Comment.Math.Univ.Carolin. 37,4 (1996)825–841

In quest of weaker connected topologies

M.G. Tkačenko, V.V. Tkachuk, V.V. Uspenskii, R.G. Wilson

Abstract. We study when a topological space has a weaker connected topology. Various sufficient and necessary conditions are given for a space to have a weaker Hausdorff or regular connected topology. It is proved that the property of a space of having a weaker Tychonoff topology is preserved by any of the free topological group functors. Examples are given for non-preservation of this property by "nice" continuous mappings.

The requirement that a space have a weaker Tychonoff connected topology is rather strong, but we show that it is difficult to construct spaces which would contain no infinite subspaces with a weaker connected $T_{3\frac{1}{2}}$-topology.

Keywords: connected, locally connected, free topological group, condensation, connectification

Classification: Primary 54H11, 54C10, 22A05, 54D06; Secondary 54D25, 54C25

0. Introduction

Let X be a topological space. If we assume no separation axioms, then it evidently has a weaker connected topology, namely the indiscrete one. In Section 2 we show that any T_0-space has a weaker connected T_0-topology and a non-one-point T_1-space has a weaker connected T_1-topology if and only if it is infinite — these results are easy to prove.

The nontrivial situations arise when we require the weaker connected topology of X to be Hausdorff or regular. For example, no compact Hausdorff disconnected space has a weaker Hausdorff connected topology. On the other hand, every regular (or Tychonoff) space X is a retract of a regular (Tychonoff) space with a weaker connected topology (see Corollary 2.3).

The problem of finding an inner characterization for the existence of a weaker connected topology seems to be too difficult to be solved in the general case. We give a characterization for locally connected spaces and a number of sufficient conditions for having, as well as for not having, a weaker connected topology.

1. Notation and terminology

All spaces under consideration are Hausdorff if no other separation axioms are assumed explicitly. If X is a space, then $\mathcal{T}(X)$ is its topology. If $A \subset X$,

Research supported by Mexican National Council of Science and Technology (CONACYT) grant 400200-5-4874E

then $\mathcal{T}(A, X) = \{U \in \mathcal{T}(X) : A \subset U\}$ and $\mathcal{T}(x, X) = \mathcal{T}(\{x\}, X)$. A "map" means a "continuous function". A map $f : X \to Y$ is called *a condensation* if it is a bijection. We also say that f condenses X onto Y. A space is called T_i-*subconnected* if it can be condensed onto a connected T_i-space or, equivalently, has a weaker connected T_i-topology. We say that a space X is T_i-closed if X is a T_i-space and it is closed in any T_i-space containing X. The terms "regular space" and "normal space" mean "regular T_1-space" and "normal T_1-space" respectively. If (X, ν) is a space and $A \subset X$, then $cl_\nu(A)$ (or simply $cl(A)$ if it does not lead to a misunderstanding) is the closure of A in the topology ν. A filter ξ on the space (X, ν) is called *open* if $\xi \cap \nu$ is a base of ξ. If for every $U \in \xi$ there is a $V \in \xi \cap \nu$ such that $cl_\nu(V) \subset U$, then ξ is called *a regular open* filter. If ξ is an open (or a regular open) filter, then it is called an open (or regular open) ultrafilter, if it is not properly contained in any open (resp. regular open) filter.

All other notions are standard and can be found in [4].

2. On the existence and non-existence of weaker connected topologies

We start with the simplest case of T_0- and T_1-spaces.

2.1 Proposition. (1) *Any T_0-space has a weaker connected T_0-topology;*

(2) *a T_1-space consisting of more than one point has a weaker connected T_1-topology if and only if it is infinite.*

Proof: Let X be a T_0-space. To prove (1) consider two cases.

(a) No finite subset of X is dense in X. Let the base of a new topology $\mathcal{U}$ be the family of all sets $X \backslash \overline{A}$ where A is a finite subset of X. It is clear that $\mathcal{U}$ is a topology on X weaker than the original one. Each pair of nonempty open subsets of $(X,\mathcal{U})$ intersect so that $(X,\mathcal{U})$ is a connected space.

If we have two different points of X, then the closure of one of them in the original topology does not contain the other. The complement of this closure belongs to $\mathcal{U}$ and T_0-separates $\{x, y\}$ which proves that $(X,\mathcal{U})$ is a T_0-space.

(b) If $X = \overline{\{x_1, \dots, x_n\}}$, then let $A_i = \overline{\{x_i\}}$ for all $i = 1, \dots, n$. The set A_i is connected so the space X is a union of $\leq n$ of its clopen components, say $X = C_1 \cup \ldots \cup C_k$. If $k = 1$, then there is nothing to prove. If $k > 1$, then let $U \in \mathcal{U}$ iff U is an open subset of $C_2 \cup \ldots \cup C_k$ or if U is an open subset of X containing $C_2 \cup \cdots \cup C_k$.

It is straightforward to check that $\mathcal{U}$ is a weaker connected T_0-topology on X, so (1) is proved.

If X is a finite T_1-space, then any weaker T_1-topology on X is discrete. If it is infinite, then the cofinite topology on X is connected, T_1 and is weaker than the original one, which proves (2). □

2.2 Proposition. *Let X be a T_i-space for some $i \in \{2, 3, 3\frac{1}{2}\}$. If H is a connected T_i-space, which is not T_i-closed, then $X \times H$ has a weaker connected T_i-topology.*

Proof: Choose a T_i-space G containing H as a non-closed subspace. Let $g \in G\backslash H$. The space $X \times H$ is a subspace of Y where Y is a quotient space of $X \times (H \cup \{g\})$ obtained by identifying the points of the set $X \times \{g\}$. The space Y is connected and $Y\backslash(X \times H)$ consists of one point y. Now choose any point x in $X \times H$ and identify the points x and y in Y. It is clear that X condenses onto the resulting space Z and that Z is a connected T_i-space. □

2.3 Corollary. *Let X be a topological space. Assume that $\mathcal{P} \in$ {Hausdorff spaces, regular spaces, Tychonoff spaces, normal spaces, collectionwise normal spaces, perfectly normal spaces, paracompact spaces, Lindelöf spaces, pseudocompact spaces, countably compact spaces}. If X belongs to the class $\mathcal{P}$, then there is a topological space Y with the following properties:*

(1) *Y is subconnected and belongs to $\mathcal{P}$;*
(2) *the space X is homeomorphic to a retract of Y;*
(3) *if X is a T_i-space, then Y is T_i-subconnected.*

Proof: Let D be a discrete space of power continuum. If $\mathcal{P}$ is any of the properties above except for countable compactness, pseudocompactness or Lindelöf property, then $Y = X \times D$ will have $\mathcal{P}$. It is clear that X is a retract of Y. The space Y can be condensed onto $X \times \mathbb{R}$ and the latter space is subconnected by Proposition 2.2. It is evident that the axioms of separation are preserved in this case.

