\mu$ for all $t \in [0,T]$ and $x(0) = x_0$ (provided $x_0 \in \mathbf{X}_\mu$).*

Proof: From assumptions we get the following estimate

$$\begin{aligned}
\|x(t) - x(s)\| &= \Big\| U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau,x(\tau))\,d\tau \\
&\quad - U(s,0)x_0 - \int_0^s U(s,\tau)f(\tau,x(\tau))\,d\tau \Big\| \\
&= \Big\| U(t,0)x_0 - U(s,0)x_0 + \int_0^s [U(t,\tau)f(\tau,x(\tau)) \\
&\quad - U(s,\tau)f(\tau,x(\tau))]\,d\tau + \int_s^t U(t,\tau)f(\tau,x(\tau))\,d\tau \Big\| \\
&\leq ME|t-s| + \|U(t,0)x_0 - U(s,0)x_0\| \\
&\quad + \int_0^s \|U(t,\tau)f(\tau,x(\tau)) - U(s,\tau)f(\tau,x(\tau))\|\,d\tau.
\end{aligned} \tag{3.5}$$

Let us put

$$P = \sup_{\tau \in [0,T]} \|A(\tau)\|, \tag{3.6}$$

and

$$D = \exp\Big(\int_0^t \|A(\tau)\|\,d\tau\Big). \tag{3.7}$$

Then by theorem of mean value in view of (3.4) and (3.6) we have

$$\begin{aligned} \|U(t,0)x_0 - U(s,0)x_0\| &= |t-s|\|A(\xi)U(\xi,0)x_0\| \\ &\le |t-s|\cdot MP\|x_0\|, \end{aligned} \tag{3.8}$$

where $\xi \in [s,t]$.

Now, by theorem of mean value and (3.4), (3.6) and (3.7) we obtain

$$\begin{aligned} &\int_0^s \|U(t,\tau)f(\tau,x(\tau)) - U(s,\tau)f(\tau,x(\tau))\|\,d\tau \\ &\le E\int_0^s \|U(t,\tau)-U(s,\tau)\|\,d\tau \\ &\le E|t-s|\int_0^s \|A(\xi_1)\|\|U(\xi_1,\tau)\|\,d\tau \\ &\le EPT|t-s|\exp\int_0^t \|A(\tau)\|\,d\tau = EPDT|t-s| \end{aligned} \tag{3.9}$$

whenever $\xi_1 \in [s,t]$.

Hence, using (3.5), (3.8) and (3.9) we derive

$$\|x(t)-x(s)\| \le K|t-s|, \tag{3.10}$$

where $K = ME + MP\|x_0\| + EPDT$.

Denote by $X_0 \subset C$ the set of all functions x satisfying the condition (3.10) and such that $x(0) = x_0$. Obviously X_0 is bounded, closed, equicontinuous and convex.

It is easy to show that the transformation

$$(\mathbf{F}x)(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau,x(\tau))\,d\tau,$$

maps continuously X_0 into itself so our problem is equivalent to the existence of a fixed point of $\mathbf{F}$.

Next, let us denote $X_{i+1} = \operatorname{Conv}\mathbf{F}X_i$, $i = 0,1,2,\ldots$. Observe that all these sets are of the same type as X_0 and $X_{i+1} \subset X_i$. Let us put $u_i(t) = \mu(X_i(t))$. Obviously $0 \le u_{i+1}(t) \le u_i(t)$, $i = 0,1,2,\ldots$. Thus the sequence $u_i(t)$ converges uniformly to function $u_\infty(t)$.

Put $y(t) = U(t,0)x_0 + U(t,0)f(0,x_0)t$ and $x \in X_1$. Then we have

$$\begin{aligned}
\|x(t) - y(t)\| &= \left\| \int_0^t U(t,\tau) f(\tau, x(\tau))\, d\tau - \int_0^t U(t,0) f(0,x_0)\, d\tau \right\| \\
&= \left\| \int_0^t [U(t,\tau) f(\tau, x(\tau)) - U(t,\tau) f(0,x_0)]\ d\tau \right. \\
&\quad \left. + \int_0^t [U(t,\tau) f(0,x_0) - U(t,0) f(0,x_0)]\ d\tau \right\| \\
&\le M \int_0^t \|f(\tau, x(\tau)) - f(0,x_0)\|\, d\tau \\
&\quad + \int_0^t \|U(t,\tau) f(0,x_0) - U(t,0) f(0,x_0)\|\, d\tau \\
&\le Mta(t) + tb(t),
\end{aligned}$$

where

$$\begin{aligned}
a(t) &= \sup\{\|f(0,x_0) - f(\tau,x)\| : \tau \le t, \\
&\qquad \|x - x_0\| \le EM\tau + \|U(\tau,0)x_0 - U(0,0)x_0\|\} \\
&= \sup\{\|f(0,x_0) - f(\tau,x)\| : \tau \le t,\ \|x - x_0\| \le EM\tau + MP\|x_0\|\}, \\
b(t) &= \sup\{\|U(t,\tau) f(0,x_0) - U(t,0) f(0,x_0)\| : \tau \le t\}.
\end{aligned}$$

