Comment.Math.Univ.Carolin. 37,4 (1996)867–872

On a theorem of Fermi

V.V. Slavskii

Abstract. Conformally flat metric $\bar{g}$ is said to be Ricci superosculating with g at the point x_0 if $g_{ij}(x_0) = \bar{g}_{ij}(x_0)$, $\Gamma^k_{ij}(x_0) = \bar{\Gamma}^k_{ij}(x_0)$, $R^k_{ij}(x_0) = \bar{R}^k_{ij}(x_0)$, where R_{ij} is the Ricci tensor. In this paper the following theorem is proved:

If γ is a smooth curve of the Riemannian manifold M (without self-crossing), then there is a neighbourhood of γ and a conformally flat metric $\bar{g}$ which is the Ricci superosculating with g along the curve γ.

Keywords: conformal connection, development

Classification: 53A30, 53C20

Let $g = g_{ij}dx^i dx^j$ and $\bar{g} = \bar{g}_{ij}dx^i dx^j$ be two metrics on the n-dimensional manifold M. The metric $\bar{g}$ is said to be tangent, respectively osculating with the metric g at the point $x_0 \in M$ ([1]), if

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \text{respectively,}$$

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \frac{\partial g_{ij}}{\partial x^k}(x_0) = \frac{\partial \bar{g}_{ij}}{\partial x^k}(x_0) \left(\text{or } \Gamma^k_{ij}(x_0) = \bar{\Gamma}^k_{ij}(x_0)\right),$$

where Γ^k_{ij} are the Cristoffel symbols.

If γ is a smooth curve of the manifold M, then there is a local Euclidean metric $\bar{g}$ which is osculating with the g along the curve (this theorem is due to Fermi [1]).

The Riemannian curvature tensor can be computed with the help of the first and second derivatives of the metric tensor g_{ij}; hence, there exists no local-Euclidean metric which is superosculating of degree two with g. But if we consider a class of conformally flat metrics, there is a conformally flat metric which has some equal combinations of first and second derivatives.

Definition. Conformally flat metric $\bar{g}$ is said to be Ricci superosculating with g at the point x_0 if

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \Gamma^k_{ij}(x_0) = \bar{\Gamma}^k_{ij}(x_0), \quad R_{ij}(x_0) = \bar{R}_{ij}(x_0),$$

where R_{ij} is the Ricci tensor.

Author is partially supported by a grant from the International Science Foundation (grant # JF3100) and a grant from the St.-Petersburg University

Theorem. *If γ is a smooth curve of the Riemannian manifold M (without self-crossing), then there is a neighbourhood of γ and a conformally flat metric $\bar{g}$ which is the Ricci superosculating with g along the curve γ.*

To prove the theorem we use the conformal connection and a conformal development of the curve ([2]–[4]). Let $R^{n+2} = R^{n+1} \times R$ be a pseudo-Euclidean Minkowski space with a Lorentz inner product given by formula ([5]):

$$\langle z_1, z_2 \rangle = (x_1, x_2) - \zeta_1 \zeta_2,$$

where $z_i = (x_i, \zeta_i) \in R^{n+2}$, $i = 1, 2$; (x_1, x_2) is an inner product for the Euclidean space R^{n+1}. Let $C^+ = \{z = (x, \zeta) : \langle z, z \rangle = 0, \zeta > 0\}$ denote the light cone of the Minkowski space. We consider the basis F for the space R^{n+2} as the column of vectors of R^{n+2}. Let $GL(n+1, 1)$ be a manifold of all the basis of the kind:

$$F^t = \{e_1, \dots, e_n, z, z^*\},$$

where $\{e_1, \dots, e_n\}$ are spacelike vectors (i.e. $\|\bar{g}_{ij}\| = \|\langle e_i, e_i \rangle\|$ is a positively definite matrix),

$$\langle z, e_i \rangle = \langle z^*, e_i \rangle = 0, \quad \langle z, z^* \rangle = -1, \quad \langle z \rangle^2 = \langle z^* \rangle^2 = 0. \tag{1}$$

Let us fix the basis $F_e \in GL(n+1, 1)$, then any other basis $F \in GL(n+1, 1)$ can be identified with the linear transformation $\mathcal{F}$ such that $F = \mathcal{F} \cdot F_e$. The matrix-valued 1-form Φ is defined by

$$\Phi = d\mathcal{F} \cdot \mathcal{F}^{-1} = \|\omega_{i.}^{\cdot j}\|, \quad i, j = 1, \dots, n+2,$$

or

$$dF = d(\mathcal{F} \cdot F_e) = d\mathcal{F} \cdot \mathcal{F}^{-1} \cdot \mathcal{F} \cdot F_e = \Phi \cdot F. \tag{2}$$