If X is a Lindelöf space, then $Y = X \times \mathbb{R}$ is also Lindelöf (since $\mathbb{R}$ is σ-compact) and T_i-subconnected. Let G be a Σ-product lying in some uncountable power of the unit segment $[0, 1]$. If X is countably compact (or pseudocompact), then $Y = X \times G$ is T_i-subconnected (by Proposition 2.2) and countably compact (or pseudocompact respectively), because in G the closure of every countable set is compact. □

2.4 Proposition. *Suppose that X is a Hausdorff space which can be densely embedded in a connected Hausdorff space Y in such a way that*

(1) *$Y\backslash X$ is closed and discrete;*
(2) *there is a bijection φ between $Y\backslash X$ and some $A \subset X$ such that A is closed and discrete in Y.*

Then X is T_2-subconnected.

Proof: For each $z \in Y\backslash X$ identify z and $\varphi(z)$. The resulting quotient space Z is Hausdorff and connected. It is evident that X can be condensed onto Z. □

2.5 Corollary. *Let X be a Hausdorff non-countably compact space without open H-closed subspaces. If $\pi w(X) \leq \omega$, then X is T_2-subconnected.*

Proof: It was proved in [11] that X is countably connectifiable, i.e. it embeds densely into a connected Hausdorff space Y with $Y\backslash X$ countable. Fix any countably infinite closed discrete subset A of X. Simple modifications of the construction in [11, Theorem 3.5] make the set A closed and discrete in Y. Now use Proposition 2.4. □

Recall that a space X is called feebly compact if every locally finite family of non-empty open subsets of X is finite.

We shall need the following fact from [11].

2.6 Fact. (1) *A countable space is H-closed iff it is feebly compact.*
(2) *No countable space without isolated points is H-closed.*

2.7 Corollary. *Every countable Hausdorff space X without isolated points is T_2-subconnected.*

Proof: The topology of X can be weakened to a Hausdorff second countable topology ν ([2, Chapter 2, Problem 148]). It is evident that ν does not have isolated points and hence (X, ν) has no open H-closed subsets by Fact 2.6. Now use Corollary 2.5. □

2.8 Proposition. *A countable space X is not H-closed if and only if it condenses onto a dense in itself space.*

Proof: Suppose that X condenses onto a dense in itself space Y. Since Y is countable it cannot be H-closed by Fact 2.6. Since the property of being H-closed is preserved by continuous maps ([4, Problem 3.12.5 (b)]), the space X cannot be H-closed and we have established the sufficiency.

If X is not H-closed, then let us consider two cases.

Case 1. The set D of isolated points of X has a non-H-closed closure. Then $cl(D)$ is not feebly compact by Fact 2.6. Let $\{U_n : n \in \omega\}$ be a locally finite family of non-empty open subsets of $cl(D)$. Let $x_n \in U_n \cap D$ for each $n \in \omega$. The set $E = \{x_n : n \in \omega\}$ is closed, infinite and consists of isolated points of X. This means X is homeomorphic with $X \oplus (\omega \times \omega)$, where ω is considered with the discrete topology. The space $\omega \times \omega$ condenses onto $X \times \omega$ so that X also condenses onto $Y = X \times \omega$. Let Z be any connected space with the underlying set ω (see [11] for examples of such spaces). Evidently, Y condenses onto $X \times Z$ and since the space Z is not H-closed, we can use Proposition 2.2 to conclude that $X \times Z$ condenses onto a connected space which, of course, has no isolated points. The composition of these condensations now gives a condensation of X onto a dense in itself space.

Case 2. The set $K = cl(D)$ is H-closed. The space X being non-H-closed, there is a locally finite family $\gamma = \{U_n : n \in \omega\}$ of non-empty open subsets of X. Only finitely many of them can intersect K, so we may assume $U_n \subset X \backslash K$ for all $n \in \omega$. Choosing smaller sets if necessary we can make the family γ disjoint (see [11, Lemma 2.1]).

The set $X \backslash K$ is dense in itself, so none of the U_n's has isolated points. Hence for every $n \in \omega$ there exists a free open ultrafilter ξ_n in X such that $U_n \in \xi_n$. It is clear that $\xi_n \neq \xi_m$ if $n \neq m$. Fix an injection $p : D \to \omega$ and for every pair of different points $x, y \in X$ pick disjoint open sets $V_{x,y}$ and $V_{y,x}$ in the following way:

(i) if x and y are not isolated, then $V_{x,y}$ and $V_{y,x}$ are any disjoint open neighbourhoods of x and y respectively;

(ii) if $x \in D$ and $y \notin D$, then $V_{x,y} = \{x\} \cup W$, and $V_{y,x} = X \backslash cl(V_{x,y})$, where $W \in \xi_{p(x)}$ does not contain y in its closure;

(iii) if $x, y \in D$, then $V_{x,y} = \{x\} \cup U_{p(x)}$ and $V_{y,x} = \{y\} \cup U_{p(y)}$.

Let μ be the topology generated by the subbase consisting of the sets $V_{x,y}, V_{y,x}$ constructed for all pairs $x, y \in X$. It is clear that μ is Hausdorff and that the space (X, μ) does not have isolated points which proves our Proposition in Case 2 as well. □

2.9 Theorem. *A countably infinite space X is T_2-subconnected if and only if it is not H-closed.*

PROOF: If X condenses onto a connected space Y, then Y can have no isolated points. Since Y is countably infinite, it is not H-closed by Fact 2.6. Therefore X is not H-closed and we have proved the necessity.

If X is not H-closed, then it condenses onto a dense in itself space Y by Proposition 2.8. Now use Corollary 2.7 to condense Y onto a connected space. □

2.10 Corollary. *A countably infinite regular space is Hausdorff subconnected if and only if it is not compact.*

2.11 Proposition. *Let X be a σ-compact Tychonoff totally disconnected space. Then X is not T_3-subconnected.*

PROOF: Let $X = \bigcup\{F_n : n \in \omega\}$ where every F_n is compact. Each F_n is totally disconnected and hence zero-dimensional. Assume that $\varphi : X \to Y$ is a condensation with Y regular. The subspace $G_n = \varphi(F_n)$ is homeomorphic to F_n for each n and $\bigcup\{G_n : n \in \omega\} = Y$; thus Y is σ-compact and hence normal. The sets G_n are zero-dimensional so by the countable sum theorem for the covering dimension ([4, Theorem 7.2.1]) the space Y is zero-dimensional too. □

2.12 Lemma. *Let $\{C_\alpha : \alpha \in I\}$ be a family of connected non-empty spaces. Let $X = \bigoplus\{C_\alpha : \alpha \in I\}$. For each $\alpha \in I$, choose $x_\alpha \in C_\alpha$ and let ν be a connected topology on $A = \{x_\alpha : \alpha \in I\}$. We define a topology μ on X as follows:*

(i) *if $U \cap A = \emptyset$, then $U \in \mu$ iff $U \in \tau$, where τ is the topology of the discrete sum on X;*

(ii) *if $U \cap A \neq \emptyset$, then $U \in \mu$ iff $U \cap A \in \nu$ and for each $x_\alpha \in U \cap A$, $U \cap C_\alpha \in \tau$.*

Then (X, μ) is connected and $\mu \subset \tau$. Furthermore, (X, μ) is a T_i-space, if (X, τ) and (A, ν) are T_i-spaces ($i = 2, 3, 3\frac{1}{2}$).

