Comment.Math.Univ.Carolin. 37,4 (1996)707–723

Characterization of sets of determination for parabolic functions on a slab by coparabolic (minimal) thinness

Jarmila Ranošová

Abstract. Let T be a positive number or $+\infty$. We characterize all subsets M of $\mathbb{R}^n\times]0,T[$ such that

(i) $$\inf_{X\in\mathbb{R}^n\times]0,T[} u(X) = \inf_{X\in M} u(X)$$

for every positive parabolic function u on $\mathbb{R}^n\times]0,T[$ in terms of coparabolic (minimal) thinness of the set $M_\delta = \cup_{(x,t)\in M} B^p((x,t),\delta t)$, where $\delta\in(0,1)$ and $B^p((x,t),r)$ is the "heat ball" with the "center" (x,t) and radius r. Examples of different types of sets which can be used instead of "heat balls" are given.

It is proved that (i) is equivalent to the condition $\sup_{X\in\mathbb{R}^n\times\mathbb{R}^+} u(X) = \sup_{X\in M} u(X)$ for every bounded parabolic function on $\mathbb{R}^n\times\mathbb{R}^+$ and hence to all equivalent conditions given in the article [7].

The results provide a parabolic counterpart to results for classical harmonic functions in a ball, see References.

Keywords: heat equation, parabolic function, Weierstrass kernel, set of determination, Harnack inequality, coparabolic thinness, coparabolic minimal thinness, heat ball

Classification: 35K05, 35K15, 31B10

I. Preliminaries

In this paper the following notation is used: Small letters, such as x, y, will denote points in $\mathbb{R}^n$; capital letters, such as X, points in $\mathbb{R}^{n+1}$, and t denotes the "time". (We will write $X=(x,t)$ for $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$.) The set $\mathbb{R}^n\times\{0\}$ is identified with $\mathbb{R}^n$, and, when there is no danger of confusion, the point $(y,0)\in\mathbb{R}^n\times\{0\}$ is denoted by y. The Lebesgue measure in $\mathbb{R}^n$ will be denoted by λ_n.

The Green function G is defined on $\mathbb{R}^{n+1}$ by

$$\begin{aligned} G(X,Y) = G((x,t),(y,s)) &= [4\pi(t-s)]^{-n/2}\exp\Big(-\frac{\|x-y\|^2}{4(t-s)}\Big) && \text{for } t>s;\\ &= 0 && \text{for } t\leqq s. \end{aligned}$$

The symbol $B(x,r)$ denotes the closed ball centered at $x\in\mathbb{R}^n$ with radius r.

In this paper the following subsets of $\mathbb{R}^{n+1}$ will be of special interest:

Let $X \in \mathbb{R}^n \times \mathbb{R}^+$, $X = (x,t) = (x_1, x_2, \dots, x_n, t)$, $r \in \mathbb{R}^+$, $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\gamma \geqq \beta$, $a, b \in \mathbb{R}^+$, $\delta \in \mathbb{R}^+$.

Discs: $D(X,r) = B(x,r) \times \{r\}$; $\quad D_{X,\alpha,\beta} = B(x, \alpha\sqrt{t}) \times \{\beta t\}$;

Cylinders: $C(X,r,[a,b]) = B(x,r) \times [a,b]$;
$C_{X,\alpha,\beta,\gamma} = B(x,\alpha\sqrt{t}) \times [\beta t, \gamma t]$;

Parabolic balls: $B^p(X,r) = \{Z \in \mathbb{R}^{n+1} : G(X,Z) \geqq (4\pi r)^{-n/2}\} \cup \{X\}$;
$B^p_{X,\delta} = B^p(X, \delta t)$;

Coparabolic balls: $B^{cp}(X,r) = \{Z \in \mathbb{R}^{n+1} : G(Z,X) \geqq (4\pi r)^{-n/2}\} \cup \{X\}$;
$B^{cp}_{X,\delta} = B^{cp}(X, \delta t)$;

Intervals: $I(X,r) = (x_1 - r, x_1 + r) \times \dots \times (x_n - r, x_n + r) \times (t - r^2, t)$;
$I_{X,\delta} = I(X, \delta\sqrt{t})$;

Paraboloids: $P(X,a) = \{(z,s) \in \mathbb{R}^{n+1} : \|z - y\|^2 \leqq a(s-t)\}$;
$P(X,a,v) = \{(z,s) \in \mathbb{R}^{n+1} : \|z - y\|^2 \leqq a(s-t)$ and $s \leqq t + v\}$;
$P_{X,a,\delta} = P(X,a,\delta t)$.

Let $T \in]0,\infty]$, $M \subset \mathbb{R}^n \times]0,T[$ and let a set A_X be associated with every $X \in M$. Then M_A will denote the set $\cup_{X \in M} A_X \cap (\mathbb{R}^n \times]0,T[)$. We will use the obvious notation $M_{D_{\alpha,\beta}}$, $M_{C_{\alpha,\beta,\gamma}}$, $M_{B^p_\delta}$, $M_{B^{cp}_\delta}$, M_{I_δ} and $M_{P_{a,\delta}}$.

For $M \subset \mathbb{R}^{n+1}$ the set $\{(y,s) \in \mathbb{R}^{n+1}; (y,-s) \in M\}$ will be called the reflection of M and denoted by $M^\ominus$.

Let $T \in]0,\infty]$, $M \subset \mathbb{R}^n \times]0,T[$ and $Y \in \mathbb{R}^n \times \{0\}$. The set is coparabolic minimal thin at Y, if and only if M is coparabolic thin at Y. (See section III.2.) We will write M is coparabolic (minimal) thin.

Let $0 < T \leqq \infty$. A point $Y = (y,0)$ is called a parabolic limit of a sequence $\{X_k\}$, $X_k = (x_k, t_k)$, of points in $\mathbb{R}^n \times]0,T[$, if $\{X_k\}$ converges to Y and

$$\liminf_{k \to \infty} t_k \|x_k - y\|^{-2} > 0$$

(that is all X_k belong to some paraboloid of revolution with vertex Y and opening upward).

