Comment.Math.Univ.Carolin. 37,4 (1996)725–748

Random coincidence degree theory with applications to random differential inclusions

E. Tarafdar, P. Watson, Xian-Zhi Yuan*

Abstract. The aim of this paper is to establish a random coincidence degree theory. This degree theory possesses all the usual properties of the deterministic degree theory such as existence of solutions, excision and Borsuk's odd mapping theorem. Our degree theory provides a method for proving the existence of random solutions of the equation $Lx \in N(\omega, x)$ where $L : dom\, L \subset X \to Z$ is a linear Fredholm mapping of index zero and $N : \Omega \times \overline{G} \to 2^Z$ is a noncompact Carathéodory mapping. Applications to random differential inclusions are also considered.

Keywords: Carathéodory upper semicontinuous, random (stochastic) topological degree, Souslin family, measurable space

Classification: Primary 47H04, 54C60; Secondary 52A07

1. Introduction

Let X and Z be normed spaces. The study of solutions of the equation

$$Lx \in N(x)$$

where $L : dom\, L \subset X \to Z$ is a linear mapping and $N : \overline{G} \to 2^Z$ is a multivalued mapping has drawn a lot of interest due to the wide range of problems which can be expressed in this manner. For instance, the equation represents many nonlinear ordinary, partial and functional differential equations as well as optimal control problems (see Pruszko [21], Gaines and Mawhin [7] and Gaines and Peterson [8] for examples of this formulation of differential problems).

Using an equivalence theorem which reduces the problem of existence of solutions of the equation to that of fixed points of an auxiliary mapping and the Leray-Schauder degree, Mawhin [15] developed a degree called the coincidence degree for the pair (L, N). With the subsequent development of the Leray-Schauder degree by authors such as Nussbaum [17], [18], Sadovskii [24] and Ma [14], the coincidence degree has been extended to the noncompact, set-valued case by Tarafdar and Teo [28].

*The corresponding author and present mail address is: George Xian-Zhi Yuan, Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, N.S., B3H 3J5, Canada.

The purpose of this paper is to study the existence of solutions of the equation

$$Lx \in N(\omega, x)$$

where $N : \Omega \times \overline{G} \to 2^Z$ is a Carathéodory mapping and (Ω, Σ) is a measurable space. The inclusion of a randomness factor $\omega \in \Omega$ leads to wider class of problems which can be solved using the coincidence degree method.

The layout of this work is as follows: in Section 2, Carathéodory mappings are introduced along with a selection theorem for these mappings. This selection result is used throughout the rest of the paper. Section 3 summarizes the deterministic coincidence degree of Tarafdar and Teo, and in Section 4 the Stochastic Coincidence Degree is defined and its properties are given. In Section 5, applications to Random Generalized Boundary Value Problems are given.

Now in order to make the paper comprehensible we recall some definitions and notations.

An abstract measurable (Ω, Σ) is a pair where Ω is a set and Σ is a σ-algebra of subsets of Ω. If X is a topological space, the Borel σ-algebra $\mathcal{B}(X)$ is the smallest σ-algebra containing all open subsets of X.

Suppose that to each finite sequence $(i_1, i_2, \cdots, i_n)$ of positive integers corresponds a subset $S_{i_1, i_2, \cdots, i_n}$ of a given set. Then the set

$$S = \bigcup_{(i_j)_{j=1}^{\infty} \in \mathcal{I}} \bigcap_{n=1}^{\infty} S_{i_1, i_2, \cdots, i_n}$$

the union being taken over all the (uncountable) collection $\mathcal{I}$ of all infinite sequence $(i_j)_{j=1}^{\infty}$, is said to have been formed from these sets by means of the *Souslin* (see Wagner [31] and reference therein) operation. Let $\mathcal{G}$ be a family of subsets in a non-empty set X. Then $\mathcal{G}$ is said to be a Souslin family if each set obtained from $\mathcal{G}$ by Souslin operation is still in $\mathcal{G}$. We shall say that the measurable space (Ω, Σ) *admits the Souslin operation* (*or say,* Σ *is a Souslin family*) if every subset formed by Souslin operation from measurable sets is still measurable. Every measurable space derived from an outer measurable admits the Souslin operation (e.g., see Rogers [23, p. 44-49]). A more compact notation for the Souslin operation is given as follows. Let $\sigma := (i_1, i_2, \cdots)$, be an infinite sequence of positive integers, we shall denote by $\sigma|n$ the finite sequence $(i_1, i_2, \cdots, i_n)$ obtained by stopping at the *nth* term. With this notation, the formula displayed above becomes

$$S = \bigcup_{\sigma \in \mathcal{I}} \bigcap_{n=1}^{\infty} S_{\sigma|n}.$$

We note that many measurable spaces are derived from an outer measurable. For example, if the measurable space (Ω, Σ) has an outer measure, then Σ is a

Souslin family (e.g., see Saks [26, p. 50]). In particular, every finite or σ-finite complete measurable space (in which subsets of sets of measure zero are measurable) has a Souslin family Σ.

Let X and Y be topological spaces. A mapping $F : X \to 2^Y$ is said to be upper semi-continuous (u.s.c.) at the point x if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(z) \subset V$ for any $z \in U$. Note that usc is nothing else than continuity in the single valued case. Let (Ω, Σ) be an abstract measurable space. A multivalued mapping $F : \Omega \to 2^X$ is said to be Σ-measurable if the set $F^{-1}(B) := \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \Sigma$ for any closed subset B of X.

2. Carathéodory set-valued mappings and related results

Let (Ω, Σ) be a measurable space, X and Y two topological spaces. Then a mapping $F : \Omega \times X \to 2^Y$ is said to be *Carathéodory* if (a) the mapping $F_x : \Omega \to 2^Y$, $F_x(\omega) := F(\omega, x)$ is measurable for all fixed $x \in X$ and (b) the mapping $F_\omega : X \to 2^Y$, $F_\omega(x) := F(\omega, x)$ is upper semicontinuous for all fixed $\omega \in \Omega$.

We begin with a result which is a special case of Theorem 2.3 of Tan and Yuan [27]. First though, we say $F : \Omega \times X \to 2^X$ has a *random fixed point* if there exists a single valued measurable mapping $\xi : \Omega \to X$ such that $\xi(\omega) \in F(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

Theorem 2.1. *Let (Ω, Σ) be a measurable space, Σ a Souslin family and X a non-empty separable Banach space. Suppose $F : \Omega \times X \to 2^X$ is such that $Graph F \in \Sigma \otimes \mathcal{B}(X \times X)$. Then F has a random fixed point if and only if the mapping $F(\omega, \cdot) : X \to 2^X$ has a fixed point for each fixed $\omega \in \Omega$.*

2A. On $k - \phi$-contractions

Let X be an arbitrary topological vector space. By $K(X)$ we shall mean the family of all (nonempty) closed convex subsets of X, and $CK(X)$ shall denote the family of all compact convex sets in X. By $\overline{A}$ we mean the closure of the set A in X and ∂A we mean the boundary of A. For details of the following, see Petryshyn and Fitzpatrick [19], Sadovskii [24] or Tarafdar and Thompson [29].

Definition 2.2. Let C be a lattice with a minimal element which we denote by zero, 0. A mapping $\phi : 2^X \to C$ where 2^X denotes the power set of X, is called a *measure of noncompactness* if, for any $A \subset X$, $B \subset X$, it satisfies the following properties:

(i) $\phi(\overline{co}\, A) = \phi(A)$,
(ii) $\phi(A) = 0$ if and only if A is precompact,
(iii) $\phi(A \cup B) = \max\{\phi(A), \phi(B)\}$.

From (iii) we see that $A \subset B \Longrightarrow \phi(A) \leq \phi(B)$.

Definition 2.3. Let ϕ be a measure of noncompactness and we additionally assume that the lattice C has the property that, for each $c \in C$ and $\lambda \in R$ with $\lambda > 0$, there is defined an element $\lambda c \in C$. Let $W \subset X$. An u.s.c. mapping $F : \overline{W} \to CK(X)$ is called a $k - \phi$-*contraction* or a $k - \phi$-*contractive mapping* if there exists some $k > 0$ such that, for every subset A of $\overline{W}$,

$$\phi(F(A)) \leq k\phi(A).$$

If the mapping F is such that $\phi(F(A)) < \phi(A)$ for all $A \subset \overline{W}$, then F is said to be ϕ-*condensing*. It is clear that if F is $k - \phi$-contraction, then it is condensing.

The following results follow almost immediately from Definition 2.3.

Proposition 2.4. *Let ϕ be a measure of noncompactness with the additional property that for any subsets A and B of X,*

$$\phi(A + B) \leq \phi(A) + \phi(B).$$

If $F, G : \overline{W} \to CK(X)$ are k_1- and $k_2 - \phi$-contractions respectively, then the mapping $F + G : \overline{W} \to CK(X)$ defined by

$$(F + G)(x) = F(x) + G(x)$$

is a $(k_1 + k_2) - \phi$-contraction.