The proof is straightforward and left to the reader. □

2.13 Definition. *Say that a topological space is a CO-space if all its components are open.*

2.14 Theorem. (1) *Let X be a Hausdorff CO-space with an infinite number of components. Then X is T_2-subconnected.*
(2) *If X is a T_3- (Tychonoff) CO-space with at least 2^ω components, then it is T_3- ($T_{3\frac{1}{2}}$-)subconnected.*

Proof: It is clear that any CO-space is a discrete sum $\bigoplus\{C_\alpha : \alpha \in B\}$ of its components. Choose $x_\alpha \in C_\alpha$ for each $\alpha \in B$. The set B being infinite, there is a connected Hausdorff topology on $A = \{x_\alpha : \alpha \in B\}$. Now use Lemma 2.12 to conclude that (1) holds. If $|B| \geq 2^\omega$, then there exists a connected Tychonoff topology on A and hence we can use Lemma 2.12 once more to establish (2). □

2.15 Lemma. *Let (X, τ) be a regular (Tychonoff) space such that $X = Y \oplus C$, where*

1. *C is a connected subspace of X;*
2. *there exists a discrete family $\{U_n : n \in \omega\}$ of non-empty open subsets of C;*
3. *$Y = \bigoplus\{Y_\alpha : \alpha \in A\}$, where each $Y_\alpha \neq \emptyset$ is connected and $|A| \leq 2^\omega$.*

Then X is regular (Tychonoff) subconnected.

Proof: Let $m \in \omega$. For any $f \in \{0,1\}^m$ fix a non-empty open set $V_f \subset U_m$ in such a way that $\overline{V_f} \cap \overline{V_g} = \emptyset$ if $f, g \in \{0,1\}^m$, $f \neq g$. It is possible because C is connected and infinite. Pick a point $x_f \in V_f$ for each $f \in \{0,1\}^m$ and $m \in \omega$. Let $y_\alpha \in Y_\alpha$ for all $\alpha \in A$. It follows from (3) that there exists an injection $\varphi : A \to \{0,1\}^\omega$. We are going to construct a new topology μ on X changing the original one only at the points y_α.

The base of μ-open neighbourhoods at the point y_α will consist of the sets $W \cup \bigcup\{W_k : k \geq m\}$, where $W \subset Y_\alpha$ is a τ-open neighbourhood of y_α and $W_k \in \mathcal{T}(x_{f_k}, V_{f_k})$, where $f_k = \varphi(\alpha) \restriction k$ for all $k \geq m$.

The space (X, μ) is connected. Indeed, let O be a μ-clopen subset containing C. Then $y_\alpha \in cl_\mu(O)$ and consequently $Y_\alpha \subset O$ for all α so that $O = X$.

To establish that (X, μ) is regular (Tychonoff) we only need to prove regularity (the Tychonoff property) at every y_α. Let $O = W \cup \bigcup\{W_k : k \geq m\}$ be a basic μ-open neighbourhood of y_α. Choose a τ-open neighbourhood W' of y_α and τ-open neighbourhoods W'_k of x_{f_k} in such a way that $cl_\tau(W') \subset W$ and $cl_\tau(W'_k) \subset W_k$ for all $k \geq m$.

Let us prove that $cl_\mu(O') \subset O$, where $O' = W' \cup \bigcup\{W'_k : k \geq m\}$. We only need to show that $y_\beta \notin cl_\mu(O')$ for every $\beta \neq \alpha$. Take any $\beta \neq \alpha$. There is a $p \in \omega$ such that $(g = \varphi(\beta)) \restriction p \neq (f = \varphi(\alpha)) \restriction p$. Then $U = Y_\beta \cup \bigcup\{V_{g\restriction k} : k \geq p\}$ is a μ-neighbourhood of y_β and $U \cap O' = \emptyset$. This proves regularity of (X, μ) in case (X, τ) is regular.

If (X, τ) is a Tychonoff space, then there is a τ-continuous function $h : X \to \mathbb{R}$ such that $h(y_\alpha) = 1 = h(x_{f_k})$ and $h \restriction X \backslash O \equiv 0$. It is straightforward that h is μ-continuous and our lemma is proved. □

2.16 Theorem. *If (X,τ) is a T_3 (Tychonoff) CO-space with an infinite set of infinite components, then X is T_3-subconnected ($T_{3\frac{1}{2}}$-subconnected respectively).*

Proof: If X has at least 2^ω components, then the result follows from the previous theorem. Thus we suppose that the number of components is less than 2^ω. Let $\{C_n : n \in \omega\}$ be different non-trivial components of X and let $C = \cup\{C_n : n \in \omega\}$. For each $n \in \omega$, choose $x_n \in C_n$ and sequences $S_n = \{x_{m,n} : m \in \omega\}$ and $\gamma_n = \{V_{m,n} : m \in \omega\}$ such that

(i) $V_{m,n}$ is open in X and $x_{m,n} \in V_{m,n} \subset C_n$;
(ii) $\overline{V_{m,n}} \cap \overline{V_{p,n}} = \emptyset$ if $m \neq p$;
(iii) $x_n \notin \overline{\cup\gamma_n}$ for all $n \in \omega$.

Such a selection is possible, because each set C_n is infinite. We define a new topology μ on C as follows. A basic open neighbourhood of x_n is of the form:

$$V \cup \bigcup\{U_m : m \geq k\},$$

where k is a natural number, $V \subset C_n$ is a τ-neighbourhood of x_n, and $U_m \subset V_{n,m}$ is an open τ-neighbourhood of $x_{n,m}$.

The μ-neighbourhoods of any $x \in C\backslash\{x_n : n \in \omega\}$ are defined to be its τ-neighbourhoods.