Let $M \subset \mathbb{R}^n \times]0,T[$. A point $Y \in \mathbb{R}^n \times \{0\}$ is called a parabolic limit point of the set M, if there exists a sequence $\{X_k\}$ such that every $X_k \in M$ and Y is a parabolic limit of $\{X_k\}$.

II. The main results

Theorem. *Let $0 < T \leqq \infty$ and $M \subset \mathbb{R}^n \times]0,T[$. Then the following statements are equivalent:*

(i)
$$\inf_{X \in \mathbb{R}^n \times]0,T[} u(X) = \inf_{X \in M} u(X)$$
for all bounded parabolic functions u on $\mathbb{R}^n \times]0,T[$;

(ii)
$$\inf_{X \in \mathbb{R}^n \times]0,T[} u(X) = \inf_{X \in M} u(X)$$
for all positive parabolic functions u on $\mathbb{R}^n \times]0,T[$;

(iii) *there exist $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\gamma \geqq \beta$ such that the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{C_{\alpha,\beta,\gamma}}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

(iv) *for any $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\gamma \geqq \beta$ the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{C_{\alpha,\beta,\gamma}}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

(v) *the set of points of $\mathbb{R}^n \times \{0\}$ which are not parabolic limit points of M has Lebesgue measure zero;*

(vi) *there exists $\delta \in (0,1)$ such that the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{B^p_\delta}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

(vii) *for any $\delta \in (0,1)$ the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{B^p_\delta}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

Remark 1. A set satisfying condition (i) will be called a set of determination.

Remark 2. The equivalence of (i), (ii), (v) and (vi) was announced in the abstract.

Remark 3. The "cylinder" conditions (iii) and (iv) include "disc" conditions, because $C_{X,\alpha,\beta,\gamma} = D_{X,\alpha,\beta}$ for $\beta = \gamma$.

Corollary. *Conditions* (ix)–(xiii) *are equivalent to* (i) *from previous Theorem:*

(ix) *there exists $\delta \in \mathbb{R}^+$ such that the set of points of $\mathbb{R}^n \times \{0\}$ in which $M_{B^{cp}_\delta}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

(x) *for any $\delta \in \mathbb{R}^+$ the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{B^{cp}_\delta}$ is (minimal) coparabolic thin has Lebesgue measure zero;*

(xi) *there exists* $\delta \in (0,1)$ *such that*
the set of points of $\mathbb{R}^n \times \{0\}$ *at which* M_{I_δ} *is* (*minimal*) *coparabolic thin has Lebesgue measure zero;*

(xii) *for any* $\delta \in (0,1)$
the set of points of $\mathbb{R}^n \times \{0\}$ *at which* M_{I_δ} *is* (*minimal*) *coparabolic thin has Lebesgue measure zero;*

(xiii) *there exist* $a, \delta \in \mathbb{R}^+$ *such that*
the set of points of $\mathbb{R}^n \times \{0\}$ *at which* $M_{P_{a,\delta}}$ *is* (*minimal*) *coparabolic thin has Lebesgue measure zero;*

(xiv) *for any* $a, \delta \in \mathbb{R}$
the set of points of $\mathbb{R}^n \times \{0\}$ *at which* $M_{P_{a,\delta}}$ *is* (*minimal*) *coparabolic thin has Lebesgue measure zero.*

In Part III, results from parabolic and coparabolic potential theory needed in this paper are summarized.

The proof of Theorem is given in Part IV. First equivalence of (i), (ii), (iii), (iv) and (v) for $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\gamma \geqq \beta$ and $\beta > 1$ will be proved.

The implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are trivial. The equivalence (i) $\Leftrightarrow$ (v) was established in [7] for $T = \infty$. But any bounded parabolic function u on $\mathbb{R}^n \times]0,T[$ is a restriction of a bounded parabolic function on $\mathbb{R}^n \times \mathbb{R}^+$ and for any u parabolic on $\mathbb{R}^n \times \mathbb{R}^+$: $\inf\limits_{X \in \mathbb{R}^n \times \mathbb{R}^+} u(X) = \inf\limits_{X \in \mathbb{R}^n \times]0,T[} u(X)$ (both assertions follow immediately from Theorem III.1), so (i) $\Leftrightarrow$ (v) is true on $\mathbb{R}^n \times]0,T[$ as well. We will prove (iii) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iv).

Then the assumption $\beta > 1$ will be removed and in the end the rest of Theorem and Corollary will be proved.

III. Parabolic and coparabolic potential theory

1. The case of $\mathbb{R}^{n+1}$.

Parabolic and coparabolic function (see [4, p. 263]).
A real function u on an open set $D \subset \mathbb{R}^{n+1}$ having continuous partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i^2}$ for $i = 1, \ldots, n$, and satisfying the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \qquad \Big(\text{resp. } \frac{\partial u}{\partial t} = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\ \Big)$$

on D is called parabolic (resp. coparabolic) on D.

A function $(x,t) \to u(x,t)$ is coparabolic on D, if and only if the function $(x,t) \to u(x,-t)$ is parabolic on $D^{\ominus}$.

The Green function of $\mathbb{R}^{n+1}$ (see [4, p. 266]) **.**
Let b be a function on $\mathbb{R}^{n+1}$ defined as

$$\begin{aligned} b(x,t) &= (4\pi t)^{-n/2}\exp(-\frac{\|x\|^2}{4t}) && \text{for } t>0,\\ &= 0 && \text{for } t \leqq 0. \end{aligned}$$

The Green function G is defined on $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ by

$$G(X,Y) = G((x,t),(y,s)) = b(x-y,t-s).$$

The function $G(.,Y)$ is the Green function with pole Y for the heat equation. This function is positive, parabolic on $\mathbb{R}^{n+1}\setminus\{Y\}$ and vanishes below Y and the limit at the point ∞ is zero.

The function $G(X,.)$ is the Green function with pole X for the adjoint equation. This function is positive, coparabolic on $\mathbb{R}^{n+1}\setminus\{X\}$ and vanishes above X and the limit at the point ∞ is zero.