Proposition 2.5. *Let ϕ be a measure of noncompactness, $F : \overline{W} \to CK(X)$ be a $k_1 - \phi$-contraction and $G : X \to X$ be linear, continuous single valued mapping such that for each subset A of X, we have*

$$\phi(G(A)) \leq k_2\phi(A).$$

Then $GF : \overline{W} \to CK(X)$ defined by

$$GF(x) = \{G(y) : y \in F(x)\}$$

is a $k_1k_2 - \phi$-contraction.

Proposition 2.6. *If F and G are $k - \phi$-contractions, then so is $\lambda F + (1 - \lambda)G$ for any $\lambda \in [0, 1]$.*

2B. A 'Selection' Theorem

Lemma 2.7. *Suppose that X is a Banach space and $\{G_i : i = 1, 2, \dots\}$ is a family of bounded, nonempty subsets of X such that $G_{n+1} \subset G_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \phi(G_n) = 0$ where ϕ is a measure of noncompactness defined above. Then $\bigcap_{n=1}^{\infty} \overline{G_n} \neq \emptyset$ and is compact.*

Proof: Theorem 6.1.2 of Lloyd [13]. $\square$

Lemma 2.8. *Let X be a topological space and $F : X \to CK(X)$ be a set-valued mapping. Then we have that*

(i) *$F : X \to CK(X)$ is upper semicontinuous if and only if $F^+(G) = \{x : F(x) \subset G\}$ is open for any open set $G \subset X$;*
(ii) *the composition of two upper semicontinuous maps is an upper semicontinuous map.*

PROOF: See Berge [2]. □

Proposition 2.9. *Let X be a topological space and Y be a normal topological space. If $F : X \to 2^Y$ is upper semicontinuous, then the mapping $\overline{F} : X \to 2^Y$ defined by $\overline{F}(x) := \overline{F(x)}$ for each $x \in X$ is also upper semicontinuous.*

PROOF: According to the definition of upper semicontinuity, for each open neighborhood V in Y, it suffices to prove that the set $\overline{F}^+(V) = \{x \in X : \overline{F}(x) \subset V\}$ is open in X. Now suppose $x_0 \in \overline{F}^+(V)$, then $\overline{F}(x_0) \subset V$. As $\overline{F}(x_0)$ is closed and Y is normal, there exists a non-empty open neighborhood V_1 of V such that $F(x_0) \subset \overline{F}(x_0) \subset V_1 \subset \overline{V_1} \subset V$. As F is upper semicontinuous and note that $F(x_0) \subset V_1$, so that there exists a non-empty open neighborhood $N(x_0)$ of x_0 such that $F(w) \subset V_1$ for all $w \in N(x_0)$. Therefore we have $\overline{F}(w) \subset \overline{V_1} \subset V$. Hence we have proved that $\overline{F}(w) \subset V$ for all $w \in N(x_0)$, which means that $N(x) \subset \overline{F}^+(V)$. Thus $\overline{F}^+(V)$ is open in X and the conclusion follows. □

Proposition 2.10. *Let X be a topological vector space and Y be a locally convex topological vector space. If $F : X \to 2^Y$ is upper semicontinuous, then the mapping $co F : X \to 2^Y$ defined by $co F(x) := co(F(x))$ for each $x \in X$ is upper semicontinuous.*

PROOF: Similar to the arguments above, it suffices to prove that for each open neighborhood V of Y, the set $(co F)^+(V) = \{x \in X : co F(x) \subset V\}$ is open in X. Now suppose $x_0 \in (co F)^+(V)$, then $co F(x_0) \subset V$. As Y is a locally convex topological vector space, without loss of generality, we may assume that V is convex. As F is upper semicontinuous and $F(x_0) \subset co F(x_0) \subset V$, there exits a non-empty open neighborhood $N(x_0)$ of x_0 such that $F(w) \subset V$ for all $w \in N(x_0)$. Thus $co f(w) \subset co(V) = V$ for all $w \in N(x_0)$ as V is convex. Therefore $N(x_0) \subset (co F)^+(V)$, which implies that $(co F)^+(V)$ is open in X. Therefore the conclusion follows and the proof is complete. □

In joining Propositions 2.9 and 2.10, we have the following theorem:

Theorem 2.11. *Let X be a topological space and Y be a non-empty subset of a normed space. If $F : X \to 2^Y$ is upper semicontinuous, then the mapping $\overline{co F} : X \to 2^Y$ defined by $\overline{co F}(x) := \overline{co F(x)}$ for each $x \in X$ is upper semicontinuous.*

PROOF: Note that each subset of normed space is normal and the composition of upper semicontinuous mappings is upper semicontinuous. Thus it is clear that the conclusion follows from Propositions 2.9 and 2.10. □

Now we present the following selection result which was first given in our paper [30]; for the convenience of the reader, we give an outline of its proof here.

Theorem 2.12. *Let (Ω, Σ) be a measurable space, X a non-empty metrizable and separable space and Y be a metrizable and separable convex set in a locally convex Hausdorff topological vector space E. Suppose $F : \Omega \times X \to 2^Y$ is a Carathéodory mapping with non-empty compact, convex values such that $F_\omega : X \to 2^Y$ is ϕ-condensing for any fixed $\omega \in \Omega$. Then there exists a set-valued mapping $S : \Omega \times X \to 2^Y$ with the following properties:*

(1) *S is Carathéodory such that S_ω is ϕ-condensing for any fixed $\omega \in \Omega$. Also S has nonempty compact convex values and $S(\omega, x) \subset F(\omega, x)$ for each $(\omega, x) \in \Omega \times X$;*

and

(2) *S is jointly measurable from $\Omega \times X$ to 2^Y (i.e. the set $S^{-1}(B) = \{(\omega, x) \in \Omega \times X : S(\omega, x) \cap B \neq \emptyset\} \in \Sigma \otimes \mathcal{B}(X)$ for each closed subset $B \in 2^Y$), and hence the graph of S is measurable.*

Proof: As assumptions on Y, the mapping F and the graph of $F(\omega, \cdot)$ for each fixed $\omega \in \Omega$ remain unchanged in the completion of Y, without loss of generality, we may assume that the space Y is complete. Since X is metrizable and separable, we shall denote by d the metric of X and let X_0 be a countable dense subset of X. We define $S : \Omega \times X \to 2^Y$ by

$$S(\omega, x) = \bigcap_{i=1}^{\infty} \overline{co\,[\cup_{\{z \in X_0 : d(z,x) < \frac{1}{n}\}} F(\omega, z)]}$$

for each $(\omega, x) \in \Omega \times X$. It is clear that

$$S(\omega, x) = \bigcap_{i=1}^{\infty} \overline{co\,[\cup_{\{z \in X_0 : d(z,x) < \frac{1}{n}\}} F(\omega, z)]} = \bigcap_{i=1}^{\infty} \overline{co\,[\cup_{\{z \in X_0 : d(z,x) \leq \frac{1}{n}\}} F(\omega, z)]}$$

for each $(\omega, x) \in \Omega \times X$. Now we shall first claim that $S(\omega, x) \neq \emptyset$ for each $(\omega, x) \in \Omega \times X$. For each fixed $\omega \in \Omega$, as the mapping $F_\omega(\cdot)$ is condensing, it follows that for each $x \in X$ and $n \in \mathbb{N}$, we have that $\phi(\overline{co\,[\cup_{\{z \in X_0 : d(z,x) < \frac{1}{n}\}} F(\omega, z)]}) \leq \phi(B_{\frac{1}{n}}(x))$. Note that $\lim_{n\to\infty} \phi(B_{\frac{1}{n}}(x)) = 0$, so that $\lim_{n\to\infty} \phi(\{\overline{co\,[\cup_{\{z \in X_0 : d(z,x) < \frac{1}{n}\}} F(\omega, z)]}\}) = 0$. Then Lemma 2.7 implies that $\bigcap_{n=1}^{\infty} \overline{co\,[\cup_{\{z \in X_0 : d(z,x) < \frac{1}{n}\}} F(\omega, z)]} \neq \emptyset$. Thus $S(\omega, x)$ is non-empty and compact for each $(\omega, x) \in \Omega \times X$.

