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On the gonality of curves in $\mathbf{P}^n$

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Abstract. Here we study the gonality of several projective curves which arise in a natural way (e.g. curves with maximal genus in $\mathbf{P}^n$, curves with given degree d and genus g for all possible d, g if $n = 3$ and with large g for arbitrary (d, g, n)).

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0. Introduction

Consider the following classical theorem of M. Noether (see [7] or [3] for references to modern complete proofs). A smooth complex degree $d \geq 5$ plane curve $C \subset \mathbf{P}^2$ has no g^1_x with $x < d-2$, every g^1_{d-1} is induced by the pencil of lines through a point $P \in C$ and the hyperplane line bundle $\boldsymbol{O}_C(1)$ is the unique g^2_d on C. This result was generalized in several directions (e.g. to singular plane curves ([7], [3]), to complete intersection curves, to 0-loci of sections of vector bundles, and so on). But it is still a very challenging problem to relate the Brill-Noether theory of special divisors on a curve C to the existence of a suitable embedding of C into $\mathbf{P}^n$. Here we consider this problem (essentially the existence and non existence of pencils) for very large classes of curves in $\mathbf{P}^n$. For the elementary background on special divisors, rational surfaces and projective curves, see [2], [10] and [11]. We just recall the notion of Clifford index $\mathrm{Cliff}\,(C)$ of a smooth complete curve C of genus ≥ 3; if $L \in \mathrm{Pic}\,(C)$, the Clifford index $\mathrm{Cliff}\,(L)$ of L is $\mathrm{Cliff}\,(L) := \deg(L) - 2(h^0(C, L)) + 2$; set $\mathrm{Cliff}\,(C) := \min\{\mathrm{Cliff}\,(L)$, with $L \in \mathrm{Pic}\,(C), h^0(C, L) \geq 2, h^1(C, L) \geq 2\}$. Here we will use two methods of proofs. One method is applied only once to prove Theorem 3.1. It is based on reducible curves and the study of limits of linear systems. Here the difficulty is that the reducible curve is not of "compact type". Here we obtain only lower bounds for $\mathrm{Cliff}\,(C)$, not the exact value. The other method will be used to prove all other results of this paper. This method is just the use of Bogomolov-Reider technique on a surface S to give conditions for the existence of linear series on a curve $C \subset S$ (see 1.2 and [3]). For all our results we will use several constructions of projective curves considered in the literature ([5], [10, Chapter III], [9], [12], [13], [14], [15]). In a few key points we will use also the proofs of the corresponding results in the literature (giving detailed explanations and quotations). Hence this paper is not self contained. For instance, one of the main results here is the following theorem.

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Theorem 0.1. *Fix integers $k \geq 2$, $h = 0, 1$ or 2 and set $n := 3k + h$. Let $d_0(n)$, $\phi(d,n)$, $\pi_k(d,n)$ be the functions defined in* [C2, § 1]*; fix integers d, g with $d > \max(d_0(n), 8k+17)$ and $\phi(d,n) < g \leq \pi_k(d,n)$. Then there are integers b and $b(i)$, $1 \leq i \leq 6-h$, with $b(i) \geq b(j)$ for $i \geq j$, $b(6-h) \geq 0$, $b > b(1)$ and $b \geq b(1)+b(2)+b(3)$, a smooth degree $4k-1+h$ surface $S_k \subset \mathbf{P}^n$, S_k blowing up of $\mathbf{P}^2$ at $6-h$ general points $P(1), \dots, P(6-h)$ and a smooth non degenerate curve $C \subset S_k$ with degree $d := (k+2)b - kb(1) - \sum_{2 \leq j \leq 6-h} b(j)$, genus $g := 1 + [b^2 - \sum_{1 \leq j \leq 6-h} b(j)^2 - d + (k-1)(b-b(1))]/2$ and such that C has no pencil of degree $< b-b(1)$ and the only degree $b-b(1)$ pencils on C are induced by the pencil of lines of $\mathbf{P}^2$ through one of the points $P(j)$ with $b(j) = b(1)$. Hence* $\mathrm{Cliff}(C) = b - b(1) - 2$. *In particular we have* $(n+6-h) \cdot (\mathrm{Cliff}(C)+2) \leq 3d$.

We will work always over an algebraically closed field of characteristic 0. For the positive characteristic case, see Remark 2.5.