If μ is a measure on $\mathbb{R}^{n+1}$ the functions $G\mu$ and μG defined by

$$G\mu(X) = \int\limits_{\mathbb{R}^{n+1}} G(X,Y)\,d\mu(Y) \qquad \text{and} \qquad \mu G(Y) = \int\limits_{\mathbb{R}^{n+1}} G(X,Y)\,d\mu(X)$$

will be called potential and copotential on $\mathbb{R}^{n+1}$, respectively.

The Green function of an interval (see [4, p. 272])**.**
Let

$$I = (a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)\times(t_1,t_2)$$

and $c_j = b_j - a_j$.
The Green function G_I is defined by

$$G_I(X,Y) = \prod_{j=1}^{n}\sum_{i=-\infty}^{\infty}[b(2ic_j - x_j + y_j, t-s) - b(2ic_j + 2a_j - x_j - y_j, t-s)]$$

for $X=(x_1,x_2,\dots,x_n,t)$ and $Y=(y_1,y_2,\dots,y_n,s)$ and $-\infty<t_1<t<t_2<\infty$.

Parabolic measure for an interval (see [4, p. 273])**.**
Let $X\in I$, $\mu_I(X,.)$ be supported by the part of ∂I strictly below X:
- on the lower boundary $\mu_I(X,.)$ is absolutely continuous relative to λ_n with continuous density

$$Y \to G_I(X,Y);$$

- on the part of the lateral boundary with jth coordinate b_j

$$Y \to -\frac{\partial}{\partial y_j} G_I(X,Y);$$

- on the part of the lateral boundary with jth coordinate a_j

$$Y \to \frac{\partial}{\partial y_j} G_I(X,Y).$$

Parabolic averages (see [4, p. 275]).

Let $X \in \mathbb{R}^{n+1}$ and $\delta > 0$. Recall that

$$I(X,\delta) = (x_1-\delta, x_1+\delta) \times (x_2-\delta, x_2+\delta) \times \cdots \times (x_n-\delta, x_n+\delta) \times (t-\delta^2, t).$$

If u is a Borel measurable function on $\partial I(X,\delta)$, we define

$$L(u,X,\delta) = \mu_{I(X,\delta)}(X,u).$$

If u is a parabolic function on D and $\overline{I}(X,\delta) \subset D$, then

$$u(X) = L(u,X,\delta).$$

Superparabolic, subparabolic, cosuperparabolic and cosubparabolic functions (see [4, p. 277]).

A function u from an open set D into $]-\infty,\infty]$ is called superparabolic if

(a) u is lower semicontinuous;

(b) u is finite on a dense subset of D;

(c) $u(X) \geqq L(u,X,\delta)$ if $\overline{I}(X,\delta) \subset D$.

A subparabolic function is defined as the negative of a superparabolic function.

A cosuperparabolic (resp. cosubparabolic) function is defined as a function on an open set D for which the function $(x,t) \to u(x,-t)$ is superparabolic (resp. subparabolic) on $D^{\ominus}$.

Let u be a function defined on D. The greatest subparabolic minorant, if there is one, is denoted by $GM_D u$. If u is a superparabolic function which has a subparabolic minorant, then $GM_D u$ exists and is parabolic. (See [4, p. 295].)

The Green function of an open set D (see [4, p. 298]).

Let D be a nonempty open subset of $\mathbb{R}^{n+1}$ and Y a point of D. The parabolic Green function with pole Y is defined on D by

$$G_D(.,Y) = G(.,Y) - GM_DG(.,Y).$$

The function $G_D(.,Y)$ is positive and superparabolic on D, parabolic on $D\setminus\{Y\}$ and differs from $G(.,Y)$ by a continuous function and $GM_DG_D(.,Y) = 0$.

The Riesz decomposition (see [4, p. 305]).

Let D be a nonempty open subset of $\mathbb{R}^{n+1}$. If v is a superparabolic function on D which has a subparabolic minorant on D, then there exist a parabolic function u on D and a measure μ on D such that $v = G_D\mu + u$ on D.

Parabolic reduction operation (see [4, p. 310]) .

Let D be an open subset of $\mathbb{R}^{n+1}$ and $M \subset D$. Let v be a positive superparabolic (resp. cosuperparabolic) function on D.

The superparabolic (resp. cosuperparabolic) reduction of v on M is defined as

$$R_v^M = \inf\{u; u \text{ is positive superparabolic function on } D, u \geqq v \text{ on } M\};$$
$$(\text{resp.}^*R_v^M = \inf\{u; u \text{ is positive cosuperparabolic function on } D, u \geqq v \text{ on } M\}).$$

The smooth reduction $\|v\|^M(x,t)$ (resp. ${}^*\|v\|^M(x,t)$) is defined by

$$\|v\|^M(x,t) = \liminf_{(y,s)\to(x,t)} R_v^M(y,s);$$
$$(\text{resp.}^*\|v\|^M(x,t) = \liminf_{(y,s)\to(x,t)} {}^*R_v^M(y,s)).$$

If the set D is not specified it is supposed that $D = \mathbb{R}^{n+1}$.

Theorem. *Let $M \subset \mathbb{R}^{n+1}$, G be the Green function. Then*

$${}^*\|G(X,.)\|^M(Y) = \|G(.,Y)\|^M(X).$$

The common value will be denoted by $G^M(X,Y)$.

If μ is a measure on $\mathbb{R}^{n+1}$, then

$$\|G\mu\|^M = G^M\mu \qquad \text{and} \qquad \|\mu G\|^M = \mu G^M.$$

Proof: (See [4, p. 342].) □

Parabolic and coparabolic thinness (see [4, p. 346]).

A set $M \subset \mathbb{R}^{n+1}$ is said to be parabolic (resp. coparabolic) thin at X (resp. at Y), if

$$G^M(X,.) \neq G(X,.) \text{ on the set } \{G(X,.) > 0\};$$
$$(\text{resp. } G^M(.,Y) \neq G(.,Y) \text{ on the set } \{G(.,Y) > 0\}).$$

This definition is equivalent to the "usual" definition of parabolic and coparabolic thinness. (See [4, p. 346].)