Secondly, by the upper semicontinuity of $F_\omega(\cdot)$, we shall show that $S(\omega, x) \subset F(\omega, x)$ for each $(\omega, x) \in \Omega \times X$ (so that $S(\omega, \cdot)$ is condensing for each fixed $\omega \in \Omega$). For each $(\omega, x) \in \Omega \times X$, let $O(\omega, x) := \{U_\alpha\}_{\alpha \in I}$ be the family of all open neighborhoods of $F(\omega, x)$. Note that $F(\omega, \cdot)$ is upper semicontinuous, for each $U_\alpha \in O(\omega, x)$, there exists $n \in \mathbb{N}$ such that for each $z \in X_0$ with $d(z,x) < \frac{1}{n}$, we have $F(\omega, z) \subset U_\alpha$. Hence for each $z \in X_0$ with $d(z,x) < \frac{1}{n}$, $S(\omega,x) \subset \overline{[co[\cup_{\{z\in X_0:d(z,x)<\frac{1}{n}\}}F(\omega,z)]} \subset \overline{U_\alpha}$. Therefore $S(\omega,x) \subset \bigcap_{\alpha\in I}\overline{U_\alpha}$. As $F(\omega,x)$ is closed, we have that $\bigcap_{\alpha\in I}\overline{U_\alpha} = F(\omega,x)$. Thus $S(\omega,x) \subset F(\omega,x)$ for each $(\omega,x) \in \Omega\times X$. It is not difficult to show that the mapping $x \mapsto \{z \in X_0 : d(z,x) \le \frac{1}{n}\}$ from X to X_0 is upper semicontinuous (e.g., see Proposition 5 in [30]), and so the composition mapping $x \mapsto \overline{co[\cup_{\{z\in X_0:d(z,x)\le\frac{1}{n}\}}F(\omega,x)]}$ is also upper semicontinuous for each $n \in \mathbb{N}$. In order to prove that $x \mapsto S_\omega(\cdot)$ is upper semicontinuous, it suffices to show that the set $V := \{x \in X : S(\omega,x) \subset U\}$ is open for each non-empty open subset U of Y. Now assume that $x_0 \in V$, i.e. $S(\omega,x_0) \subset U$. We claim that there exists $n_0 \in \mathbb{N}$ such that $S'_{n_0}(\omega,x_0) := \overline{co[\cup_{\{z\in X_0:d(z,x)\le\frac{1}{n}\}}F(\omega,x_0)} \subset U$ by the fact that the measure ϕ of noncompactness of decreasing closed subset $\{S'_n(\omega,x_0)\}_{n\in N}$ converges to zero as $n\to\infty$ since

$$\phi(S_n(\omega,x_0)) < \phi(\{z\in X_0 : d(x,z) \le \frac{1}{n}\}) \le \phi(B_{\frac{1}{n}}(x_0)) \to 0$$

when $n\to\infty$.

As $S'_{n_0}(\omega,\cdot)$ is upper semicontinuous, there exists a non-empty open neighborhood $N(x_0)$ of x_0 in X such that $S'_{n_0}(\omega,z) \subset U$ for all $z \in N(x_0)$. Note that $S(\omega,x) \subset S'_n(\omega,x)$ for all $(\omega,x)\in\Omega\times X$. Thus $N(x_0)\subset V$, and we have proved that V is open in X. Hence $S(\omega,\cdot) : X \to 2^Y$ is upper semicontinuous with non-empty closed and convex values.

Finally we show that S is jointly measurable. Let B be any non-empty closed subset B of Y. We can show that

$$\{(\omega,x)\in\Omega\times X : S(\omega,x)\cap B \ne \emptyset\} = \bigcap_{n\in\mathbb{N}}\{(\omega,x)\in\Omega\times X : S_n(\omega,x)\cap B\ne\emptyset\},$$

where for each $n\in\mathbb{N}$, S_n is a set-valued mapping from $\Omega\times X$ to Y defined as

$$S_n(\omega,x) = \overline{co[\cup_{\{z\in X_0:d(z,x)<\frac{1}{n}\}}F(\omega,z)]}$$

for each $(\omega,x)\in\Omega\times X$. Let $W_1 := \{(\omega,x)\in\Omega\times X : S(\omega,x)\cap B\ne\emptyset\}$ and $W_2 := \bigcap_{n\in\mathbb{N}}\{(\omega,x)\in\Omega\times X : S_n(\omega,x)\cap B\ne\emptyset\}$. It is clear that $W_1\subset W_2$. We now show that $W_2\subset W_1$. Let $(\omega,x)\in W_2$. Then $S_n(\omega,x)\cap B\ne\emptyset$ for all $n\in\mathbb{N}$. Note that $\{S_n(\omega,x)\cap B\}_{n\in\mathbb{N}}$ is a decreasing non-empty closed subsets of Y and $\lim_{n\to\infty}\phi(S_n(\omega,x)\cap B) \le \lim_{n\to\infty}\phi(S_n(\omega,x)) = 0$. By Lemma 2.7 again, we have that $\bigcap_{n\in\mathbb{N}} S_n(\omega,x)\cap B\ne\emptyset$, so that $(\omega,x)\in W_1$ and we have proved that $W_1 = W_2$. For each $n\in\mathbb{N}$, note that the set

$$\begin{aligned}\{(\omega,x)\in\Omega\times X : (&\bigcup_{z\in X_0}\{F(\omega,z) : d(z,x)<\frac{1}{n})\cap B\ne\emptyset\}\\ &= \bigcup_{z\in X_0}\{\omega\in\Omega : F(\omega,z)\cap B\ne\emptyset\}\times\{x\in X : d(x,z)<\frac{1}{n}\}\end{aligned}$$

belongs to $\Sigma \otimes \mathcal{B}(Y)$. As each measurable mapping is weakly measurable, Theorem 9.1 of Himmelberg in [11] implies that the mapping S_n is jointly weakly measurable. Note that

$$\{(\omega, x) \in \Omega \times X : S(\omega, x) \cap B \neq \emptyset\}$$
$$= \bigcap_{n \in \mathbb{N}} \{(\omega, x) \in \Omega \times X : S_n(\omega, x) \cap B \neq \emptyset\} \in \Sigma \otimes \mathcal{B}(X).$$

Hence S is also jointly weakly measurable. Since the mapping S has non-empty compact values, Theorem 3.1 of Himmelberg in [11] shows that S is jointly measurable and the proof is complete.

□

3. Deterministic coincidence degree theory

Here we shall summarize some results obtained in Tarafdar and Teo [28].

3A. Ultimately compact mappings

Let X denote a separated locally convex topological vector space over the reals with the additional property that for each compact subset A of X, there is a retraction of X onto the convex closure of A, $\overline{co}\, A$.

Let $\Omega \subset X$ be open and let $F : \overline{\Omega} \to K(X)$ be u.s.c. We define a transfinite sequence of sets $\{K_\alpha\}$ so that the limit of the sequence necessarily contains all the fixed points of F. Let $K_0 = \overline{co}\, F(\overline{\Omega})$ and suppose K_β has been defined for all ordinals β less that the ordinal α. If α is an ordinal of the first kind, let $K_\alpha = \overline{co}\, F(\overline{\Omega} \cap K_{\alpha-1})$ and if α is an ordinal of the second kind, let $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$.

Note the following properties of the transfinite family $\{K_\alpha\}$:

(i) each K_α is closed and convex with $K\alpha \subseteq K_\beta$ for all $\alpha \geq \beta$;

(ii) $F(\overline{\Omega} \cap K_\alpha) \subseteq K_\alpha$ for any ordinal α.

Furthermore, as the transfinite sequence is nonincreasing, there is an ordinal γ such that $K_\gamma = K_{\gamma+1}$, and so $K_\gamma = K_\beta$ for all $\beta \geq \gamma$. We define $K = K(F, \overline{\Omega}) := K_\gamma$. Then it is clear that $F(\overline{\Omega} \cap K) \subset K$ and indeed $\overline{co}\, F(\overline{\Omega} \cap K) = K$.

Definition 3.1. An u.s.c. mapping $F : \overline{\Omega} \to K(X)$ is said to be *ultimately compact* if either $K \cap \overline{\Omega} = \emptyset$ or, if $K \cap \overline{\Omega} \neq \emptyset$, then $F(\overline{\Omega} \cap K)$ is relatively compact. If F is ultimately compact, then we say $I - F$ is an ultimately compact vector field, where I is the identity mapping on X.

Definition 3.2 (The degree for ultimately compact vector fields)**.**
Let $\Omega \subset X$ be open and let $F : \overline{\Omega} \to K(X)$ be ultimately compact with $0 \notin x - F(x)$ for each $x \in \partial\Omega$. If $K \cap \Omega$ is empty, define the degree of $I - F$ on Ω with respect to zero, denoted by $d(I - F, \Omega, 0)$ to be zero. If $K \cap \Omega \neq \emptyset$, let ϱ be a retraction of X onto K and define

$$d(I - F, \Omega, 0) = d_c(I - F\varrho, \varrho^{-1}(\Omega), 0),$$

where the right-hand term is the degree for compact set-valued vector fields as studied by Ma [14].

To see that this degree is well defined and has the usual properties of a degree, please see Petryshyn and Fitzpatrick [19].

The following result is a simple consequence of Lemmas 3.2 and 3.4 of Petryshyn and Fitzpatrick [19].

Proposition 3.3. *Let $\phi : 2^X \to R^+ = \{t \in R : t \geq 0\} \cup \{\infty\}$ be a measure of noncompactness and suppose that $F : \overline{\Omega} \to CK(X)$ is a $k - \phi$-contraction with $0 < k < 1$ and $\phi(F(\overline{\Omega})) < \infty$. If either X is quasi-complete or $\overline{\Omega}$ is complete, then F is ultimately compact.*

3B. The set-valued coincidence degree

Let X and Z be two vector spaces and let $L : domL \subset X \to Z$ be a linear mapping, where $domL$ represents the domain of L. Similarly we shall denote the kernel of L by $kerL$, the range of L by ImL and the quotient space Z/ImL, the cokernel of L, by $cokerL$.

Given a vector subspace of a vector space E, there always exists a projection (a linear and idempotent operator) P of E onto Y and E is the direct sum of $ImP = Y$ and $kerP$.