It is easy to see that the topology μ differs from τ only at the points x_n and that for each n, x_n has a local base of μ-neighbourhoods which miss $\{x_m\} \cup \{x_{m,k} : k \in \omega\}$ for each $m \neq n$.

Now fix a basic neighbourhood $W = V \cup \bigcup\{U_m : m \geq k\}$ of the point x_n for some $n \in \omega$. Since C is regular, there exist τ-open sets V' and U'_m with $x_n \in V' \subset cl_\tau(V') \subset V$ and $x_{n,m} \in U'_m \subset cl_\tau(U'_m) \subset U_m$ for all $m \geq k$. It is immediate, that the μ-closure of the μ-open neighbourhood $W' = V' \cup \bigcup\{U'_m : m \geq k\}$ of the point x_n is contained in W. This proves regularity of (C,μ).

If C is Tychonoff, then there is a τ-continuous function $f : C \to \mathbb{R}$ such that $f(x_n) = 1 = f(x_{n,m})$ for all $m \geq k$ and $f \restriction (C\backslash W) \equiv 0$. It is immediate, that f is μ-continuous, so that (C,μ) is Tychonoff if C is.

Furthermore, a μ-clopen subset W of C would be a τ-clopen subset of C and hence $W = \bigcup\{C_n : n \in A\}$ for some $A \subset \omega$. Without loss of generality the set A could be assumed infinite (otherwise consider the τ-clopen set $C\backslash W$). Since for each μ-neighbourhood of any point $x_n \in C_n$ meets all but finitely many τ-components C_m, it follows that $x_n \in \overline{W}$ for each $n \in \omega$. Thus $W = C$ and this proves that (C,μ) is connected.

Consider the components $\{Y_\alpha : \alpha \in A\}$ of $X\backslash C$. Now $X = C \oplus Y$, where $Y = \bigoplus\{Y_\alpha : \alpha \in A\}$. Let U_n be a τ-open set with $\overline{U_n} \subset C_n\backslash(\overline{\gamma_n} \cup \{x_n\})$. It is clear that U_n is μ-open and the family $\{U_n : n \in \omega\}$ is μ-discrete. Now apply Lemma 2.15. $\square$

2.17 Corollary. *If X is a locally connected T_i-space with an infinite set of infinite components, then X is T_i-subconnected ($i = 2, 3, 3\frac{1}{2}$).*

Proof: Every locally connected space is a CO-space, so we may use Theorem 2.16. $\square$

2.18 Examples. (1) *Let C_α be the Cantor set for each $\alpha \in A$. The space $\bigoplus\{C_\alpha : \alpha \in A\}$ is T_3-subconnected if and only if it is Tychonoff subconnected and the latter occurs iff $|A| \geq \omega_1$. It is T_2-subconnected iff $|A| \geq \omega$.*
(2) *Let I_α be the unit segment $[0,1]$ for each $\alpha \in A$. The space $\bigoplus\{I_\alpha : \alpha \in A\}$ is T_3-subconnected if and only if $|A| \geq \omega$.*

Proof: If the set A is countable, then Proposition 2.11 shows that $X = \bigoplus\{C_\alpha : \alpha \in A\}$ is not T_3-subconnected. Suppose that $|A| \geq \omega_1$. It suffices to construct a weaker connected topology on X when $|A| = \omega_1$. Indeed, if $|A| > \omega_1$, then represent A as an infinite union of its subsets of cardinality ω_1. Introduce a weaker connected topology on the corresponding subsets of X and use Theorem 2.16.

Without loss of generality we can assume that the index set A coincides with ω_1. Choose a subset $\{p_\alpha : \alpha < \omega_1\} \subset [0,1]$ such that $p_\alpha \neq p_\beta$ if $\alpha \neq \beta$. For each $\alpha < \omega_1$ it is easy to construct a map $f_\alpha : C_\alpha \to [0,1]^{\omega_1}$ in such a way that

(i) $\pi_\alpha(f_\alpha(C_\alpha)) = [0,1]^\alpha$, where $\pi_\alpha(f) = f \restriction \alpha$ for all $f \in [0,1]^{\omega_1}$;
(ii) f_α is an embedding for all $\alpha < \omega_1$;
(iii) $g(\beta) = p_\alpha$ for any $g \in f_\alpha(C_\alpha)$ and $\beta > \alpha + 1$.

It is evident that $f_\alpha(C_\alpha) \cap f_\beta(C_\beta) = \emptyset$ if $\alpha \neq \beta$. The union of the maps f_α is a condensation of X onto the subset $C = \bigcup\{f_\alpha(C_\alpha) : \alpha < \omega_1\}$ of $[0,1]^{\omega_1}$. The subspace C is connected, because its projections cover all countable faces of $[0,1]^{\omega_1}$ ([9]). This proves that X is $T_{3\frac{1}{2}}$-subconnected and (1) is established.

To prove (2), observe that if A is finite, then $\bigoplus\{I_\alpha : \alpha \in A\}$ is a compact disconnected space and hence is not T_2-subconnected. If A is infinite, use Theorem 2.16. $\square$

We now turn to the case in which X has a finite number of components.

2.19 Definition. *Given a natural number n let us call a space X Hausdorff (resp. regular or Tychonoff) n-extendable if there exists a Hausdorff (resp. regular or Tychonoff) space Y such that X is a dense subspace of Y and $|Y \backslash X| = n$.*

2.20 Theorem. *If a Hausdorff (regular or Tychonoff) space (X,τ) has a finite number of components $\{C_i : 1 \leq i \leq n\}$, then X is Hausdorff (regular or Tychonoff respectively) subconnected if and only if it is Hausdorff (resp. regular or Tychonoff) $(n-1)$-extendable.*

Proof: The sufficiency is proved by induction on n. If we have a Hausdorff (resp. regular or Tychonoff) extension Y of the space X such that $Y \backslash X = \{y_1, \dots, y_{n-1}\}$, then $y_j \in \overline{C}_i$ for some i, j. Take any point $x \in X \backslash C_i$ and define the topology at x to be a τ-neighbourhood of x union with the trace of some neighbourhood of y_j on X. The weaker topology thus defined has $\leq (n-1)$ components and is Hausdorff (resp. regular or Tychonoff) $(n-2)$-extendable. Now use the inductive hypothesis.

For the necessity, suppose that (X, τ) has n components and is not $(n-1)$-extendable. The proof is by induction on n. If $n = 2$, then X is H-closed (regular closed or compact, respectively) and so each component of X is H-closed (regular closed or compact, respectively). However, if μ is some Hausdorff (regular or Tychonoff) topology on X with $\mu \subset \tau$, then each τ-component of X with the relative μ-topology is still H-closed (regular closed or compact, respectively) because all these properties are preserved by continuous maps. Thus (X, μ) is not connected.