Parabolic (resp. coparabolic) thinness is a local property. It means: Let $r \in \mathbb{R}^+$. A set M is parabolic (resp. coparabolic) thin at X, if and only if $M \cap B(X,r)$ is parabolic (resp. coparabolic) thin at X.

It is clear that the set $M \subset \mathbb{R}^{n+1}$ is parabolic thin at $X = (x,t)$, if and only if $M^{\ominus}$ is coparabolic thin at $(x,-t)$.

2. The case of the slab.

Let $0 < T \leqq \infty$. The Green function for a slab $\mathbb{R}^n \times]0,T[$ is the restriction of G to $(\mathbb{R}^n \times]0,T[) \times (\mathbb{R}^n \times]0,T[)$.

The Weierstrass kernel for $\mathbb{R}^n \times]0,T[$ with the pole at y in $\mathbb{R}^n$ is given by

$$p(X,y) = (4\pi t)^{-n/2} \exp(-\frac{\|x-y\|^2}{4t}),$$

where $X = (x,t) \in \mathbb{R}^n \times]0,T[$.

Clearly, $p(X,y) = b(x-y,t) = G(X,(y,0))$.

Theorem 1. *A function u on $\mathbb{R}^n \times]0,T[$ is a difference of two positive parabolic functions, if and only if there is a signed measure μ_u on $\mathbb{R}^n$ for which*

$$\int_{\mathbb{R}^n} \exp(-\frac{\|y\|^2}{4t})\, d|\mu_u|(y) < \infty$$

for all $t \in]0,T[$ and

$$u(X) = \int_{\mathbb{R}^n} p(X,y)\, d\mu_u(y), \;\; X \in \mathbb{R}^n \times]0,T[.$$

The map $u \to \mu_u$ is a one-to-one linear order-preserving map from the class of parabolic functions satisfying these conditions onto the vector lattice of charges on $\mathbb{R}^n$ satisfying the above inequality.

The function u is bounded, if and only if there exists $f_u \in L_\infty(\mathbb{R}^n)$ such that $\mu_u = f_u \lambda$.

Proof: (See [4, p. 290].) □

From this theorem $\inf\limits_{X\in\mathbb{R}^n\times]0,T[} u(X) = \operatorname*{ess\,inf}\limits_{y\in\mathbb{R}^n} f_u(y)$ for any bounded parabolic function u on $\mathbb{R}^n\times]0,T[$.

Coparabolic minimal thinness on $\mathbb{R}^n\times]0,T[$.

Let $M \subset \mathbb{R}^n\times]0,T[$, $Y \in \mathbb{R}^n \times \{0\}$. The set M is said to be coparabolic minimal thin at Y, if

$$\|p(.,y)\|^M \neq p(.,y) \text{ on the set } \{p(.,y) > 0\}.$$

(Compare [4, p. 378].)

The reduction is, of course, taken with respect to $D = \mathbb{R}^n\times]0,T[$. But:

The restriction of any function superparabolic on $\mathbb{R}^{n+1}$ to $\mathbb{R}^n\times]0,T[$ is superparabolic on $\mathbb{R}^n\times]0,T[$.

If v is a positive function superparabolic on $\mathbb{R}^n\times]0,T[$, there exists a positive parabolic function u on $\mathbb{R}^n\times]0,T[$ and a measure μ on $\mathbb{R}^n\times]0,T[$ such that

$$v = G_{\mathbb{R}^n\times]0,T[}\mu + u \text{ on } \mathbb{R}^n\times]0,T[.$$

But knowing that $G_{\mathbb{R}^n\times]0,T[} = G$ and $p(X,y) = G(X,(y,0)$ on $\mathbb{R}^n\times]0,T[$ and using the representation of u guaranteed by previous Theorem, we have

$$v = G\mu + G\mu_u \text{ on } \mathbb{R}^n\times]0,T[.$$

The function $G\mu + G\mu_u$ is a positive superparabolic function on $\mathbb{R}^{n+1}$.

So the reduction can be taken with respect to $\mathbb{R}^{n+1}$.

As $p(.,y) = G(.,Y)$, we have for any $M \subset \mathbb{R}^n\times]0,T[$:

$$\|p(.,y)\|^M = \|G(.,Y)\|^M = G^M(.,Y).$$

It means that if $M \subset \mathbb{R}^n \times \mathbb{R}^+$ and $Y \in \mathbb{R}^n \times \{0\}$, M is coparabolic thin at Y, if and only if $M \cap (\mathbb{R}^n\times]0,T[)$ is coparabolic minimal thin at Y. So we will write M is coparabolic (minimal) thin at Y.

Theorem 2. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$. If the set of points of $\mathbb{R}^n \times \{0\}$ at which M is coparabolic* (*minimal*) *thin has Lebesgue measure equal to zero, then $\|1\|^M = 1$ on $\mathbb{R}^n \times \mathbb{R}^+$.*

Proof: We have $1 = \int\limits_{\mathbb{R}^n} p(X,y)\,d\lambda(y)$ on $\mathbb{R}^n \times \mathbb{R}^+$ and

$$\|1\|^M = \|\int\limits_{\mathbb{R}^n} p(X,y)\,d\lambda(y)\|^M = \|\int\limits_{\mathbb{R}^{n+1}} G(X,Y)\,d\lambda(Y)\|^M = \|G\lambda\|^M = G^M\lambda$$

on $\mathbb{R}^n \times \mathbb{R}^+$.