Now (1) implies

$$d\bar{g}_{ij} = \omega_{i.}^{\cdot k} \bar{g}_{kj} + \omega_{j.}^{\cdot k} \bar{g}_{ki},$$
$$\omega_{n+1.}^{\cdot n+2} = \omega_{n+2.}^{\cdot n+1} = 0, \quad \omega_{n+1.}^{\cdot n+1} + \omega_{n+2.}^{\cdot n+2} = 0,$$
$$\omega_{n+1.}^{\cdot k} \bar{g}_{ki} - \omega_{i.}^{\cdot n+2} = 0, \quad \omega_{n+2.}^{\cdot k} \bar{g}_{ki} - \omega_{i.}^{\cdot n+1} = 0.$$

Let us set

$$\omega_{n+1.}^{\cdot i} = \omega^i, \quad \omega_{i.}^{\cdot n+2} = \omega_i, \quad \omega_{n+2.}^{\cdot i} = \omega^{*i}, \quad \omega_{i.}^{\cdot n+1} = \omega_i^*, \quad \alpha = \omega_{n+1.}^{\cdot n+1}.$$

The equation of Cartan-Maurera $d\Phi + \frac{1}{2}[\Phi, \Phi] = 0$ for Φ is written

$$d\omega_{i.}^{\cdot j} = \sum_{k=1}^{n} \omega_{i.}^{\cdot k} \wedge \omega_{k.}^{\cdot j} + \omega_{i.}^{*} \wedge \omega^j + \omega_i \wedge \omega^{*j}, \quad d\alpha = \omega_i \wedge \omega^{*i},$$
$$d\omega^i = \omega^k \wedge \omega_{k.}^{\cdot i} + \alpha \wedge \omega^i, \quad d\omega^{*i} = \omega^{*k} \wedge \omega_{k.}^{\cdot i} - \alpha \wedge \omega^{*i}.$$

We now give a construction of the conformal Cartan connection. Let $g = g_{ij}dt^i dt^j$ be the Riemannian metric on the manifold M. Let Φ be the differential matrix 1-forms defined by the formulas:

$$\omega^i = dt^i, \quad \omega_{i.}^{\cdot j} = \Gamma_{ik}^j dt^k, \quad \omega_i^* = \frac{1}{n-2}\left(R_{ik}dt^k - \frac{Rg_{ik}dt^k}{2(n-1)}\right) = A_{ik}dt^k, \quad \alpha = 0,$$

where Γ_{ik}^j are the Cristoffel symbols, R_{ik} is the Ricci tensor, A_{ik} is the reduced Ricci tensor, R is the scalar curvature. The form Φ is called the normal conformal Cartan connection of the Riemannian manifold M ([2]–[4]). We denote this matrix of differential 1-forms by $\Phi = \{\omega^i, \omega_{i.}^{\cdot j}, \omega_i^*, 0\}$.

Definition. Let $\{M, g\}$ be the Riemannian manifold; and let α be a differential 1-form on M. The extended conformal Cartan connection matrix $\widetilde{\Phi} = \{\omega^i, \widetilde{\omega}_{i.}^{\cdot j}, \widetilde{\omega}_{i.}^*, \alpha\}$ on the manifold M, associated with α, is the matrix defined by the formulas:

$$\widetilde{\omega}_{i.}^{\cdot j} = \omega_{i.}^{\cdot j} - \alpha_i\omega^j + \alpha^j\omega_i = \left(\Gamma_{ik}^j - \alpha_i\delta_k^j + \alpha^j g_{ik}\right)dt^k,$$

$$\widetilde{\omega}_i^* = \omega_i^* - \alpha_i\alpha - D\alpha_i + \frac{|\alpha|^2}{2}\omega_i = \left(A_{ij} - \alpha_i\alpha_j - \alpha_{i,j} + \frac{|\alpha|^2}{2}g_{ij}\right)dt^j,$$

where $\alpha = \alpha^i\omega_i = \alpha_i\omega^i$, $D\alpha_i$ is the covariant derivative, $|\alpha|^2 = \alpha^i\alpha_i$.