Definition 3.4. If X, Z, and L are as above, let P and Q be projections on X and Z respectively such that

$$ImP = kerL \text{ and } kerQ = ImL.$$

Such a pair of projections (P, Q) will be called *exact with respect to L.*

Definition 3.5. Let L_P denote the restriction of L to $kerP \cap domL$. Then L_P is an isomorphism from $kerP \cap domL$ to ImL. Let $K_P : ImL \to kerP \cap domL$ be the inverse of L_P. K_P is then called the *pseudo inverse of L associated with P.*

Let $\pi : Z \to cokerL$ be the canonical surjection, that is $\pi(z) = z + ImL$ for each $z \in Z$. It is easy to verify that the restriction of π to ImQ is an algebraic isomorphism. Furthermore, if Z is a topological vector space and $cokerL$ is given the quotient topology, then π is continuous.

The following results are almost immediate:

$$\begin{gathered} PK_P = 0, \\ LK_P = L_PK_P = I, \\ K_PL = K_PL(I - P) = I - P, \\ Qz = 0 \Leftrightarrow z \in ImL \Leftrightarrow \pi(z) = 0. \end{gathered}$$

The next two results can be found in Mawhin [15].

Proposition 3.6. *Let (P, Q) and (P', Q') be pairs of projections exact with respect to L. Then*

$$K_{P'} = (I - P')K_P$$
$$PK_{P'} + P'K_P = 0$$

where K_P and $K_{P'}$ denote the pseudo-inverses of L associated with P and P' respectively.

Proposition 3.7. *Let P, P' be projections of X onto $ker L$ and let $P'' = aP + bP'$ for some real numbers a and b. Then P'' is a projection onto $ker L$ if and only if $a + b = 1$. If this necessary and sufficient condition holds, the pseudo-inverse of L associated with P'' is given by*

$$K_{P''} = aK_P + bK_{P'}.$$

In determining the existence of solutions to the equation $Lx \in Nx$, we need only consider the existence of a fixed point of an auxiliary mapping. Formally we have:

Theorem 3.8. *Let X and Z be two vector spaces over the same scalar field. Let $L : dom L \subset X \to Z$ be a linear mapping and $N : A \subset X \to 2^Z$ be a set-valued mapping. Further assume that there is a linear one-to-one mapping*

$$\psi : coker L \to ker L.$$

Then $x_0 \in dom L \cap A$ is a solution of the equation

$$Lx \in Nx$$

if and only if x_0 is a fixed point of the set-valued mapping $M_\psi : A \to 2^X$ defined by

$$M_\psi x = Px + [\psi\pi + K_P(I - Q)]Nx$$

for every pair (P, Q) of exact projections with respect to L.

The proof can be found in Tarafdar and Teo [28], Theorem 3.1.

Assumptions I.

(a) X is a real Banach space and Z is a real normed linear space.

(b) $L : dom L \subset X \to Z$ is a linear Fredholm mapping of index zero defined on a subspace $dom L$ of X, that is L is linear, $Im L$ is closed and

$$dim\, ker L = dim\, coker L < \infty.$$

(c) Ω is a bounded open set in X and $N : \overline{\Omega} \to CK(Z)$ is a set-valued mapping.

(d) N is upper semi-continuous with $\pi N(\overline{\Omega})$ bounded in $coker L$.

(e) Let (P, Q) be an exact pair of projections with respect to L and let K_P be the pseudo-inverse of L associated with P. Let ϕ be a measure of noncompactness defined on 2^X such that ϕ satisfies the subadditivity condition of Proposition 2.4 and takes values in $R^+ = \{t \in R : t \geq 0\} \cup \{\infty\}$. We assume that with such a measure of noncompactness ϕ, $K_P(I - Q)N$ is a $k - \phi$-contraction with $0 < k < 1$ and that $\phi(K_P(I - Q)N(\overline{\Omega})) < \infty$. In this case we also assume that K_P is continuous.

(f) $0 \notin (L - N)(dom L \cap \partial\Omega)$.

Remark 3.9.
(1) From assumption (b), it follows that the exact pair of projections (P, Q) may be assumed continuous. Furthermore (b) guarantees the existence of a linear isomorphism $\psi : coker L \to ker L$.

(2) Suppose assumptions (a) to (d) hold. Let (P, Q) and (P', Q') be exact pairs of continuous projections with respect to L, and suppose that (P, Q) satisfy assumption (e). Then the pair (P', Q') also satisfies assumption (e). (Proposition 3.1 of Tarafdar and Teo [28].)

(3) Suppose assumptions (a) to (e) are satisfied and let $\psi : coker L \to ker L$ be a continuous isomorphism. Then the image of a point $x \in \overline{\Omega}$ under the mapping M_ψ as defined in Theorem 3.15 is a compact convex subset of X. Furthermore, M_ψ is a $k - \phi$-contraction (Proposition 3.2 of Tarafdar and Teo [28]). From Propositions 3.8 and 3.9 we see that M_ψ is an ultimately compact mapping and from assumption (f) and Theorem 3.15, $0 \notin (I - M_\psi)(dom L \cap \partial\Omega)$. Thus the degree of the ultimately compact vector field $I - M_\psi$ with respect to zero is well defined.

Definition 3.10. Let $\mathcal{L}_L$ denote the set of all continuous isomorphisms from $coker L$ to $ker L$. ψ and ψ' are said to be *homotopic in* $\mathcal{L}_L$ if there exists a continuous mapping $\overline{\psi} : coker L \times [0, 1] \to ker L$ such that $\overline{\psi}(\cdot, 0) = \psi$, $\overline{\psi}(\cdot, 1) = \psi'$ and for any $\lambda \in [0, 1]$, $\overline{\psi}(\cdot, \lambda) \in \mathcal{L}_L$.

Note that to be homotopic is an equivalence relation which partitions $\mathcal{L}_L$ into equivalence classes called homotopy classes.

Definition 3.11. $\psi : coker L \to ker L$ is said to be *orientation preserving* if $\{\psi a_1, \dots, \psi a_n\}$ belongs to the orientation chosen in $ker L$ where $\{a_1, \dots, a_n\}$ is a basis for $coker L$ belonging to a certain chosen orientation. Otherwise ψ is said to be orientation reversing.

Indeed, $\mathcal{L}_L$ is partitioned into two homotopy classes, those continuous isomorphisms which are orientation preserving, and those which are orientation reversing. (See Gaines and Mawhin [7].)

Theorem 3.12. *Let assumptions* (a) *to* (f) *be satisfied. Then* $d(I - M_\psi, \Omega, 0)$ *as defined in Definition* 3.2 *depends only on* L, N *and the homotopy class of* ψ *in* $\mathcal{L}_L$.

PROOF: See Theorem 3.2 of Tarafdar and Teo [28]. □

Definition 3.13 (The set-valued coincidence degree)**.**
Suppose that assumptions (a) to (f) are satisfied and ψ is an orientation preserving continuous isomorphism from $coker L$ to $ker L$. Then the *coincidence degree of* L *and* N *in* Ω, denoted by $d[(L,N),\Omega]$, is defined by

$$d[(L,N),\Omega] = d(I - M_\psi, \Omega, 0)$$

where $M_\psi : \Omega \to CK(X)$ is defined by

$$M_\psi = P + [\psi\pi + K_P(I - Q)]N$$

and the right-hand term is the degree for the set-valued ultimately compact vector field $I - M_\psi$ as defined in Definition 3.2.

The reader is referred to Tarafdar and Teo [28] for the properties of this coincidence degree.

4. Stochastic topological degree theory

Let $L : dom L \subset X \to Z$ and $N : \Omega \times G \to 2^Z$, where (Ω, Σ) is a measurable space, X is a Banach space and Z is a normed linear space, both of which are over the reals.

Definition 4.1. A *random* (*or stochastic*) solution to the equation $Lx \in N(\omega, x)$ is a measurable mapping $\varphi : \Omega \to X$ such that

$$L\varphi(\omega) \in N(\omega, \varphi(\omega))$$

for all $\omega \in \Omega$.

Theorem 4.2 (Equivalence Theorem)**.**
Let X *and* Z *be two vector spaces over the same scalar field. Let* $L : dom L \subset X \to Z$ *be a linear mapping and* $N : \Omega \times G \to 2^Z$ *be a set-valued mapping where* $G \subset X$. *Further, assume that there is a linear one-to-one mapping*

$$\psi : coker L \to ker L.$$

Then $\varphi : \Omega \to dom L \cap G$ *is a random solution of the equation*

$$Lx \in N(\omega, x)$$

if and only if $\varphi : \Omega \to dom L \cap G$ *is a random fixed point of the set-valued mapping* $M_\psi : \Omega \times G \to 2^X$ *defined by*

$$M_\psi(\omega, x) = Px + [\psi\pi + K_P(I - Q)]N(\omega, x)$$

for every pair of exact projections (P, Q) *with respect to* L*, where* π *and* K_P *have their meaning as explained in Section* 3*.*

Proof: Suppose $\varphi : \Omega \to dom L \cap G$ is a random solution of $Lx \in N(\omega, x)$. Then for $\omega \in \Omega$ fixed, $\varphi(\omega) \in dom L \cap G$ satisfies $L\varphi(\omega) \in N(\omega, \varphi(\omega)) := N_\omega\varphi(\omega)$. By Theorem 3.15 we see that $\varphi(\omega)$ is a fixed point of the mapping $M_\psi(\omega, \cdot) : G \to 2^X$. As this is true for all $\omega \in \Omega$ and $\varphi : \Omega \to dom L \cap G$ is measurable, then φ is a random fixed point of $M_\psi : \Omega \times G \to 2^X$.