Because $G^M(.,Y) = G(.,Y)$ on $\mathbb{R}^n \times \mathbb{R}^+$ for λ-almost all Y we have $G^M\lambda = G\lambda = 1$ on $\mathbb{R}^n \times \mathbb{R}^+$. $\square$

Theorem 3 (Harnack inequality)**.** *Let $n \in \mathbb{N}$. Then there exists a constant c_H such that for any $T \in]0, \infty]$ and any (x_1, t_1), (x_2, t_2) belonging to $\mathbb{R}^n \times]0, T[$ such that $t_2 > t_1$ and for any positive parabolic function u on $\mathbb{R}^n \times]0, T[$*

$$u(x_1, t_1) \leqq u(x_2, t_2).e^{c_H(\frac{\|x_2 - x_1\|}{t_2 - t_1} + 1)} \left(\frac{t_2}{t_1}\right)^{c_H}.$$

Proof: (See [6, p. 104].) □

3. Parabolic capacity.

Let $K \subset \mathbb{R}^{n+1}$ be a compact set. The parabolic capacity of K is defined by

$$\gamma(K) = \sup\{\mu(\mathbb{R}^{n+1}); \mu \in \mathcal{M}^+(K), G\mu \leqq 1 \text{ in } \mathbb{R}^{n+1}\},$$

where $\mathcal{M}^+(K)$ is the set of Borel measures supported by K.

Let $M \subset \mathbb{R}^{n+1}$ be an arbitrary set. Then

$$\gamma_\star(M) = \sup\{\gamma(K); K \subset M, K \text{ compact}\}$$

is called the inner parabolic capacity of M and

$$\gamma^\star(M) = \inf\{\gamma_\star(G); G \supset M, G \text{ open}\}$$

the outer parabolic capacity of M.

Lemma 1. *Let F be a Borel subset of $\mathbb{R}^{n+1}$, $t_0 \in \mathbb{R}$. Then*

$$\gamma(F \times \{t_0\}) = \lambda_n(F).$$

Proof: See [8, p. 355]. □

Theorem. *Let $M \subset \mathbb{R}^{n+1}$, $Y \in \mathbb{R}^{n+1}$. Then M is parabolic thin at Y, if and only if*

$$\int_0^1 \gamma^\star(M \cap B^p(Y, r)) r^{-\frac{n}{2} - 1} \, dr < \infty.$$

Proof: See [3, p. 99]. □

From this and from what was said about the relation between parabolic and coparabolic thinness and the relation between coparabolic minimal thinness on $\mathbb{R}^n \times]0, T[$ and coparabolic thinness on $\mathbb{R}^{n+1}$ follows:

Theorem 4. *Let $M \subset \mathbb{R}^n \times \mathbb{R}^+$ and $Y \in \mathbb{R}^n \times \{0\}$. Then M is coparabolic (minimal) thin at Y, if and only if*

$$\int_0^1 \gamma^\star(M^\ominus \cap B^p(Y,r)) r^{-\frac{n}{2}-1}\, dr < \infty.$$

4. Geometrical properties of the heat ball, the coparabolic ball and the paraboloid.

Lemma 2. *Let $X \in \mathbb{R}^{n+1}$, $r \in \mathbb{R}^+$.*
Then

$$B^p(X,r) \subset C(X, \sqrt{\frac{2n}{e}}\sqrt{r}, [t, t-r]);$$

$$B^p(X,r) \supset D((x, t-\frac{r}{e}), \sqrt{\frac{2n}{e}}\sqrt{r});$$

and

$$B^{cp}(X,r) \subset C(X, \sqrt{\frac{2n}{e}}\sqrt{r}, [t, t+r]);$$

$$B^{cp}(X,r) \supset D((x, t+\frac{r}{e}), \sqrt{\frac{2n}{e}}\sqrt{r}).$$

Lemma 3. *Let $a \in \mathbb{R}^+$, then there exists a number a_1 such that for all $X \in \mathbb{R}^{n+1}$, $r \in \mathbb{R}^+$ and $v \leqq a_1 r$ is $P(X,a,v) \subset B^{cp}(X,r)$.*

Lemma 4. *Let $a_1, \alpha, \beta \in \mathbb{R}^+$.*

Then there exists a number a_2 such that for any $Y \in \mathbb{R}^{n+1}$, and for any $X \in P(Y, c_1)$, $X \neq Y$, the disc $D_{X,\alpha,\beta}$ is a subset of $P(Y, a_2)$.

Proofs of these lemmas are elementary.

IV. Proof of Theorem

1. In this part it will be proved that (iii) ⇒ (ii) for $\beta > 1$.

Theorem 1. *Let $n \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{R}^+$ and $\gamma \geqq \beta > 1$. Then there exists a positive constant c such that for every $T \in]0, \infty]$ and $M \subset \mathbb{R}^n \times]0, T[$, and for every positive parabolic function u on $\mathbb{R}^n \times]0, T[$,*

$$\inf_{X \in M_{C_{\alpha,\beta,\gamma}}} u(X) \geqq c \inf_{X \in M} u(X).$$

Proof: Let $M \subset \mathbb{R}^n \times]0, T[$ and $X = (x,t)$, $X \in M$. Since $\beta > 1$, $T \geqq \gamma t \geqq s \geqq \beta t > t$ whenever $(y,s) \in C_{X,\alpha,\beta,\gamma} \cap (\mathbb{R}^n \times]0, T[)$, we have by the Harnack inequality:

$$u(y,s).e^{c_H(\frac{\|y-x\|^2}{s-t}+1)}.\left(\frac{s}{t}\right)^{c_H} \geqq u(x,t).$$

Using the fact that $\gamma t \geqq s \geq \beta t$ and $\|y-x\| \leqq \alpha\sqrt{t}$ we arrive at

$$u(y,s).e^{c_H(\frac{\alpha^2 t}{\beta t - t}+1)}.\left(\frac{\gamma t}{t}\right)^{c_H} \geqq u(x,t)$$

or

$$u(y,s).e^{c_H(\frac{\alpha^2}{\beta-1}+1)}.\gamma^{c_H} \geqq u(x,t),$$

thus

$$u(y,s) \geqq e^{-c_H(\frac{\alpha^2}{\beta-1}+1)}.\gamma^{-c_H} u(x,t).$$

From here the theorem immediately follows. □

Theorem 2. *Let* $0 < T \leqq \infty$ *and* $M \subset \mathbb{R}^n \times]0,T[$ *for which there exist* $\alpha,\beta,\gamma \in \mathbb{R}^+$, $\gamma \geqq \beta > 1$ *such that the set of points of* $\mathbb{R}^n \times \{0\}$ *at which* $M_{C_{\alpha,\beta,\gamma}}$ *is (minimal) coparabolic thin has Lebesgue measure zero.*

Then there exists a constant c *depending only on* α,β,γ,n *such that*

$$\inf_{X\in\mathbb{R}^n\times]0,T[} u(X) \geqq c \inf_{X\in M} u(X)$$

for all positive parabolic functions u *on* $\mathbb{R}^n\times]0,T[$.