The curvature matrix of the extended Cartan connection is defined by the formula:

$$d\widetilde{\Phi} + \frac{1}{2}[\widetilde{\Phi}, \widetilde{\Phi}] = \begin{pmatrix} \widetilde{W}_{i.}^{\cdot j} & | & \widetilde{S}_i & | & 0 \\ — & & — & & — \\ 0 & | & 0 & | & 0 \\ — & & — & & — \\ \widetilde{S}^j & | & 0 & | & 0 \end{pmatrix},$$

where $\widetilde{W}_{i.}^{\cdot j} = W_{i.}^{\cdot j}$ are the forms of the Weyl tensor (it is independent of the choice of α), $\widetilde{S}^i = S^i + \alpha^s W_{s.}^{\cdot j}$ are the forms of the Schouten-Weyl tensor ([6]).

Definition. Let $\gamma = \{x^i(t) : a \le t \le b\}$ be a smooth curve of the manifold $\{M, g, \alpha\}$ and $F_0 = \{e_1^0, \ldots, e_n^0, z^0, z^{0*}\} \in GL(n+1, 1)$ be a starting basis such that:

$$g_{ij}(\gamma(a)) = \langle e_i^0, e_j^0 \rangle.$$

The lifting $F : [a, b] \to GL(n+1, 1)$ is the solution of the differential matrix equation $\dot{F} = \varPhi(\dot{\gamma})F, \quad F(a) = F_0$. The vector function $z : [a, b] \to C^+$ (the component of F) is called the conformal development of γ.

Remark 1. The development z does not depend on the choice of the local basis $\{e_1, \ldots, e_n\}$ and the form α on the manifold M (for the proof see [3]).

Remark 2. In the case of the continuous curve γ, the lifting of the curve, for an arbitrary connection, was defined in [7].

Lemma. *Let $Z : D \to C^+$ be an immersion of the domain $D \subset R^n$ into the light cone $C^+ \subset R^{n+2}$ of the Minkowski space R^{n+2}, and let $\bar{g} = \langle dZ, dZ \rangle = \bar{g}_{ij} dx^i dx^j$ denote the conformally flat metric* ([5]), *then*

$$d^2 Z = Z_{ij} dx^i dx^j = Z_k \bar{\Gamma}^k_{ij} dx^i dx^j + Z^* \bar{g}_{ij} dx^i dx^j + Z \bar{A}_{ij} dx^i dx^j,$$

where $\bar{\Gamma}^k_{ij}$ are the Cristoffel symbols, $\bar{A}_{ik}$ is the reduced Ricci tensor of the metric $\bar{g}$, the vector Z^* is defined by the formulas:

$$\langle Z^*, Z_i \rangle = 0, \quad \langle Z, Z^* \rangle = -1, \quad \langle Z^*, Z^* \rangle = 0,$$

where $Z_i = \frac{dZ}{dx^i}, \quad Z_{ij} = \frac{\partial^2 Z}{dx^i dx^j}$.

Proof: The vectors $\{Z_1, \dots, Z_n, Z, Z^*\}$ form a basis, hence

$$Z_{ij} = Z_k M^k_{ij} + Z^* N_{ij} + Z P_{ij}. \tag{3}$$

It is easy to see, from (1), that

$$M^k_{ij} = \bar{\Gamma}^k_{ij}, \quad N_{ij} = \bar{g}_{ij}.$$

If we differentiate (3) with respect to x^s and equate the mixed partial derivatives Z_{ijs} and Z_{isj}, we obtain $P_{ij} = \bar{A}_{ij}$. □

Remark. The tensors $\bar{g}_{ij}$ and $\bar{A}_{ij}$ completely define the second fundamental form of the surface $\{Z(x)\}$ in the Minkowski space.

Proof of the theorem: Let $\{x^1, x^2, \dots, x^n\}$ be the Fermi co-ordinate system ([8]) in the tube about the curve

$$\gamma = \left\{ x : x^2 = \dots = x^n = 0,\, a \le x^1 \le b \right\},$$

that is:

(a) the tangent vectors $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ form the orthonormal frame field along the curve γ,

(b) the curves $\{x^1 = const, x^i = c^i s : s > 0, i \ge 2\}$ are geodesics, where $\{c^i\}_{i=2}^n$ are constants.