Now suppose that the result is true for any space with k components, and suppose that X has $(k+1)$ components and is not k-extendable within the class of T_i-spaces, $i = 2, 3, 3\frac{1}{2}$. If there is a weaker connected T_i topology ν on X, then one of the τ-components C_j of X is not ν-closed; let $x \in cl_\nu(C_j) \cap C_l$. Define a new topology μ on X by:

(i) if $x \notin U$, then $U \in \mu$ iff $U \in \tau$;

(ii) open μ-neighbourhoods of x are of the form $U \cap (C_l \cup C_j)$, where $U \in \nu$.

Clearly, $\nu \subset \mu \subset \tau$ and (X, μ) is a T_i-space with k components. Now suppose that (X, μ) has a T_i-extension (Z, ξ) with $|Z \backslash X| = k-1$. Change the topology ξ at the point x by defining the new base of open neighbourhoods of x to be its τ-neighbourhoods. A routine verification shows that the resulting topology ζ on Z is T_i and (Z, ζ) is a $(k-1)$-extension of (X, τ).

If $i \in \{2, 3\}$, consider the traces of the μ-open neighbourhoods of x on C_j. They form a free τ-open (resp. regular τ-open) filter which is contained in some non-convergent τ-open (resp. regular τ-open) ultrafilter $\mathcal{F}$ on (X, τ). However, no point of $Z \backslash X$ is a limit point of $\mathcal{F}$ (neither in ξ nor in ζ) because Z is a T_i-extension of (X, μ). Consequently, we can adjoin the filter $\mathcal{F}$ to Z as a new point in a standard way obtaining thus a k-extension of (X, τ) (with the necessary axiom of separation) which is a contradiction.

If we consider the Tychonoff case, then it suffices to prove that (Z, ζ) is not compact. But the traces of the μ-open neighbourhoods of x on C_j form a regular open filter on (Z, ζ), which, since it has empty intersection, can have no limit. Hence (Z, ζ) is at least 1-extendable. But this extension would give a k-extension of (X, τ) and we have a contradiction in this case as well.

Thus, for any $i = 2, 3, 3\frac{1}{2}$ we may apply the inductive hypothesis, concluding that (X, μ) is not T_i-subconnected, which contradicts the existence of the connected T_i-topology ν on X.

$\square$

2.21 Corollary. *If a Hausdorff* (*regular*) *space X has a finite number n of components, then X is Hausdorff* (*regular*) *subconnected if and only if it has at least* $(n-1)$ *open* (*regular open*) *ultrafilters.*

2.22 Proposition. *Let (X,τ) be a separable metrizable non-compact space. Then X admits a weaker separable metrizable topology μ which is nowhere locally compact.* (*In particular,* (X,μ) *has no open compact subsets.*)

Proof: It suffices to define a condensation $g:(X,\tau)\to Y\subset I^\omega$ such that both Y and $I^\omega\setminus Y$ are dense in I^ω. Here $I=[0,1]$ is the unit segment with its natural topology. Identifying X and Y we obtain a weaker separable metrizable topology μ on X which is nowhere locally compact.

Since (X,τ) is not compact, there exists an infinite closed discrete subset $K=\{x_n:n\in\omega\}$ of X with $x_n\neq x_m$ for $n\neq m$. Choose two disjoint countable dense subsets $S=\{s_n:n\in\omega\}$ and $T=\{t_n:n\in\omega\}$ of $(0,1]^\omega$. Obviously, both S and T are dense in I^ω.

For every $n\in\omega$ we will define a continuous function $g_n:Z\to I$ satisfying

$$(*)\qquad g_n(x_k)=p_n(s_k)\ \text{ and }\ g_n(k)=p_n(t_k)\ \text{ for each }\ k\in\omega,$$

where $p_n:I^\omega\to I_n$ is the projection onto the n-th factor I_n.

Let $\mathcal{B}$ be a countable base in X such that $|\overline{U}\cap K|\leq 1$ for any $U\in\mathcal{B}$. Denote by $\mathcal{P}$ the family of all pairs (U,V) of elements of $\mathcal{B}$ such that $\overline{U}\cap\overline{V}=\emptyset$ and let $\mathcal{P}=\{P_n:n\in\omega\}$ be a faithful enumeration of elements of $\mathcal{P}$.

Now for every $n\in\omega$ construct a function g_n which satisfies $(*)$ and also

$$(**)\qquad \begin{array}{l} \text{if } P_n=(U,V) \text{ and } \overline{U}\cap K=\emptyset=\overline{V}\cap K \\ \text{then } g_n(U)=\{0\} \text{ and } g_n(V)=\{1\}; \\ \text{if only one of the sets } \overline{U},\overline{V} \text{ intersects } K,\text{ say } \overline{U}\cap K=\{x_i\}, \\ \text{then } g_n(U)=\{g_n(x_i)\}=\{p_n(s_i)\} \text{ and } g_n(V)=\{0\}. \end{array}$$

If $\overline{U}$ and $\overline{V}$ both intersect K, then g_n is defined to satisfy $(*)$. That such a function g_n exists, is a simple consequence of Tietze's theorem. The functions g_n having been constructed for all $n\in\omega$ let $g=\Delta\{g_n:n\in\omega\}$ be the diagonal product of g_n's. Clearly, g is a continuous map of Z to I^ω. From $(*)$ it follows that $g(x_n)=s_n$ and $g(y_n)=t_n$ for all $n\in\omega$, that is $g(K)=S$. In particular, $g(X)$ is dense in I^ω. Let us show that $T\subset I^\omega\setminus g(X)$.

If $x\in X\setminus K$, then there is a pair $P_n=(U,V)\in\mathcal{P}$ such that $x\in U$ and $\overline{U}\cap K=\emptyset=\overline{V}\cap K$. Therefore $g_n(x)=p_n(g(x))=0$ while $p_n(T)\not\ni 0$ so that $g(x)\notin T$. If $x=x_n\in K$, then $g(x)\in S\subset I^\omega\setminus T$. Thus, both sets $g(X)$ and $I^\omega\setminus g(X)$ are dense in I^ω.

It remains to verify that the mapping g is one-to-one. Let $x,y\in X\setminus K$. There exist $U,V\in\mathcal{B}$ such that $x\in U$, $y\in V$ and $\overline{U}\cap\overline{V}=\emptyset$, $\overline{U}\cap K=\emptyset=\overline{V}\cap K$. Then $(U,V)=P_n$ for some n and $g_n(x)=0$, $g_n(y)=1$ by $(**)$, that is $g(x)\neq g(y)$. If $x\in K$ and $y\notin K$, then there exist $P_n=(U,V)$ such that $x\in U$, $y\in V\subset\overline{V}\subset X\setminus K$. By $(**)$ we have $g_n(y)=0\neq g_n(x)$ because $g_n(x)\in p_n(S)\not\ni 0$. Therefore $g(y)\neq g(x)$.