Obviously

$$\lim_{t\to 0} a(t) = \lim_{t\to 0} b(t) = 0.$$

Moreover

$$\begin{aligned}
X_1(t) &\subset K(y(t), (a(t)M + b(t)t)) \\
&= U(t,0)x_0 + U(t,0)f(0,x_0)t + t(a(t)M + b(t))K(\theta,1).
\end{aligned} \tag{3.11}$$

Because $x_0 \in \mathbf{X}_\mu$, $f(0,x_0) \in \mathbf{X}_\mu$ and in virtue of the fact that μ is sublinear measure of noncompactness, by Lemma 3.2 and (3.11) we have

$$\begin{aligned}
u_1(t) &= \mu(X_1(t)) \\
&\le \mu(U(t,0)x_0) + t\mu(U(t,0)f(0,x_0)) + t(Ma(t) + b(t))\mu(K(\theta,1)) \\
&= t(Ma(t) + b(t))\mu(K(\theta,1)).
\end{aligned} \tag{3.12}$$

Now using the inequality

$$u_\infty(t) \le u_1(t),$$

by (3.12) we obtain

$$\lim_{t\to 0_+} \frac{u_\infty}{t} = 0.$$

Applying Lemma 3.2 and (3.4) for any arbitrary fixed $t, \bar{t} \in [0, T]$, $t \le \bar{t}$ we obtain

$$
\begin{aligned}
u_{n+1}(\bar{t}) - u_{n+1}(t) &= \mu\Big(U(\bar{t}, 0)x_0 + \int_0^{\bar{t}} U(\bar{t}, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&\quad - \mu\Big(U(t, 0)x_0 + \int_0^{t} U(t, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&= \mu\Big(\int_0^{\bar{t}} U(\bar{t}, \tau) f(\tau, X_n(\tau))\, d\tau\Big) - \mu\Big(\int_0^{t} U(t, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&\le \mu\Big(\int_t^{\bar{t}} U(\bar{t}, \tau) f(\tau, X_n(\tau))\, d\tau\Big) + \mu\Big(\int_0^{t} U(\bar{t}, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&\quad - \mu\Big(\int_0^{t} U(t, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&= \mu\Big(\int_t^{\bar{t}} U(\bar{t}, \tau) f(\tau, X_n(\tau)\, d\tau)\Big) \\
&\quad + \mu\Big(U(\bar{t}, t)\int_0^{t} U(t, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&\quad - \mu\Big(\int_0^{t} U(t, \tau) f(\tau, X_n(\tau))\, d\tau\Big) \\
&\le \int_t^{\bar{t}} \mu(U(\bar{t}, \tau) f(\tau, X_n(\tau)))\, d\tau \\
&\le \int_t^{\bar{t}} \omega(\tau, u_n(\tau))\, d\tau.
\end{aligned}
$$

Hence, passing to the limit with $n \to \infty$ we have

$$
u_\infty(\bar{t}) - u_\infty(t) \le \int_0^{\bar{t}} \omega(\tau, u_\infty(\tau))\, d\tau.
$$

Thus $u_\infty(t) \equiv 0$ and consequently

$$
\lim_{n\to\infty} \{\max\{u_n(t) : t \in [0, T]\}\} = 0.
$$

This implies that the set $\mathbf{X}_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty, convex, closed $\mathbf{X}_\infty \subset \mathbf{X}_\mu$. Moreover $\mathbf{F}$ maps $\mathbf{X}_\infty$ into itself and the Schauder fixed point theorem completes the proof of Theorem 3.7. □

Example 3.9. Let $1 < p < \infty$ and Ω be a bounded domain with the smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Consider the initial value problem

$$
\begin{cases}
\frac{\partial u}{\partial t} + A(t, x, D)u = f(t, x, u) & \text{in } \Omega \times [0, T] \times L^p(\Omega) \\
D^\alpha u(t, x) = 0,\ |\alpha| < m & \text{on } \partial\Omega \times [0, T], \\
u(0, x) = u_0(x) & \text{in } \Omega,
\end{cases}
\tag{3.13}
$$

where

$$A(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x) D^\alpha.$$

An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is called a multiindex and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{for} \quad x = (x_1, x_2, \dots, x_n).$$

Denoting $D_k = \frac{\partial}{\partial x_k}$ and $D = (D_1, D_2, \dots, D_n)$ we have

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

We will make the following assumptions:

(1) The operators $A(t,x,D)$, $t \ge 0$ are uniformly strongly elliptic in Ω i.e. there is a constant $c > 0$ such that

$$(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} a_\alpha(t,x) \xi^\alpha \ge c|\xi|^{2m}$$

for every $x \in \bar{\Omega}$, $0 \le t \le T$ and $\xi \in \mathbb{R}^n$.