Proof: This theorem is obtained by combining the previous Theorem and Theorem III.2.

□

Theorem 3. *Let* $0 < T \leqq \infty$ *and* $M \subset \mathbb{R}^n\times]0,T[$. *Then the following statements are equivalent:*

(i)

$$\inf_{X\in\mathbb{R}^n\times]0,T[} u(X) = \inf_{X\in M} u(X)$$

for all positive parabolic functions u *on* $\mathbb{R}^n\times]0,T[$;

(ii) *there exists* $c > 0$ *such that*

$$\inf_{X\in\mathbb{R}^n\times]0,T[} u(X) \geqq c \inf_{X\in M} u(X)$$

for all positive parabolic functions u *on* $\mathbb{R}^n\times]0,T[$.

Proof: (i) $\Rightarrow$ (ii) is clear, put $c = 1$.
(ii) $\Rightarrow$ (i) Let us suppose that there exists a set M satisfying (ii), but not (i). Then c in (ii) belongs to $(0,1)$.

Let u be a positive parabolic function for which (i) is not true.

$$\text{Denote} \quad \inf_{X\in\mathbb{R}^n\times]0,T[} u(X) = c_1 \quad \text{and} \quad \inf_{X\in M} u(X) = c_2.$$

We suppose that $c_2 > c_1 \geqq c.c_2$. Let ε be a positive number and $v(X) = u(X) - c_1 + \varepsilon$ for $X \in \mathbb{R}^n \times]0,T[$.

Then v is a positive parabolic function and

$$\inf_{X \in \mathbb{R}^n \times]0,T[} v(X) = c_1 - c_1 + \varepsilon = \varepsilon, \quad \text{and} \quad \inf_{X \in M} v(X) = c_2 - c_1 + \varepsilon.$$

It follows from (ii) that $\varepsilon \geqq c(c_2 - c_1 + \varepsilon)$ for every $\varepsilon > 0$, which is a contradiction. □

Theorem 4. *Let $0 < T \leqq \infty$ and $M \subset \mathbb{R}^n \times]0,T[$ for which there exist $\alpha, \beta, \gamma \in \mathbb{R}^+$, $\gamma \geqq \beta > 1$ such that the set of points of $\mathbb{R}^n \times \{0\}$ at which $M_{C_{\alpha,\beta,\gamma}}$ is (minimal) coparabolic thin has Lebesgue measure zero.*

Then

$$\inf_{X \in \mathbb{R}^n \times]0,T[} u(X) = \inf_{X \in M} u(X)$$

for all positive parabolic functions u on $\mathbb{R}^n \times]0,T[$.

Proof: The result is obtained by combining two previous Theorems. □

The implication (iii) $\Rightarrow$ (ii) is proved.

2. In this part the implication (v) $\Rightarrow$ (iv) will be proved.

Lemma 1. *Let $\{\alpha_k\}_{k=0}^{\infty}$ be a decreasing sequence of strictly positive numbers with limit zero.*

Then

$$\sum_{k=1}^{\infty} (1 - \frac{\alpha_k}{\alpha_{k-1}}) = \infty.$$

Proof: The infinite product $\prod_{k=1}^{\infty} \frac{\alpha_k}{\alpha_{k-1}}$ obviously diverges to 0. Consequently, the above sum diverges. □

Theorem 5. *Let $0 < T \leqq \infty$ and $M \subset \mathbb{R}^n \times]0,T[$, $Y \in \mathbb{R}^n \times \{0\}$ and $\alpha, \beta \in \mathbb{R}^+$. If Y is a parabolic limit point of M, then $M_{D_{\alpha,\beta}}$ is not coparabolic (minimal) thin at Y.*

Proof: Theorem III.4 will be used and so we are interested in the set $M_{D_{\alpha,\beta}}^{\ominus} \cap B^p(Y,r)$, which is clearly equal to the set $(M_{D_{\alpha,\beta}} \cap B^{cp}(Y,r))^{\ominus}$.

Let $a_1 \in \mathbb{R}^+$ and $\{X_k\}_{k=1}^{\infty}$ be a sequence of points of M for which

$$X_k = (x_k, t_k), \quad \lim_{k \to \infty} x_k = y, \quad \lim_{k \to \infty} t_k = 0, \quad \|x_k - y\| \leqq a_1 t_k.$$

We can suppose that $t_k \downarrow 0$.

By Lemma III.4, there exists a constant $a_2 \in \mathbb{R}^+$ depending on a_1, α, β such that $D_{X_k,\alpha,\beta}$ is a subset of $P(Y, a_2)$ for all X_k.

Having a_2, there exists, by Lemma III.3, a positive constant a_3, such that $P(Y, a_2, v) \subset B^{cp}(Y, r)$ for all $v \leqq a_3 r$.

Now it is clear that $D_{X_k,\alpha,\beta} \subset B^{cp}(Y, r)$ for all $X_k = (x_k, t_k)$ satisfying $\beta t_k \leqq a_3 r$. Let us denote $r_k = \frac{\beta}{a_3} t_k$. Without loss of generality we can suppose that $r_1 \leqq 1$ and put $r_0 = 1$. Of course, $r_k \downarrow 0$.

Now we have that $D_{X_k,\alpha,\beta} \subset B^{cp}(y, r)$ for all $k \in \mathbb{N}$ such that $r \geqq r_k$.