Exactly the same argument gives the converse result. □

Assumptions II.

(â) X is a real Banach space, Z is a real normed linear space, (Ω, Σ) is a measurable space where Σ is a Suslin family and $G \subset X$ is bounded and open.

($\hat{\text{b}}$) $L : dom L \subset X \to Z$ is a linear Fredholm mapping of index zero defined on a subspace $dom L$ of X.

(ĉ) The set-valued mapping $N : \Omega \times \overline{G} \to CK(Z)$ is Carathéodory upper semi-continuous and the image of each (ω, x) in $\Omega \times \overline{G}$ under N is a nonempty compact convex subset of Z.

For each $\omega \in \Omega$;

($\hat{\text{d}}$) $\pi N(\omega, \overline{G})$ is bounded in $coker L$.

(ê) Let (P, Q) be an exact pair of projections with respect to L and let K_P be the pseudo-inverse of L associated with P. Let ϕ be a measure of noncompactness defined on 2^X such that ϕ satisfies the subadditivity condition of Proposition 2.4 and takes values in $R^+ = \{t \in R : t \geq 0\} \cup \{\infty\}$. We assume that with such a measure of noncompactness ϕ, $K_P(I - Q)N(\omega, \cdot)$ is a $k - \phi$-contraction with $0 < k < 1$ and that $\phi(K_P(I - Q)N(\omega, \overline{G})) < \infty$. In this case we also assume that K_P is continuous.

($\hat{\text{f}}$) $0 \notin (L - N_\omega)(dom L \cap \partial G)$.

Definition 4.3 (The stochastic coincidence degree for set-valued noncompact mappings)**.**
Suppose that assumptions (â) to ($\hat{\text{f}}$) are satisfied. Then the *stochastic coincidence degree of* L *and* N *in* G, denoted by $d_S[(L, N), G]$ is defined as

$$d_S[(L, N), G] = \{d[(L, N_\omega), G] : \text{ for all } \omega \in \Omega \text{ such that } d[(L, N_\omega), G] \text{ is defined}\}.$$

Remark 4.4. In the deterministic case, that is, say, when Ω contains a single element, the Stochastic Degree reduces to the Deterministic Degree of Tarafdar and Teo [28].

Theorem 4.5 (Properties of the stochastic degree).
Suppose that the conditions of Definition 4.3 are satisfied so that the degree is well defined. Then we have:

(a) **Existence Theorem**: *If in addition to assumption (â), X is separable and each element of $d_S[(L,N),G]$ is not equal to zero then there is a random solution to $Lx \in N(\omega,x)$.*

(b) **Excision Property**: *If $G_1 \subset G$ is an open set such that $(L-N_\omega)^{-1}(0) \subset G_1$ for all $\omega \in \Omega$ then*

$$d_S[(L,N),G] = d[(L,N),G_1].$$

(c) **Additivity**: *If G is the union of two disjoint open sets G_1 and G_2 such that there is no solution to $Lx \in N(\omega,x)$ on $\partial G_1 \cup \partial G_2$, then*

$$d_S[(L,N),G] = \{d[(L,N_\omega),G_1] + d[(L,N_\omega),G_2] : \omega \in \Omega\}.$$

(d) **Borsuk's Theorem**: *If G is a symmetric, bounded and open neighborhood of the origin and $N(\omega,-x) = -N(\omega,x)$ for all $(\omega,x) \in \Omega \times \overline{G}$ then each element of $d_S[(L,N),G]$ is odd.*

Proof: (a): It is clear that N satisfies those properties of the Selection theorem, Theorem 2.12, and indeed so does the mapping $M_\psi(\omega,x) = Px + [\psi\pi + K_P(I-Q)]N(\omega,x)$ from $\Omega \times \overline{G}$ to $CK(X)$. Let $S : \Omega \times \overline{G} \to CK(X)$ be the selection of M_ψ. By hypothesis we have for all $\omega \in \Omega$, $d(I - M_\psi(\omega,\cdot),G,0) \neq 0$. Also it is clear that $S(\omega,\cdot)$ is homotopic to $M_\psi(\omega,\cdot)$ so we observe that $S(\omega,\cdot)$ has a deterministic fixed point for all $\omega \in \Omega$. Theorem 2.1 implies that S has a random fixed point, and as S is contained in M_ψ, then M_ψ has a random fixed point. An application of Theorem 4.2 gives the result.

(b) and (c): Both follow from the definition of the Stochastic Coincidence Degree and the corresponding property of the Deterministic Coincidence Degree as defined in Tarafdar and Teo [28].

(d): From the construction of the mapping $S(\omega,x)$ as in Theorem 2.12, it is clear that if $N(\omega,-x) = -N(\omega,x)$ then $S(\omega,-x) = -S(\omega,x)$. The result follows from Theorem 3.4 of Tarafdar and Teo [28]. $\square$

Theorem 4.5 (Homotopy Invariance).
Let assumptions (â) and ($\hat{\text{b}}$) be satisfied and let G be a bounded open subset of X. Let ϕ, P, Q and K_P be as given in assumption (ê) and suppose $\widehat{N} : \Omega \times \overline{G} \times [0,1] \to CK(Z)$ satisfy the following:

For each $\omega \in \Omega$,

(i) $\widehat{N}(\omega,\cdot,\cdot)$ *is u.s.c. on* $\overline{G}\times[0,1]$ *and for* $(x,t)\in\overline{G}\times[0,1]$ *fixed,* $\widehat{N}(\cdot,x,t)$ *is measurable,*

(ii) $\pi\widehat{N}_\omega(\overline{G}\times[0,1])$ *is bounded,*

(iii) $\phi(K_P(I-Q)\widehat{N}_\omega(\overline{G}\times[0,1]))<\infty$,

(iv) *there exists* $k\in(0,1)$ *such that, for every* $A\subset G$,

$$\phi(K_P(I-Q)\widehat{N}_\omega(A\times[0,1]))<k\phi(A),$$

(v) *for each* $\lambda\in[0,1]$,

$$0\notin(L-\widehat{N}(\omega,\cdot,\lambda))(dom\,L\cap\partial G).$$

Then $d_S[(L,\widehat{N}(\cdot,\cdot,\lambda)),G]$ *is independent of* $\lambda\in[0,1]$.

Proof: Fix $\omega\in\Omega$. From Theorem 3.5 of Tarafdar and Teo [28], $d[(L,N_\omega(\cdot,\lambda)),G]$ is independent of $\lambda\in[0,1]$ and so the conclusion is clear. □

Remark 4.6. This stochastic coincidence degree can be generalized further to the case when L has non-negative index, e.g., see Akashi [1] or Gaines and Mawhin [7] for details.

Definition 4.7. A mapping $N:\Omega\times\overline{G}\to CK(Z)$ satisfying assumptions (ĉ), ($\hat{\text{d}}$) and (ê) will be called an $L-k-\phi$-contraction.

5. Applications to random differential inclusions

This section begins with a result generalizing the work of several authors including Petryshyn and Fitzpatrick [19, Theorem 3.5] and Pruszko [21, Theorem 2.9]. Then the result will be applied to the Random Generalized Boundary Value Problem; initially for first-order differential inclusions, and then for elliptic partial differential inclusions.

Theorem 5.1. *Let* $F_i:\Omega\times X\to 2^Z$ *(for* $i=1,2,3$*) be* $L-k-\phi$*-contraction mappings with* $k<1$ *such that*

(i) $Lx\in F_2(\omega,x)\Longrightarrow\|x\|<1$ *for any* $\omega\in\Omega$*; and* $F_2(\omega,kx)=kF_2(\omega,x)$ *for any* $k\in\mathbb{R}$*, that is* $F_2(\omega,\cdot)$ *is homogeneous,*

(ii) $K_P(I-Q)F_1(\omega,X)$ *and* $\pi F_1(\omega,X)$ *are bounded,*

(iii) $F_3(\omega,x)\subset F_1(\omega,x)+F_2(\omega,x)$ *for all* $(\omega,x)\in\Omega\times X$.

Then there exists a random solution to the equation $Lx\in F_3(\omega,x)$.

Proof: Fix $\omega\in\Omega$. We may choose $\beta>0$ such that $\|x-y\|\geq\beta$ for all $x\in\partial B_1(0)$ and $y\in M_\psi(\omega,x)=P(x)+[\psi\pi+K_P(I-Q)]F_2(\omega,x)$. Indeed if this is not the case, then we can choose $\{x_n\}_{n=1}^\infty\subset\partial B_1(0)$ and $y_n\in M_\psi(\omega,x_n)$ such that $\|x_n-y_n\|<\frac{1}{n}$.