Finally, if both x and y are in K, say $x=x_m$, $y=x_n$, $m\neq n$, then $g(x)=s_m$ and $g(y)=s_n$ and the conclusion follows since $s_m\neq s_n$. □

2.23 Remark. *In 2.22 we have proved even more: given a countably infinite closed discrete subspace K of (X,τ), the set K can be made dense in (X,μ).*

2.24 Theorem. *A regular disconnected space X with a countable network is T_2-subconnected if and only if it is not compact.*

PROOF: We need to prove only the sufficiency. If X is not compact, there exists a discrete family $\{U_n : n \in \omega\}$ of non-empty open subsets of X. Choose a point $x_n \in U_n$ for all $n \in \omega$. Let f_0 be a continuous function on X with $f_0(x_n) = n$ for each $n \in \omega$. Find a family $\{f_n : n = 1, 2, \dots\}$ of real-valued continuous functions on X which separates the points of X — it exists because X has a countable network. Now let $f = \Delta\{f_n : n \in \omega\}$ be the diagonal product of f_n's. The space $Y = f(X)$ is second countable and X condenses onto Y. Let p_0 be the projection of Y onto the 0-th coordinate. It is clear that $p_0(f(x_n)) = n$, so that p_0 is an unbounded real-valued continuous function on Y. Hence Y is not compact and we may use Proposition 2.22 to condense Y onto a second countable space Z without open compact subsets. Finally, apply Corollary 2.5 to conclude that Z and hence X has a weaker connected Hausdorff topology. □

2.25 Corollary. *A regular disconnected second countable space is T_2-subconnected iff it is not compact.*

2.26 Remark. *Theorem 2.24 gives another proof of Corollary 2.10. However, it does not cover Theorem 2.9 because it says nothing about the non-regular case. The interesting question then arises as to whether a Hausdorff space with countable network is T_2-subconnected as long as it is not H-closed.*

2.27 Proposition. *Let (X,τ) be a second countable T_3-space with at least one non-compact component. Then X is T_3-subconnected.*

PROOF: Let C be a non-compact component of X. There exists a countably infinite $K \subset C$ which is closed and discrete in X. Use Proposition 2.22 and Remark 2.23 to find a weaker separable metrizable topology μ on X with K dense in (X,μ). From $K \subset C$ it follows that C is a dense connected subspace of (X,μ), so that (X,μ) is connected. □

2.28 Lemma. *Let (X,τ) be a second countable T_3-space which has an infinite set of non-trivial components $\{C_n : n \in \omega\}$ such that $cl(\cup\{C_n : n \in \omega\})$ is not compact. Then (X,τ) can be condensed onto a second countable T_3-space with at least one non-compact component.*

PROOF: If (X,τ) has a non-compact component, then we are done; hence suppose that all components of (X,τ) are compact. Since (X,τ) is second countable and $cl(\cup\{C_n : n \in \omega\})$ is not compact, there exists an infinite $A \subset \omega$ and $x_n \in C_n$ for each $n \in A$ such that the set $E = \{x_n : n \in A\}$ is closed and discrete in X.

There exists a discrete family $\{U_n : n \in A\}$ of open subsets of X such that $U_n \cap E = \{x_n\}$, and since each C_n is connected and non-trivial, it follows that $U_n \cap C_n$ is infinite (in fact, has cardinality 2^ω). For each $n \in A$, we choose a disjoint family $\{V_{mn} : m \in \omega\}$ of open subsets of X and a set $\{s_{mn} : m \in \omega\} \subset X$ such that

(i) $V_{mn} \subset U_n$ for all $n \in A$, $m \in \omega$;
(ii) $V_{mn} \cap V_{pn} = \emptyset$ if $p \neq m$;
(iii) $s_{mn} \in V_{mn} \cap C_n$ for all $n \in A$, $m \in \omega$.

Let $S_k = \{s_{kn} : n \in A\}$. Clearly S_k is closed and discrete for any $k \in \omega$. We define a new topology μ on X as follows:

(iv) if $x \notin E$, then U is a μ-neighbourhood of x if and only if U is a τ-neighbourhood of x;
(v) U is a μ-neighbourhood of $x_n \in E$ if $U = V \cup \bigcup\{V_r : r \geq i\}$, where V is a τ-neighbourhood of x_n and V_r is an open subset of X such that $s_{nr} \in V_r \subset V_{nr}$ for every $r \geq i$.

It is easy to see that collectionwise normality of (X, τ) enables one to separate S_k and S_l for any distinct $k, l \in \omega$ and whence it follows that x_k and x_l have disjoint μ-open neighbourhoods. Thus, the space (X, μ) is Hausdorff. Furthermore, since (X, τ) is regular and S_n is discrete, each set V_r contains a closed neighbourhood of s_{nr} and so (X, μ) is regular. Finally, the set $C = \cup\{C_n : n \in A\}$ is connected but not compact since for each $n \in A$ it is possible to choose $z_n \in (C_n \cap U_n) \setminus \{s_{mn} : m \in \omega\}$ and then the set $\{z_n : n \in A\}$ is infinite closed and discrete in (X, μ) and lies in C.

The space (X, μ) is not necessarily second countable. But it has a countable network so that it is possible to condense it onto a second countable regular space Y in exactly the same way as was done at the beginning of the proof of Theorem 2.24 to achieve the image C' of C to be closed and non-compact. Now the component of Y containing C' will be non-compact and we are done. □

2.29 Theorem. *Let (X, τ) be a σ-connected (that is, the family of the components of X is countable), disconnected, second countable T_3-space. Then X is T_3-subconnected if and only if the closure of the union of its non-trivial components is not compact.*

Proof: The sufficiency follows from Proposition 2.27 and Lemma 2.28. The necessity follows from the fact that if the closure D of the union of non-trivial components is compact, then it must be compact and hence closed in any weaker regular topology μ. If D coincides with X, the X is a compact disconnected space and hence it is not even T_2-subconnected. Therefore we may assume that $X \setminus D$ is non-empty. It is obvious, that $X \setminus D$ is open, countable, and thus zero-dimensional in μ. Thus (X, μ) is not connected. □

2.30 Corollary. *A second countable locally connected regular space X is Tychonoff subconnected if and only if it has an infinite number of nontrivial components or has a non-compact component.*

It is clear that subconnectedness is preserved by arbitrary products and by strengthening the topology. The following proposition shows that the free topological group functors preserve it as well.