(2) The coefficients $a_\alpha(t,x)$ are smooth functions of the variables x in $\bar{\Omega}$ for every $0 \le t \le T$ and satisfy for some constants $c_1 > 0$ and $0 \le \beta < 1$

$$|a_\alpha(t,x) - a_\alpha(s,x)| \le c_1 |t-s|^\beta$$

for $x \in \bar{\Omega}$, $0 \le s$, $t \le T$ and $|\alpha| \le 2m$.

The conditions (1) and (2) provide Proposition 3.7. (see [8]).

With the family $A(t,x,D)$, $t \in [0,T]$ of strongly elliptic operators, we associate a family of linear operators $A_p(t)$, $t \in [0,T]$, in $L^p(\Omega)$, $1 < p < \infty$.

This is done as follows:

$$D(A_p(t)) \equiv D = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

and

$$A_p(t)u = A(t,x,D)u, \quad \text{for} \quad u \in D.$$

If $u_0 \in L^p(\Omega)$ and $f(t,x) \in L^p(\Omega)$ for every $0 \le t \le T$ then a classical solution u of the abstract initial value problem

$$\begin{cases} \frac{du}{dt} + A_p(t)u = f(t,u) \\ u(0) = u_0 \end{cases} \tag{3.14}$$

in $L^p(\Omega)$ is defined to be a generalized solution of the initial value problem (3.13). Recall that such a generalized solution u, if it exists, satisfies:

$$u(t,x) \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \quad \text{for} \quad t > 0;$$

$\frac{du}{dt}$ exists in the sense of $L^p(\Omega)$ and is continuous on $(0,T]$, u itself is continuous on $[0,T]$ and satisfies (3.14) in $L^p(\Omega)$.

We can deduce the following result.

Theorem 3.10. *Assume that the family $A(t,x,D)$, $0 \le t \le T$, satisfies the conditions* (1) *and* (2), *$f(t,x,u) \in L^p(\Omega)$ for $t \in [0,T]$, and f satisfies the Kamke comparison condition of the form* (1.2) *with respect to sublinear measure μ. Then for every $u_0(x) \in L^p(\Omega)$ the system* (3.13) *has at least one solution such that $u(t,x) \in L^p(\Omega)$.*

The proof is analogous to the proof of Theorem 3.8.

4. Applications

In this section we illustrate how the result obtained in this paper provides a tool for control theory.

Assume that $(E, \|\cdot\|_E)$, $(U, \|\cdot\|_U)$ are two real Banach spaces. Let us consider a linear system described by a linear differential equation

$$\frac{dx}{dt} = A(t)x + S(t)u, \qquad a \le t \le b < +\infty, \tag{4.1}$$

with the initial condition

$$x(a) = x_0, \tag{4.2}$$

where $x(t) \in E$ and $u(t) \in U$ for all t.

Assume that $\{A(t)\}$ is a family of closed bounded operators satisfying condition of Proposition 3.7. Then we have the existence of an evolution operator $\{U(t,s)\}$, $a \le s \le t \le b$.

Regarding $u(t)$ and $S(t)$ we shall assume that they are measurable with values in U and in $L(U,E)$, respectively, and such that $S(t)u(t)$ is an integrable function with values in E.

Observe that the above assumptions ensure the existence of the mild solution for the system (4.1) and (4.2):

$$x(t) = U(t,a)x_0 + \int_a^t U(t,s)S(s)u(s)\, ds. \tag{4.3}$$

Function $u(\cdot)$ will be called control, while the space of all functions $u(\cdot)$ will be called the space of controls and denoted by $\hat{U}$.

Definition 4.1. We say that the system described by the differential equation (4.1) is *controllable from* 0 *at the time* T if, for arbitrary $x_1 \in E$, we can find a control $u(\cdot) \in \hat{U}$ such that the mild solution $x(\cdot)$ corresponding to the control $u(\cdot)$ and to the limit condition $x(a) = 0$ satisfies the final condition $x(T) = x_1$, where $T \in [a, b]$.

Observe that for sublinear measure of noncompactness μ and (4.3) we obtain

$$\begin{aligned}\mu(X(t)) &= \mu\Big(U(t,a)x_0 + \int_a^t U(t,s)S(s)\hat{U}(s)\,ds\Big)\\ &\le \mu(U(t,a)x_0) + \int_a^t \mu(U(t,s)S(s)\hat{U}(s))\,ds\\ &= \int_a^t \mu(S(s)\hat{U}(s))\,ds,\end{aligned}$$

whenever $x_0 \in E_\mu$.

Analogously arguing as in proof of Theorem 3.8 we can formulate the following theorem.

Theorem 4.2. *The system* (4.1) *with* (4.2) *is controllable from* 0 *at the time* T *if the set* $S(s)\hat{U}(s)$ *is relatively compact in the space* E.

□

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Department of Mathematics, Maria Curie-Skłodowska University, pl. M. Curie-Skłodowskiej 1, PL–20–031 Lublin, Poland

E-mail: akarcz@golem.umcs.lublin.pl

Department of Mathematics, Technical University, ul. W. Pola 2, PL–35–959 Rzeszów, Poland

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