Now $\{x_n\} \subset \{y_n\} + \{x_n - y_n\}$ so $\phi(\{x_n\}) \leq \phi(\{y_n\}) + \phi(\{x_n - y_n\}) = \phi(\{y_n\})$ as $\|x_n - y_n\| < \frac{1}{n}$. Using the fact that $M_\psi(\omega, \cdot)$ is a k-set contraction (as $F_2(\omega, \cdot)$ is), we get $\phi(\{x_n\}) < \phi(\{x_n\})$ which means that the sequence $\{x_n\}$ is relatively compact. Without loss of generality, let us assume that $x_n \to x_0$. Clearly $x_0 \in \partial B_1(0)$ and $y_n \to x_0$, and so we conclude that $x_0 \in M_\psi(\omega, x_0)$. But by virtue ot the Equivalence Theorem this contradicts assumption (i).

From condition (ii), we can choose $r > 0$ such that $[\psi\pi + K_{P,Q}]F_1(\omega, X)$ is contained in $B_r(0)$. Choose $s > 0$ such that $s\beta > r$. Then if $\|x\| = s$ and $y \in M_\psi(\omega, x)$ for any $\omega \in \Omega$, then

$$\|x - y\| = s\|x/s - y/s\| \geq s\beta > r$$

because $y/s \in M_\psi(\omega, x/s)$.

Define the homotopy $H : [0,1] \times \Omega \times \overline{B_s(0)} \to CK(Z)$ by

$$H(t, \omega, x) = F_2(\omega, x) + t(F_3(\omega, x) - F_2(\omega, x)).$$

It is easy to verify that H satisfies conditions (i) to (iv) of Theorem 4.5. We now show that $Lx \notin H(t, \omega, x)$ for all $(t, \omega, x) \in [0,1] \times \Omega \times \partial B_s(0)$.

Note that $Lx \in H(t, \omega, x)$ for some $x \in \partial B_s(0)$ if and only if the sets

$$\{x - z : x \in \partial B_s(0), z \in M_\psi(\omega, x)\}$$

and

$$\{t(y - z) : t \in [0,1], y \in [\psi\pi + K_{P,Q}]F_3(\omega, x), z \in [\psi\pi + K_{P,Q}]F_2(\omega, x)\}$$

have a nonempty intersection.

But $\{x - z : x \in \partial B_s(0), z \in M_\psi(\omega, x)\} \cap \overline{B_r(0)} = \emptyset$ because if $x \in \partial B_s(0)$ and $z \in M_\psi(\omega, x)$ then $\|x - z\| > r$.

Furthermore $\{t(y - z) : t \in [0,1], y \in [\psi\pi + K_{P,Q}]F_3(\omega, x), z \in [\psi\pi + K_{P,Q}]F_2(\omega, x)\} \subset B_r(0)$. This follows since $ty = tz + t(y - z) \in t[\psi\pi + K_{P,Q}]F_3(\omega, x) \subset t[\psi\pi + K_{P,Q}]F_2(\omega, x) + tB_r(0)$ and $tz \in t[\psi\pi + K_{P,Q}]F_2(\omega, x)$ together imply $t(y - z) \in B_r(0)$.

Hence the intersection is empty so $Lx \notin H(t, \omega, x)$ for all $(t, \omega, x) \in [0,1] \times \Omega \times \partial B_s(0)$.

Consequently,

$$\begin{aligned} d[(L, F_3(\omega, \cdot)), B_s(0)] &= d[(L, H(1, \omega, \cdot)), B_s(0)] \\ &= d[(L, H(0, \omega, \cdot)), B_s(0)] \\ &= d[(L, F_2(\omega, \cdot)), B_s(0)] \\ &\neq 0 \end{aligned}$$

as $F_2(\omega,\cdot)$ is odd. From the definition of the Coincidence Degree this is equivalent to $d(I-N_\psi(\omega,\cdot),B_s(0),0)\neq 0$ where $N_\psi(\omega,x)=Px+[\psi\pi+K_P(I-Q)]F_3(\omega,x)$. As before, it is easy to verify that this mapping satisfies those properties in Theorem 2.12, so let $S:\Omega\times\overline{G}\to CK(X)$ be the selection of N_ψ. Note that $d(I-S(\omega,\cdot),B_s(0),0)\neq 0$ because S is homotopic to N_ψ and so for each $\omega\in\Omega$, $S(\omega,\cdot)$ has a fixed point which implies that S has a random fixed point because of Theorem 2.1 and the fact that S has measurable graph. As $S(\omega,x)\subset N_\psi(\omega,x)$, then N_ψ has a random fixed point and from Theorem 4.2, we have that F_3 has a random fixed point. □

Theorem 5.1 generalizes Theorem 3.7 of Petryshyn and Fitzpatrick [19] in the following ways: (i) the mappings are random (as opposed to being deterministic) and (ii) the fixed point case has been extended to the coincidence case.

Although Pruszko considered the coincidence case, the mappings in Theorem 2.9 of [21] are compact and deterministic.

Instead of condition (i) both Petryshyn and Fitzpatrick and Pruszko had the stronger condition

$$\text{(i}'\text{)}\qquad Lx\in F_2(\omega,x)\Longrightarrow x=0.$$

Example A.

Let us consider the following random boundary value differential inclusion:

$$(\star)\qquad \begin{cases} x'(t)\in\Phi(\omega,t,x(t)) & \text{where } t\in T,\\ l_0(x)+l_1(x)=0 \end{cases}$$

where $\Phi:\Omega\times T\times\mathbb{R}^k\to CK(\mathbb{R}^k)$, T is some interval $[a,b]$, and $l_0,l_1:C(T;\mathbb{R}^k)\to\mathbb{R}^k$.

A mapping $\xi:\Omega\times T\to\mathbb{R}^k$ is a random solution of $(\star)$ if it is measurable in ω, absolutely continuous in t and for all $\omega\in\Omega$,

$$\xi'(\omega,t)\in\Phi(\omega,t,\xi(\omega,t))$$

and

$$l_0(\xi(\omega,t))+l_1(\xi(\omega,t))=0.$$

By $C(T;\mathbb{R}^k)$ we mean the separable Banach space of continuous functions from T to $\mathbb{R}^k$ with the norm $\|x\|=\max\{|x(t)|:t\in T\}$ and by $L^1(T;\mathbb{R}^k)$ we mean the separable Banach space of Lebesgue integrable functions with the norm $\|x\|=\int_T|x(t)|\,dt$.

Let $g:\Omega\times T\to\mathbb{R}^+$ and $h:\Omega\times T\times\mathbb{R}\to\mathbb{R}^+$ be single valued mappings, where $\mathbb{R}^+=[0,\infty)$. We make the following assumptions:

(1) the σ-algebra Σ is complete;
(2) g is measurable in ω and integrable in t;

(3) h is $\Sigma \otimes \mathcal{B}_T$ measurable, homogeneous and continuous in $x \in \mathbb{R}$ and for any $A \subset \mathbb{R}$ bounded, there exists a function $f_A \in L^1(T; \mathbb{R})$ such that

$$|h(\omega, t, x)| \leq f_A(t) \text{ for all } x \in A;$$

(4) Φ is $\Sigma \otimes \mathcal{B}_T$-measurable and u.s.c. in $x \in \mathbb{R}^k$;
(5) $|\Phi(\omega, t, u)| = \sup\{\|z\| : z \in \Phi(\omega, t, u)\} \leq g(\omega, t) + h(\omega, t, |u|)$;
(6) $l_0 : C(T; \mathbb{R}^k) \to \mathbb{R}^k$ is continuous and homogeneous, and l_1 is continuous and bounded.

Proposition 5.2. *The mapping* $(l_0 + l_1) : C(T; \mathbb{R}^k) \to \mathbb{R}^k$ *is a compact mapping.*

Proof: As l_0 is continuous at the point zero, it is bounded in a certain neighborhood of zero. So if $A \subset C(T; \mathbb{R}^k)$ is bounded, then so is $l_0(A)$ from the homogeneity of l_0. Clearly then $(l_0 + l_1)(A)$ is bounded in $\mathbb{R}^k$. □

Let $C_a(T; \mathbb{R}^k) \subset C(T; \mathbb{R}^k)$ denote the space of all absolutely continuous functions. We now formulate an equivalent problem.

Let $\Lambda : \Omega \times C(T; \mathbb{R}^k) \to 2^{L^1(T; \mathbb{R}^k)}$ be defined by

$$\Lambda(\omega, x) = \{z \in L^1 : z(t) \in \Phi(\omega, t, x(t)) \text{ for all } \ t \in T\}$$

and $N : \Omega \times C(T; \mathbb{R}^k) \to 2^{L^1 \times \mathbb{R}^k}$ be defined by

$$N(\omega, x) = \begin{pmatrix} \Lambda(\omega, x) \\ l_0(x) + l_1(x) \end{pmatrix}.$$

Let $L : C_a(T; \mathbb{R}^k) \to L^1(T; \mathbb{R}^k) \times \mathbb{R}^k$ be defined by

$$L(x) = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}.$$

It is easy to verify that $\xi : \Omega \times T \to \mathbb{R}^k$ is a solution of $(\star)$ if and only if ξ is a random solution of $Lx \in N(\omega, x)$. □

Proposition 5.3. *Under these assumptions,* L *is a linear Fredholm mapping of index zero,* N_x *is measurable and* N_ω *is an* L*-compact mapping.*

Proof: The following is clear:

$$\begin{aligned} \mathit{ker}\, L &\approx \mathbb{R}^k \\ \mathit{Im}\, L &= L^1 \times \{0\} \\ \mathit{coker}\, L &= L^1 \times \mathbb{R}^k \,/\, L^1 \times \{0\} \end{aligned}$$

and also L is linear. Thus the first part of the proof is complete.