2.31 Proposition. *Let X be a Tychonoff subconnected space, $|X| > 1$. Then (1) the free Graev topological group $F_\Gamma(X)$ and the free Graev abelian topological group $A_\Gamma(X)$ admit a weaker connected Hausdorff topological group topology; (2) the free topological group $F(X)$ and the free abelian topological group $A(X)$ are $T_{3\frac{1}{2}}$-subconnected.*

Proof: Let $\varphi : X \to Y$ be a condensation of X onto a connected Tychonoff space Y.

(1) Denote by $\hat{\varphi}$ a continuous homomorphism of $F_\Gamma(X)$ to $F_\Gamma(Y)$ extending φ. It is clear that $\hat{\varphi}$ is an algebraic isomorphism between these groups. By Fact (A) of [5, Section 6], the group $F_\Gamma(Y)$ is connected, which proves our claim about the group $F_\Gamma(X)$. The same reasoning applies to the group $A_\Gamma(X)$ and gives a continuous one-to-one homomorphism $\psi : A_\Gamma(X) \to A_\Gamma(Y)$ extending the mapping φ. The same Fact (A) of [5, Section 6] guarantees the connectedness of the group $A_\Gamma(Y)$.

(2) The problem here is that the groups $F(Y)$ and $A(Y)$ are not connected, so the proof is not so straightforward as in (1). For every element $g \in F(Y)$, let $n = l(g)$ be the length of g, that is, the number of letters in the irreducible word g written in the alphabet $Y \cup Y^{-1}$. If $g = y_1^{\varepsilon_1} \dots y_n^{\varepsilon_n}$ with $y_1, \dots, y_n \in Y$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, +1\}$, denote by $l_+(g)$ (respectively $l_-(g)$) the number of indices $i \le n$ such that $\varepsilon_i = 1$ (resp. $\varepsilon_i = -1$). Put

$$G = \{g \in F(Y) : l_+(g) = l_-(g)\}.$$

It is clear that G is a closed subgroup of $F(Y)$ and the quotient group $F(Y)/G$ is topologically isomorphic to the discrete group of integers $\mathbf{Z}$. Indeed, let $p : Y \to \mathbf{Z}$ be a constant mapping defined by $p(y) = 1$ for each $y \in Y$. Extend p to a continuous homomorphism $\hat{p} : F(Y) \to \mathbf{Z}$. Then $ker(\hat{p}) = G$, which implies the facts that G is closed in $F(Y)$ and $F(Y)/G \cong \mathbf{Z}$. Since Y is connected, Assertion 1.1 of [10] implies that the group G is connected (to see that directly, one can use natural "product" mappings $i_{n,\varepsilon} : Y^{2n} \to G$ defined by $i_{n,\varepsilon}(y_1, y_2, \dots, y_{2n}) = y_1^{\varepsilon_1} \cdot y_2^{\varepsilon_2} \cdot \dots \cdot y_{2n}^{\varepsilon_{2n}}$ for each point $y = (y_1, y_2, \dots, y_{2n}) \in Y^{2n}$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n}) \in \{-1, +1\}^{2n}$, to cover G by a countable family of connected subspaces containing the identity of G). Thus, the space $F(Y)$ is the free topological sum of a countably infinite family of connected subspaces homeomorphic to G. Apply now Theorem 2.16 to conclude that $F(Y)$ is $T_{3\frac{1}{2}}$-subconnected. Since $\hat{p}$ is a condensation of $F(X)$ onto $F(Y)$, the space $F(X)$ is also $T_{3\frac{1}{2}}$-subconnected.

An analogous reasoning shows that the space $A(X)$ is $T_{3\frac{1}{2}}$-subconnected. □

It is well known that connectedness is invariant under any continuous map. The following examples show that the subconnectedness can be destroyed by as good a map as one can imagine.

2.32 Examples. (1) *If* $i \in \{2, 3, 3\frac{1}{2}\}$, *then* T_i*-subconnectedness is not preserved by condensations;*
(2) *If* $i \in \{2, 3, 3\frac{1}{2}\}$, *then* T_i*-subconnectedness is not preserved by open two-to-one maps;*
(3) *Tychonoff* (*and regular*) *subconnectedness is not preserved by perfect open maps.*

Proof: A discrete space of power continuum is Tychonoff subconnected. However, it is condensable onto the discrete union of two unit segments, which is not Hausdorff subconnected being compact and disconnected. This proves (1).

To establish (2), observe that the set

$$X = ((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\}) \cup (2, 3] \times \{0\}) \cup ([2, 3) \times \{1\}) \subset \mathbb{R} \times \mathbb{R},$$

with the topology induced from the plane is Tychonoff subconnected by Corollary 2.21. The projection of X onto the x-axis yields a two-to-one open map of X onto $[0, 1] \cup [2, 3]$ which is not Hausdorff subconnected. Consequently, we have proved (2).

To prove (3) consider the projection map of $X = [0, 1] \times \omega$ onto ω, where ω has the discrete topology. It is open and perfect. The space X is Tychonoff subconnected by Corollary 2.17, while ω is not because it is countable. □

Finally we look at the spaces which have an infinite subconnected subspace. As one would expect, this class turns out to be much wider than the class of subconnected spaces. For example, every infinite Hausdorff space has an infinite discrete and hence T_2-subconnected subspace. Hence we only consider Tychonoff spaces.

2.33 Proposition. *The following conditions are equivalent for every Tychonoff space* X*:*

(1) X *has an infinite Tychonoff subconnected subspace;*
(2) X *has a subspace which can be condensed onto the unit segment* $[0, 1]$*;*
(3) X *has a subspace that maps continuously onto* $[0, 1]$*;*
(4) X *has a subspace that maps continuously onto an infinite connected Tychonoff space.*

Proof: The implications (2) ⇒ (3) ⇒ (4) are evident. If a subspace of X maps onto a connected space Y, then choose a point in the inverse image of each point of Y. The resulting subspace condenses onto Y and the implication (4) ⇒ (1) is established.

Now let $f : Z \to Y$ be a condensation, where $Z \subset X$ and Y is an infinite connected Tychonoff space. Take different points $y_0, y_1 \in Y$ and a map $h : Y \to [0, 1]$ with $h(y_0) = 0$ and $h(y_1) = 1$. Since Y is connected we have $h(Y) = [0, 1]$. For every $t \in [0, 1]$ take a point $x_t \in f^{-1}(h^{-1}(t))$. The set $\{x_t : t \in [0, 1]\}$ condenses onto $[0, 1]$ and we have established (1) ⇒ (2). □

2.34 Proposition. *The following classes of spaces have infinite Tychonoff subconnected subspaces:*

(1) *any Tychonoff space X with $ind(X) > 0$;*
(2) *any compact non-scattered space;*
(3) *any space X which has a discrete subspace of power continuum;*
(4) *any Čech-complete space without isolated points.*

Proof: (1) Take a point $x \in X$ such that the dimension at it is greater than zero. This means there is a $U \in \mathcal{T}(x, X)$ such that there is no clopen set containing x and lying inside U. Pick a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and $f \restriction X \backslash U \equiv 0$. Then $f(X) = [0, 1]$, because if $t \in [0, 1] \backslash f(X)$, then $W = f^{-1}([t, 1])$ would be a clopen set such that $x \in W \subset U$ which is a contradiction. Now use Proposition 2.33 (3).