Let $B_\varepsilon(\gamma) = \{x : \sum_{i=2}^n (x^i)^2 < \varepsilon^2,\, a \le x^1 \le b\}$ denote the (solid) tube of the radius ε about γ. We will determine the immersion

$$Z : B_\varepsilon(\gamma) \to C^+,$$

such that $\bar{g} = \langle dZ, dZ \rangle$ will be the required conformally flat metric.

Let $x_0 = (a, 0, \dots, 0) \in B_\varepsilon(\gamma)$ be the starting point. Any point $x = (x^1, \dots, x^n) \in B_\varepsilon(\gamma)$ can be connected with x_0 by the curve (open polygon) l_x consisting of the two arcs. The first arc is a line segment along the axis of $B_\varepsilon(\gamma)$:

$$x(t) = \{a(1-t) + tx^1, 0, \dots, 0\}, \quad 0 \le t \le 1,$$

the second arc is the geodesic segment which is orthogonal to the axis

$$x(t) = \{x^1, (t-1)x^2, \dots, (t-1)x^n\}, \quad 1 \le t \le 2.$$

The curve $l_x : [0,2] \to B_\varepsilon(\gamma)$ depends smoothly on the terminal point x. Let $\mathcal{L}$ denote the family of all such curves l_x.

Let us fix the basis $F_0 \in GL(n+1,1)$ and associate l_x with the lifting

$$F_x(t) = \{e_1(t)_x, \dots, e_n(t)_x, z(t)_x, z^*(t)_x\},$$

into the group $GL(n+1,1)$. Then we may consider the map

$$\mathbf{F} : x \in B_\varepsilon(\gamma) \to F_x(2) \in GL(n+1,1).$$

Let $\bar\Phi = \{\bar\omega^i, \bar\omega_{i.}^{\cdot j}, \bar\omega_{i.}^{*}, \alpha\}$ be a matrix 1-forms corresponding to the map $\mathbf{F}$ by the formula (2). Let $Z(x) = z(2)_x$ be a component of the map $\mathbf{F}$, that is the terminal point of the development l_x.

Since $\bar\Phi = \Phi\big|_{\mathcal{L}}$, then the forms $\bar\Phi$ and Φ are equal along the axis $B_\varepsilon(\gamma)$ (*on the tangent space of the manifold* M). We have

$$dZ = Z_i\bar\omega^i = Z_i dx^i,$$
$$dZ_i = \bar\omega_{i.}^{\cdot j} Z_j + \bar\omega_i Z^* + \bar\omega_i^* Z,$$

for any point of γ. Then

$$d^2Z = \bar\omega_{i.}^{\cdot j} dx^i Z_j + \bar\omega_i dx^i Z^* + \bar\omega_i^* dx^i Z = \omega_{i.}^{\cdot j} dx^i Z_j + \omega_i dx^i Z^* + \omega_i^* dx^i Z,$$

for any point of γ. The application of the Lemma gives us the required result.

Other applications of the conformal development curves can be find in [3]. □

References

[1] Cartan E., *Leçons sur la Géometrie des Espaces de Riemann*, Gauthier-Villars, Paris, 1928, p. 242.

[2] Cartan E., *Les espaces à connection conforme*, Ann. Soc. Po. Math. **2** (1923), 171–221.

[3] Slavskii V.V., *Conformal development of the curve on the Riemannian manifold in the Minkowski space*, Siberian Math. Journal **37** (1996), no. 3, 676–699.

[4] Akivis M.A., Konnov V.V., *Sense local aspects of the theory of conformal structure*, Russian Math. Surveys **48** (1993), 3–40.

[5] Slavskii V.V., *Conformally flat metrics and the geometry of the pseudo-Euclidean space*, Siberian Math. Jour. **35** (1994), no. 3, 674–682.
[6] Besse A.L., *Einstein Manifolds*, Erg. Math. Grenzgeb. 10, Berlin-Heidelberg-New York, 1987.
[7] Reshetnyak Yu.G., *On the lifting of the non-regular path in the bundle manifold and its applications*, Siberian Math. Jour. **16** (1975), no. 3, 588–598.
[8] Gray A., Vonhecke L., *The volumes of tubes about curves in a Riemannian manifold*, Proc. London Math. Soc. **44** (1982), no. 2, 215–243.

Altai State University, Dimitrov Street 66, 656099 Barnaul-99, Russia
E-mail: mathan@math.mezon.altai.su

(*Received* January 22, 1996)