The fact that N_x is measurable follows from Nowak [14, p. 419] as we assume that Σ is complete. Using Proposition 1.4 of Pruszko [21] we see that $\Lambda_\omega(\cdot)$ is a weakly compact mapping and hence so is $N(\omega, \cdot)$. In order to show that $N_\omega(x)$ is L-compact, it is enough to show that $\psi\pi N_\omega(x)$ and $K_P(I-Q)N_\omega(x)$ are compact. But $\psi\pi$ is a linear continuous mapping, and if $A \subset C(T;\mathbb{R}^K)$ is bounded, then from assumption (5), $N_\omega(A)$ is bounded in $L^1 \times \mathbb{R}^k$ so that $\psi\pi N_\omega(A)$ is bounded in a finite dimensional space. Using Proposition 1.7 of Pruszko [21], we see that $\psi\pi N_\omega(x)$ is a compact mapping. Now we come to proving that $K_P(I-Q)N_\omega(x)$ is a compact mapping. The projection $Q : L^1 \times \mathbb{R}^k \to L^1 \times \mathbb{R}^k$ is given by $Q(z,c) = (0,c)$ so that $Im L = ker Q$. We now find an explicit form for K_P.

Let $P : C(T;\mathbb{R}^k) \to \mathbb{R}^k$ be defined by $P(x(t)) = x(a)$. Then $Im P = \mathbb{R}^k = ker L$ and the pseudo-inverse of L associated with P is given by the mapping $K_P : L^1 \times \{0\} \to C_a(T;\mathbb{R}^k) \cap ker P$ defined by

$$K_P(z,0)(t) = \int_a^t z(s)\, ds.$$

See Gaines and Mawhin [7].

Let $A \subset C(T;\mathbb{R}^k)$ be bounded. From Proposition 5.2 we know that $l_0 + l_1$ is a compact mapping. Using Proposition 1.7 of Pruszko [21] it is enough to show that the subset of $C_a(T;\mathbb{R}^k)$ consisting of those functions defined by $\int_a^t z(s)\, ds$ where $z \in \Lambda_\omega(A)$ is relatively compact. But using assumption (5) and the Arzelà-Ascoli theorem, we see that indeed this is the case, and the proof is complete. $\square$

Theorem 5.4. *If the function $\xi(\omega,t) \equiv 0$ is the only solution of the problem*

$$(\star\star) \qquad \begin{cases} |x'(t)| \le h(\omega,t,|x(t)|) \\ l_0(x) = 0 \end{cases}$$

then problem $(\star)$ has a solution.

Proof: Define the following mappings:

$$\Lambda_1(\omega,x) = \{z \in L^1(T;\mathbb{R}^k) : |z(t)| \le g(\omega,t) \text{ for all } t \in T\}$$
$$\Lambda_2(\omega,x) = \{z \in L^1(T;\mathbb{R}^k) : |z(t)| \le h(\omega,t,|x(t)|) \text{ for all } t \in T\}$$
$$\Lambda_3(\omega,x) = \{z \in L^1(T;\mathbb{R}^k) : z(t) \in \Phi(\omega,t,x(t)) \text{ for all } t \in T\}.$$

We aim to apply Theorem 5.1 with the functions:

$$F_1(\omega,x) = \begin{pmatrix} \Lambda_1(\omega,x) \\ l_1(x) \end{pmatrix},$$
$$F_2(\omega,x) = \begin{pmatrix} \Lambda_2(\omega,x) \\ l_0(x) \end{pmatrix}, \text{ and}$$
$$F_3(\omega,x) = \begin{pmatrix} \Lambda_3(\omega,x) \\ l_0(x) + l_1(x) \end{pmatrix}.$$

On account of the assumptions on the mappings g and h, and Proposition 5.3, we have that F_i is an $L-k-\phi$-contraction for $i=1,2,3$ (because F_i is a compact mapping). By hypothesis, $x=0$ is the only solution of $Lx \in F_2(\omega, x)$ and moreover $F_2(\omega,\cdot)$ is homogeneous. Thus (i) of Theorem 5.1 is satisfied.

Noticing that $\pi F_1(\omega, X) \equiv l_1(X)$ and recalling the form of the mappings K_P and Q as given in Proposition 5.3, part (ii) of Theorem 5.1 is easily verified.

See Pruszko [21] page 966 for the proof that $F_3 \subset F_1 + F_2$. As all the assumptions of Theorem 5.1 are satisfied, $Lx \in F_3(\omega, x)$ has a random solution, or equivalently, problem $(\star)$ has a solution. $\square$

Remark 5.5. Pruszko [21] has proved Theorem 5.4 in the deterministic case.

Example B.

Let G be a bounded domain in $\mathbb{R}^n$ whose boundary ∂G is C^∞. We will consider real-valued functions of the following type $u : G \to \mathbb{R}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and a function $u : G \to \mathbb{R}$, the symbol

$$D^\alpha u = D^{|\alpha|} u / (\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n)$$

will denote the partial derivative of u (if it exists) of the order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $C^m(G)$ be the space of all functions $u : G \to \mathbb{R}$ which are continuous together with derivatives $D^\alpha u$, $|\alpha| < m$, and let

$$\hat{C}_p^m(G) = \{u \in C^m(G) : \sum_{|\alpha| \le m} \int_G |D^\alpha u(x)|^p \, dx < \infty\}$$

for $1 \le p < \infty$. In the space $\hat{C}_p^m(G)$ we define the norm as follows:

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \le m} \int_G |D^\alpha u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

By $H_{m,p}(G)$ we will mean the Sobolev space which is the completion of $\hat{C}_p^m(G)$ with respect to the above norm, and by $C_0^\infty(G)$ we will mean the space of infinitely differentiable functions $u : G \to \mathbb{R}$ with compact support contained in G.

Let $u, v : G \to \mathbb{R}$ be two integrable functions. We say that the function v is the α-th weak derivative of u if, for every $f \in C_0^\infty(G)$,

$$\int_G u(x) D^\alpha f(x) \, dx = (-1)^{|\alpha|} \int_G v(x) f(x) \, dx$$

and we write $\hat{D}^\alpha(u)$ instead of v.

The following two facts are well known (see [6] or [9]):

(i) $H_{m,p}(G) = \{u \in L^p(G) : \hat{D}^\alpha u \in L^p(G), |\alpha| \le m\}$;

(ii) let α be such that $|\alpha| \leq m$. Then the mapping $\hat{D}^{\alpha} : H_{m,p}(G) \to L^p(G)$ is a continuous extension of the mapping $D^{\alpha} : C^m(G) \to C^0(G)$.

Let $C^m(\overline{G})$ be the space of functions $u : G \to \mathbb{R}$ which are uniformly continuous together with derivatives $D^{\alpha}(u)$ for $|\alpha| \leq m$. In this space we shall define a norm as

$$|u|_m = \sum_{|\alpha| \leq m} \sup_{x \in G} |D^{\alpha} u(x)|.$$

Let $C^{m+\mu}(\overline{G}),\ 0 < \mu < 1$ be the Hölder space with the norm

$$|u|_{m+\mu} = |u|_m + \sum_{|\alpha|=m} \sup\{\frac{|D^{\alpha}u(x) - D^{\alpha}u(y)|}{|x-y|^{\mu}} : x, y \in G, x \neq y\}$$

and note that $C^{m+\mu}(\overline{G}) \subset C^m(\overline{G})$. Indeed the embedding $\hat{i} : C^{m+\mu}(\overline{G}) \to C^m(\overline{G})$ given by $\hat{i}(u) = u$ is a compact mapping (see [6]). The Sobolev Imbedding Theorem (again see [6]) ensures that for $p > n$, the mapping $\hat{j} : H_{m,p} \to C^{m-1+\mu}$ where $\mu = \frac{n}{p}$ defined by $\hat{j}(\hat{u}) = u$ where $\hat{u} = u$ almost everywhere on G is a well defined continuous mapping.

We shall consider the elliptic operator $A_p : H_{m,p}(G) \to L^p(G)$ given by

$$A_p(u)(x) = \sum_{|\alpha| \leq m} a_{\alpha}(x) \hat{D}^{\alpha}(u)(x),$$

where $a_{\alpha}(\cdot) \in C^{\infty}(\overline{G})$ and the differential boundary operators $B_j : C^{m-1}(\overline{G}) \to C^0(\overline{G}),\ j = 1, 2, \ldots, k$ given by

$$B_j(u)(x) = \sum_{|\alpha| \leq m_j} b^j_{\alpha}(x) D^{\alpha}(u)(x)$$

where $m_j < m$ and $b^j_{\alpha} \in C^{\infty}(\overline{G})$.