For $r \in]r_k, r_{k-1}]$ let us take instead of $M_{D_{\alpha,\beta}} \cap B^{cp}(Y, r)$ its subset $D_{X_k,\alpha,\beta}$. Then

$$\int_0^1 \gamma^\star(M^{\ominus}_{D_{\alpha,\beta}} \cap B^p(Y,r)) r^{-\frac{n}{2}-1}\, dr = \int_0^1 \gamma^\star((M_{D_{\alpha,\beta}} \cap B^{cp}(Y,r))^{\ominus}) r^{-\frac{n}{2}-1}\, dr =$$

$$\sum_{k=1}^{\infty} \int_{r_k}^{r_{k-1}} \gamma^\star((M_{D_{\alpha,\beta}} \cap B^{cp}(Y,r))^{\ominus}) r^{-\frac{n}{2}-1}\, dr \geqq \sum_{k=1}^{\infty} \int_{r_k}^{r_{k-1}} \gamma^\star(D^{\ominus}_{X_k,\alpha,\beta}) r^{-\frac{n}{2}-1}\, dr =$$

$$\sum_{k=1}^{\infty} \gamma^\star(D^{\ominus}_{X_k,\alpha,\beta}) \left[-\frac{2}{n} r^{-\frac{n}{2}}\right]_{r_k}^{r_{k-1}} = \frac{2}{n} \sum_{k=1}^{\infty} \gamma^\star(D^{\ominus}_{X_k,\alpha,\beta})(r_k^{-\frac{n}{2}} - r_{k-1}^{-\frac{n}{2}}).$$

Let κ denote the volume of the unit ball in $\mathbb{R}^n$. Since $D^{\ominus}_{X_k,\alpha,\beta} = B(x_k, \alpha\sqrt{t_k}) \times \{-\beta t_k\}$, Lemma III.1. yields

$$\gamma^\star(D_{X_k,\alpha,\beta}) = \lambda_n(B(x_k, \alpha\sqrt{t_k})) = \kappa\alpha^n t_k^{\frac{n}{2}}$$

and this is equal to $\kappa\alpha^n (\frac{a_3}{\beta})^{\frac{n}{2}} r_k^{\frac{n}{2}}$.

Denoting $a_4 = \kappa\alpha^n (\frac{a_2}{\beta})^{\frac{n}{2}}$ we arrive at

$$\gamma^\star(D^{\ominus}_{X_k,\alpha,\beta}) = a_4 r_k^{\frac{n}{2}}.$$

So the series is equal to

$$\frac{2}{n} a_4 \sum_{k=1}^{\infty} r_k^{\frac{n}{2}} \left(\frac{1}{r_k^{\frac{n}{2}}} - \frac{1}{r_{k-1}^{\frac{n}{2}}}\right) = \frac{2}{n} a_4 \sum_{k=1}^{\infty} \left(1 - \left(\frac{r_k}{r_{k-1}}\right)^{\frac{n}{2}}\right)$$

and by Lemma 1 its sum is equal to ∞, finishing the proof. $\square$

The implication (v) $\Rightarrow$ (iv) immediately follows from this theorem.

3. So far we have proved the equivalence of (i), (ii), (iii), (iv), (v) for $\beta > 1$. Now this condition will be removed.

Let $0 < T \leqq \infty$ and $M \subset \mathbb{R}^n \times]0, T[$, $d \in \mathbb{R}^+$ and

$$M(d) = \{(y,s) \in \mathbb{R}^n \times]0,T[; \text{ex. } (x,t) \in M, x = y, s = d.t\}.$$

Let $X = (x,t)$, $X \in \mathbb{R}^n \times \mathbb{R}^+$. We denote $X(d)$ the point $(x, d.t)$.

Lemma 2. *Let $0 < T \leqq \infty$ and $M \subset \mathbb{R}^n \times]0,T[$, $d \in \mathbb{R}^+$, $Y \in \mathbb{R}^n \times \{0\}$. The point Y is a parabolic limit point of the set M if and only if Y is a parabolic limit point of $M(d)$.*

Proof: Let Y be a parabolic limit point of M. Then there exist $c \in \mathbb{R}^+$ and $\{X_k\}_{k=1}^{\infty}$ such that

$$X_k = (x_k, t_k) \in M, \lim_{k\to\infty} t_k = 0, \text{ and } \|x_k - y\| \leqq ct_k.$$

There exist k_0 such that for all $k > k_0$ is $dt_k < T$ and so $X_k(d) \in M(d)$.

$$\text{And } \lim_{k\to\infty} dt_k = d \lim_{k\to\infty} t_k = 0, \text{ so } \lim_{k\to\infty} X_k(d) = Y \text{ and } \|x_k - y\| \leqq \frac{c}{d}.d.t_k.$$

Thus Y is a parabolic limit of $\{X_k(d)\}_{k=k_0}^{\infty}$ and thus a parabolic limit point of $M(d)$.

As $M(d)(\frac{1}{d}) \subset M$, the opposite is true. □

Remark. From this lemma and Theorem (v) it follows that M is a set of determination if and only if $M(d)$ is a set of determination.

Lemma 3. *Let $X \in \mathbb{R}^n \times \mathbb{R}^+$, $X \in \mathbb{R}^n \times \mathbb{R}^+$, $\alpha, \beta, \gamma, d \in \mathbb{R}^+$. Then*

$$C_{X,\alpha,\beta,\gamma} = C_{X(d),\alpha,\frac{\beta}{d},\frac{\gamma}{d}} \qquad \text{and} \qquad M_{C_{\alpha,\beta,\gamma}} = M(d)_{C_{\alpha,\frac{\beta}{d},\frac{\gamma}{d}}}.$$

Proof: A straightforward calculation. □

Now we will remove the condition $\beta > 1$:

Let $\alpha, \beta, \gamma \in \mathbb{R}^+, \gamma \geqq \beta > 0$.

Using Remark to Lemma 2 we have the equivalence of these conditions:

(i) M is a set of determination;

(i_1) there exists $\beta \in \mathbb{R}^+$ such that $M(\frac{\beta}{2})$ is a set of determination;

(i_2) for any $\beta \in \mathbb{R}^+$, $M(\frac{\beta}{2})$ is a set of determination.