Every compact non-scattered space maps onto the unit segment ([8]), and therefore (2) holds; (3) is clear and (4) holds because any Čech-complete space without isolated points contains a non-scattered compact subspace and the latter maps continuously onto $[0, 1]$. □

Recall that a Luzin space is a uncountable space without isolated points such that its nowhere dense sets are countable. It is known that under CH there are Luzin subsets of the reals ([3], [6]) and that there are no Luzin spaces under Martin's axiom and the negation of the continuum hypothesis ([3], [11]).

2.35 Proposition. *Let X be a hereditarily separable Luzin space. Then X contains no Tychonoff subconnected subspace.*

Proof: Suppose that $Y \subset X$ and $f : Y \to [0, 1]$ is a condensation. The set Y is uncountable, so that $Y \backslash A$ is dense in an open set of X for some countable $A \subset Y$. The set $Y \backslash A$ is evidently a Luzin set, which condenses onto $[0, 1] \backslash f(A)$. Choose a countable $B \subset Y \backslash A \subset \overline{B}$. It follows from the definition of a Luzin set, that $(Y \backslash A) \backslash W$ is countable for every open $W \supset B$. Therefore, $[0, 1] \backslash U$ is countable for each open $U \supset f(A) \cup f(B)$. But this is impossible because $f(A) \cup f(B)$ is of measure zero and hence can be covered by an open subset U of $[0, 1]$ of measure $\leq \frac{1}{2}$; then $[0, 1] \backslash U$ has measure $\geq \frac{1}{2}$ so it cannot be countable. □

3. Unsolved problems

As usual, these are more numerous than those we have solved. The topic seems to be new, and the problems below might be easy or difficult, but all of them seem to require some new approach.

3.1 Problem. *Find an inner characterization of the regular second countable Tychonoff subconnected spaces.*

3.2 Problem. *Find an inner characterization of the Hausdorff second countable Hausdorff subconnected spaces.*

3.3 Problem. *Is it true that a Hausdorff second countable space is T_2-subconnected iff it is non-H-closed?*

3.4 Problem. *Are there Luzin spaces which condense onto the unit segment?*

3.5 Problem. *Let X be a compact space of cardinality $\geq 2^\omega$. Is it true in ZFC that X contains an infinite Tychonoff subconnected subspace?*

3.6 Problem. *Let X be a countably compact space of cardinality $\geq 2^\omega$. Is it true in ZFC that X contains an infinite Tychonoff subconnected subspace?*

3.7 Problem. *Let X be a pseudocompact space of cardinality $\geq 2^\omega$. Is it true in ZFC that X contains an infinite Tychonoff subconnected subspace?*

3.8 Problem. *Is there in ZFC a Tychonoff space of cardinality $\geq 2^\omega$ with no infinite Tychonoff subconnected subspaces?*

3.9 Problem. *Let X be a Tychonoff subconnected space. Is it true, that the Markoff free topological group over X has a weaker connected group topology?*

3.10 Problem. *Is it true in ZFC that the discrete sum of ω_1 copies of the Cantor set condenses onto a connected compact space?*

3.11 Problem. *Is Tychonoff subconnectedness invariant with respect to perfect finite-to-one maps?*

3.12 Problem. *Is Hausdorff subconnectedness invariant with respect to perfect finite-to-one maps?*

References

[1] Alas O.T., Tkačenko M.G., Tkachuk V.V., Wilson R.G., *Connectifying some spaces*, Topology and its Applications, to appear.

[2] Arhangel'skii A.V., Ponomarev V.I., *General Topology in Problems and Exercises* (in Russian), Nauka Publishing House, Moscow, 1974.

[3] Arhangel'skii A.V., *The structure and classification of topological spaces and cardinal invariants* (in Russian), Uspehi Mat. Nauk **33.6** (1978), 29–84.

[4] Engelking R., *General Topology*, PWN, Warszawa, 1977.

[5] Graev M.I., *Theory of topological groups, I* (in Russian), Uspehi Mat. Nauk **5.2** (1950), 3–56.

[6] Miller A.W., *Special subsets of the real line*, in: Handbook of Set-Theoretic Topology (Eds. K. Kunen, J.E. Vaughan), Elsevier S.P. B.V., 1984, pp. 203-233.

[7] Rudin M.E., *Martin's axiom*, in: Handbook of Mathematical Logic, edited by J. Barwise, Amsterdam, North Holland P.C., 1977, pp. 491–501.

[8] Shapirovsky B.E., *On mappings onto Tychonoff cubes* (in Russian), Uspehi Mat. Nauk **35.3** (1980), 122–130.

[9] Tkačenko M.G., *Examples of connected left separated spaces and topological groups*, Acta Math., Acad. Sci. Hungar. **38.4** (1981), 257–261.

[10] Tkačenko M.G., *On group uniformities on the square of a space and extending pseudometrics*, Bull. Austral. Math. Soc. **51.2** (1995), 309–335.
[11] Watson S., Wilson R.G., *Embeddings in connected spaces*, Houston J. Math. **19.3** (1993), 469–481.

Tkačenko, Mikhail G.
Departamento de Matemáticas, Universidad Autónoma Metropolitana,
Av. Michoacán y La Purísima, Iztapalapa, A.P. 55-532, C.P. 09340, México, D.F.

E-mail: mich@xanum.uam.mx

Tkachuk, Vladimir V.
Departamento de Matemáticas, Universidad Autónoma Metropolitana,
Av. Michoacán y La Purísima, Iztapalapa, A.P. 55-532, C.P. 09340, México, D.F.

E-mail: vova@xanum.uam.mx

Uspenskii, Vladimir V.
International Institute for Mathematical Geophysics and Earthquakes Prediction Theory, Varshavskoye sh., 79, kor. 2, Moscow 113556, Russia

E-mail: vvu@uspensky.ras.ru

Wilson, Richard G.
Departamento de Matemáticas, Universidad Autónoma Metropolitana,
Av. Michoacán y La Purísima, Iztapalapa, A.P. 55-532, C.P. 09340, México, D.F.

E-mail: rgw@xanum.uam.mx

(*Received* October 6, 1995)