For (Ω, Σ) a complete measurable space, and $\Phi : \Omega \times \overline{G} \times \mathbb{R} \times \mathbb{R} \to CK(\mathbb{R})$, we formulate the following boundary value problem:

$$(\dagger) \qquad \begin{cases} u(\omega, \cdot) \in C^{m-1}(\overline{G}) \\ A_p(u_{\omega})(x) \in \Phi(\omega, x, u(x), D^{\beta}u(x)) \text{ on } G, |\beta| < m,\ p > n \\ B_j(u_{\omega})(x) = 0 \text{ on } \partial\Omega,\ j = 1, \ldots, k. \end{cases}$$

A solution of $(\dagger)$ is a mapping $u : \Omega \times \overline{G} \to \mathbb{R}$ such that $u_x : \Omega \to \mathbb{R}$ is Σ measurable for all $x \in \overline{G}$ and $(\dagger)$ is satisfied for all $\omega \in \Omega$.

Theorem 5.6. *Suppose that a multivalued mapping $\Phi : \Omega \times \overline{G} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is Carathéodory and the values of Φ are bounded in the following sense:*

$$|\Phi(\omega, x, u, v)| \leq g(\omega, x) \quad \forall\, (\omega, x) \in \Omega \times \overline{G},\ (u, v) \in \mathbb{R}^2$$

where g_x is measurable for each fixed $x \in \overline{G}$ and $g_\omega \in L^p(G)$, $p > n$.

Assume also that $Im A_p = L^p(G)$ and $j(ker A_p) \subset C^\infty(G)$. If for each $\omega \in \Omega$, the system

$$(\ddagger) \qquad \begin{cases} A_p(u_\omega) & = 0 \\ B_j(u_\omega)(x) & = 0 \text{ for all } x \in \partial G, j = 1, \dots, k \end{cases}$$

has only the trivial solution $u(\omega, x) \equiv 0$, then problem (†) has a solution.

Proof: We shall apply Theorem 5.1. To this end, let us define the following sets: $X_1 = C^{m-1}(\overline{G})$, $X_2 = L^p(G)$ and

$$X = \{u \in X_1 : u \in H_{m,p}(G), B_j(u)(x)|_{\partial G} \equiv 0\ j = 1, \dots, k\}.$$

Let $F_1 : \Omega \times X_1 \to 2^{X_2}$ be defined by

$$F_1(\omega, u) = \{v \in X_2 : v(x) \in \Phi(\omega, x, u(x), D^\beta u(x)) \text{ for all } x \in G\}$$

and $L : dom L = X \to X_2$ by

$$Lu = A_p(u).$$

It is clear that $u : \Omega \times G \to \mathbb{R}$ is a random solution of problem (†) if and only if u is a random solution of $Lu \in F_1(\omega, u)$.

Note that $F_1(\cdot, u)$ is a measurable mapping for all $u \in X_1$ from the same reasoning as in Proposition 5.3.

As problem (‡) has only the trivial solution, we see that $A_p|_X$ is a one-to-one mapping and so from the Banach mapping theorem, $A_p|_X{}^{-1}$ is continuous. We claim that $A_p|_X{}^{-1}$ is in fact a compact map. Consider the commutative diagram:

$$\begin{tikzcd} L^p(G) \arrow[r, "A_p|_X{}^{-1}"] \arrow[d, "L^{-1}"'] & (X, \|\cdot\|_{m,p}) \arrow[d, "\hat{j}|_X"] \\ (X, |\cdot|_{m-1}) & (X, |\cdot|_{m-1+\mu}) \arrow[l, "\hat{\imath}|_X"] \end{tikzcd}$$

By virtue of the remarks above regarding the regularity of the mappings $\hat{\imath}$ and $\hat{\jmath}$, we see that L^{-1} is a completely continuous mapping. Thus the pseudo-inverse of L is exactly L^{-1} and using Proposition 1.7 of Pruszko [21] it is clear that $F_1(\omega, \cdot)$ is an L-compact mapping. So we have (i) of the Theorem satisfied with $F_2 \equiv 0$, (ii) is satisfied with F_1 as above and for $F_3 = F_1$, (iii) is satisfied. Thus there exists a random solution to the equation $Lu \in F_1(\omega, u)$, and so problem (†) has a random solution. □

Remark 5.7. This result can be generalized further to the case where (‡) has only finitely many linearly independent solutions so that the kernel of L in X is no longer simply $\{0\}$; in this case L has nonzero index, e.g., see Akashi [1].

Notice that in Theorem 5.1 there is no explicit mention of a measure on the σ-algebra Σ and consequently we do not speak of properties holding *almost everywhere*, but rather *everywhere* in Ω. In the Examples it is implicit that there exists a measure (as we assume that Σ is complete) although we continue to employ the stronger condition of property P holding everywhere in Ω. This is not necessary and a minor alteration of the proofs will give the same results for the *almost everywhere* case.

References

[1] Akashi W.Y., *Equivalence theorems and coincidence degree for multivalued mappings*, Osaka J. Math. **25.1** (1988), 33–47.

[2] Berge C., *Espaces Topologiques*, Dunod, Paris, 1959.

[3] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.

[4] Engl H., *Random fixed point theorems for multivalued mappings*, Pacific J. Math. **76** (1978), 351–360.

[5] Fan K., *Fixed points and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.

[6] Friedman A., *Partial Differential Equations*, Holt, Rinehart, Winston, New York, 1969.

[7] Gaines R.E., Mawhin J.L., *Coincidence degree and nonlinear differential equations*, Springer-Verlag Lecture Notes No. 568, 1977.

[8] Gaines R.E., Peterson J.K., *Periodic solutions to differential inclusions*, Nonlinear Analysis **5** (1981), 1109–1131.

[9] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag New York, 1983.

[10] Glicksberg I., *A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.

[11] Himmelberg C.J., *Measurable relations*, Fund. Math. **87** (1975), 53–72.

[12] Lasota A., Opial Z., *An application of the Kakutani-Ky Fan Theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. **13** (1965), 781–786.

[13] Lloyd N.G., *Degree Theory*, Cambridge University Press, 1978.

[14] Ma T.W., *Topological degree for set valued compact vector fields in locally convex spaces*, Dissertationes Math. **92** (1972), 1–43.

[15] Mawhin J., *Equivalence theorems for nonlinear operator-equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations **12** (1972), 610–636.

[16] Nowak A., *Applications of random fixed point theorems in the theory of generalized random differential equations*, Bull. Pol. Acad. Sci. **34** (1986), 487–494.

[17] Nussbaum R.D., *The fixed point index and fixed point theorems for k-set contractions*, Doctoral Dissertation, University of Chicago, Chicago, Ill., 1969.

[18] Nussbaum R.D., *The fixed point index for condensing maps*, Ann. Mat. Pura. Appl. **89** (1971), 217–258.

[19] Petryshyn W.V., Fitzpatrick P.M., *A degree theory, fixed point theorems and mapping theorems for multivalued noncompact mappings*, Trans. Amer. Math. Soc. **194** (1974), 1–25.

[20] Pruszko T., *A Coincidence degree for L-compact convex-valued mappings and its applications to the Picard problem for orientor fields*, Bull. Acad. Pol. Sci. **27** (1979), 895–902.
[21] T. Pruszko, *Topological degree methods in multivalued boundary value problems*, Nonlinear Analysis **5** (1981), 959–973.
[22] Robertson A.P., *On measurable selections*, Proc. R.S.E. (A) **72** (1972/73), 1–7.
[23] Rogers C.A., *Hausdorff Measures*, Cambridge University Press, 1970.
[24] Sadowski B.W., *Limit-compact and condensing operators*, Russian Math. Surveys **27** (1972), 85–155.
[25] Saint-Beuve M.F., *On the existence of von Neumann-Aumann's theorem*, J. Functional Analysis **17** (1974), 112–129.
[26] Saks S., *Theory of the Integral*, Dover, New York, 1968.
[27] Tan K.K., Yuan X.Z., *On deterministic and random fixed points*, Proc. Amer. Math. Soc. **119** (1993), 849–856.
[28] Tarafdar E., Teo S.K., *On the existence of solutions of the equation $Lx \in Nx$ and a coincidence degree theory*, J. Austral. Math. Soc. Ser. A. **28** (1979), 139–173.
[29] Tarafdar E., Thompson H.B., *On the solvability of nonlinear noncompact operator equations*, J. Austral. Math. Soc. Ser. A **43** (1987), 103–126.
[30] Tarafdar E., Watson P., Yuan X.Z., *Jointly measurable selections of condensing random upper semi-continuous set-valued mappings and its applications to random fixed points*, Nonlinear Analysis, T.M.A. (in press), 1996.
[31] Wagner D.H., *Survey of measurable selection theorems*, SIAM J. Control. Optim. **15** (1977), 859–903.

Department of Mathematics, University of Queensland, Brisbane, Australia 4072
E-mail: xzy@axiom.math.uq.oz.au

(*Received* September 26, 1995)