Now we will use the equivalence of (i), (iii) and (iv) of Theorem for $M(\frac{\beta}{2})_{C_{\alpha,2,2\frac{\gamma}{\beta}}}$. ($2>1$.) From the equivalence of (i) and (iii) it follows that (i_1) is equivalent with (ii_1) and from the equivalence of (i) and (iv) it follows that (i_2) is equivalent with (ii_2):

(ii_1) There exist $\alpha,\beta,\gamma\in\mathbb{R}^+,\gamma\geqq\beta$ such that the set of points of $\mathbb{R}^n\times\{0\}$ at which $M(\frac{\beta}{2})_{C_{\alpha,2,2\frac{\gamma}{\beta}}}$ is (minimal) coparabolic thin has Lebesgue measure zero;

(ii_2) for any $\alpha,\beta,\gamma\in\mathbb{R}^+,\gamma\geqq\beta$ the set of points of $\mathbb{R}^n\times\{0\}$ at which $M(\frac{\beta}{2})_{C_{\alpha,2,2\frac{\gamma}{\beta}}}$ is (minimal) coparabolic thin has Lebesgue measure zero.

From Lemma 3 we have

$$M_{C_{\alpha,\beta,\gamma}}=M(\frac{\beta}{2})_{C_{\alpha,2,2\frac{\gamma}{\beta}}}.$$

Using this equality, (ii_1) and (ii_2) can be formulated in this way:

(iii) There exist $\alpha,\beta,\gamma\in\mathbb{R}^+$, $\gamma\geqq\beta$ such that the set of points of $\mathbb{R}^n\times\{0\}$ at which $M_{C_{\alpha,\beta,\gamma}}$ is (minimal) coparabolic thin has Lebesgue measure zero;

(iv) for any $\alpha,\beta,\gamma\in\mathbb{R}^+$, $\gamma\geqq\beta$ the set of points of $\mathbb{R}^n\times\{0\}$ at which $M_{C_{\alpha,\beta,\gamma}}$ is (minimal) coparabolic thin has Lebesgue measure zero.

The condition $\beta>1$ was removed. The equivalence of (i), (ii), (iii), (iv) and (v) is proved.

4. In this part the rest of Theorem and Corollary will be proved.

Lemma 4. *Let $0<T\leqq\infty$ and $M\subset\mathbb{R}^n\times]0,T[$ and $\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2\in\mathbb{R}^+$ and let with every point of M a set A_X be associated such that*

$$D_{X,\alpha_2,\beta_2}\subset A_X\subset C_{X,\alpha_1,\beta_1,\gamma_1}.$$

Then M is a set of determination if and only if the set of points of $\mathbb{R}^n\times\{0\}$ at which M_A is (minimal) coparabolic thin has Lebesgue measure zero.

Proof: Let M be a set of determination. Then by Theorem (iv), the set of points of $\mathbb{R}^n\times\{0\}$ at which $M_{D_{\alpha_2,\beta_2}}$ is (minimal) coparabolic thin has Lebesgue measure zero. From $M_A\supset M_{D_{\alpha_2,\beta_2}}$ the assertion of the lemma follows.

If the set of points of $\mathbb{R}^n\times\{0\}$ at which M_A is (minimal) coparabolic thin has Lebesgue measure zero, then the same is true for $M_{C_{\alpha_1,\beta_1,\gamma_1}}$ (because $M_A\subset M_{C_{\alpha_1,\beta_1,\gamma_1}}$) and by Theorem (iii) the converse implication of this lemma follows. $\square$

Proof of Theorem (v), (vi) and Corollary:

Using Lemma III.2 we have

$$D_{X,\sqrt{\frac{2n}{e}\delta},1-\frac{\delta}{e}} \subset B^{p}_{X,\delta} \subset C_{X,\sqrt{\frac{2n}{e}\delta},1-\delta,1};$$

and
$$D_{X,\sqrt{\frac{2n}{e}\delta},1+\frac{\delta}{e}} \subset B^{cp}_{X,\delta} \subset C_{X,\sqrt{\frac{2n}{e}\delta},1,1+\delta}.$$

Similar properties are true for the paraboloid and the interval:

$$D_{X,\sqrt{\delta},1+\delta} \subset P_{X,a,\delta} \subset C_{X,\sqrt{\delta},1,1+\delta,};$$

and
$$D_{X,\delta,1} \subset I_{X,\delta} \subset C_{X,\delta,1-\delta^2,1}.$$

From here and from previous lemma, Theorem (v), (vi) and Corollary immediately follow.

□

Added in the proof. After having submitted the paper I found out that Theorem 5 is a known result, see Proposition 3.1 in Mair B.: *Fine and parabolic limits for solutions of second order linear parabolic equations on an infinite slab*, Trans. Amer. Math. Soc. **284** (1984), 583–599.

Theorem 5 is a consequence of Corollary 2 in Netuka I.: *Thinnesss and the heat equation*, Časopis Pěst. Mat. **99** (1974), 293–299, as well.

References

[1] Aikawa H., *Sets of determination for harmonic function in an NTA domains*, J. Math. Soc. Japan, to appear.
[2] Bonsall F.F., *Domination of the supremum of a bounded harmonic function by its supremum over a countable subset*, Proc. Edinburgh Math. Soc. **30** (1987), 441–477.
[3] Brzezina M., *On the base and the essential base in parabolic potential theory*, Czechoslovak Math. J. **40 (115)** (1990), 87–103.
[4] Doob J.L., *Classical Potential Theory and Its Probabilistic Counterpart*, Springer-Verlag, New York, 1984.
[5] Gardiner S.J., *Sets of determination for harmonic function*, Trans. Amer. Math. Soc. **338.1** (1993), 233–243.
[6] Moser J., *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **XVII** (1964), 101–134.
[7] Ranošová J., *Sets of determination for parabolic functions on a half-space*, Comment. Math. Univ. Carolinae **35** (1994), 497–513.
[8] Watson A.N., *Thermal capacity*, Proc. London Math. Soc. **37.3** (1987), 342–362.

Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

E-mail: ranosova@karlin.mff.cuni.cz

(*Received* October 12, 1995)