1. Clifford index of general Castelnuovo curves

First we give a general remark on the Clifford index (see 1.1). Then we will explain Bogomolov-Reider technique (see 1.2). Then we will compute the Clifford index of general Castelnuovo curves, i.e. of curves $C \subset \mathbf{P}^n$ with maximal genus for fixed degree. Such curves are contained in a minimal degree surface $S \subset \mathbf{P}^n$ (see [10, Chapter III], or [2, Chapter II], or [5]). We study the gonality and the Clifford index of such curves (see 1.3 for the case with S a Veronese surface, 1.4 for the case n odd and S smooth, 1.7 for the case n even and S smooth, 1.10 for the case S singular). In 1.6 and 1.9 we will consider the case of nodal curves, because if one allows nodal curves in this way one recovers all possible geometric genera below the maximal arithmetic genus (see the introduction of [5]). We will use both the additive and the multiplicative conventions for line bundles and divisors on a smooth surface S and set $\boldsymbol{O} := \boldsymbol{O}_S$ and $\boldsymbol{I}_A := \boldsymbol{I}_{A,S}$ for every closed subscheme A of S. Unless otherwise stated, cohomology groups will be computed on S. Every time we compute or bound the Clifford index of a curve we will use without further mention the following remark.

Remark 1.1. By [8, Theorem 2.3], if $\mathrm{Cliff}(C)$ is not computed by a pencil, then C has infinitely many pencils of degree $\mathrm{Cliff}(C)+3$. Hence if there is an integer $a > 0$ such that C has only finitely many pencils of degree a and C has a pencil of degree a, then $\mathrm{Cliff}(C) = a - 2$.

From now on in this section we will consider the case of curves with few nodes on a rational minimal degree surface scroll S.

1.2. (Reider-Bogomolov construction (see [4, § 1])). Let S be a smooth projective surface, $L \in \mathrm{Pic}(S)$ and $T \subset X$ finite, $T \neq \emptyset$; set $j := \mathrm{card}(T)$. Assume $h^1(S, L \otimes \boldsymbol{I}_T) > h^1(S, L \otimes \boldsymbol{I}_{T'})$ for every $T' \subset T$ with $\mathrm{card}(T') = \mathrm{card}(T) - 1$ and

$L \cdot L > 4j$. Then there is a rank 2 vector bundle E on S and two exact sequences

$$0 \to \boldsymbol{O} \to E \to L \otimes \boldsymbol{I}_T \to 0 \tag{1}$$

$$0 \to L - D \to E \to D \otimes \boldsymbol{I}_W \to 0 \tag{2}$$

with D effective, W 0-dimensional or empty, a multiple of $L - 2D$ effective and $(L - 2D) \cdot H > 0$ for every ample line bundle H and hence

$$(L - 2D) \cdot M \geq 0 \text{ for every nef line bundle } M \tag{3}$$

and $j = (L - D)D + \text{length}(W)$ and in particular

$$\Phi := (L - D) \cdot D \leq j. \tag{4}$$

Furthermore, if $W \neq \emptyset$ we have

$$L \cdot D - j \leq D \cdot D < (L \cdot D)/2 < j. \tag{5}$$

Assume that $C \subset S$ is an integral nodal curve; set $A := \text{Sing}(T)$. Let Z be the 0-locus of a general section of a base point free line bundle on C. Set $L := \boldsymbol{O}(C)$. By adjunction theory there is $A' \subset A$ such that, setting $T := A' \cup Z$ and $t := \text{card}(T)$ then we are in the previous situation; hence if $L \cdot L > 4(\text{card}(A) + \text{card}(Z))$, the inequalities (3), (4) and (5) are satisfied.

Remark 1.3. Let $S \cong \mathbf{P}^2$ be the Veronese surface in $\mathbf{P}^5$. Let $C \subset S$ be a smooth curve of degree d (hence of degree $d/2$ as smooth plane curve). Then the Clifford index of C is $-3 + (d/2)$ and it is computed exactly by the pencil of lines through a point of C. If C is nodal and there are not too many nodes (see [7], or for nodes in general position [3]), then $\text{Cliff}(C) = -4 + (d/2)$ and the Clifford index is computed by the pencil of lines through a node.

Now we will consider the case $n = 2m + 1$ with m integer and with $S \cong F_e$ smooth. Since we are just interested in showing that the Clifford index is as small as possible for curves with that degree, arithmetic genus and number of nodes on any such smooth S, we will assume $e = 0$. First we will consider the case of a smooth curve.

Proposition 1.4. *Assume that C is smooth of type (a, b) with $2 \leq a \leq b$ and $b \leq 3$. Then C has no pencil of degree $< a$; if $a < b$, then the only pencil on C of degree a is induced by the family of lines of type $(0, 1)$; if $a = b$, then the only pencils on C of degree a are induced by one of the family of lines of S. Hence* $\text{Cliff}(C) = a - 2$.

Proof: Let $z \leq a$ be the degree of base point free pencil. By 1.2 there are nonnegative integers (u, v) with $D := (u, v)$, $L := (a, b)$ with $0 \leq 2u \leq a$, $0 \leq 2v \leq b$ and satisfying

$$\Phi := u(b - v) + v(a - u) \leq z \leq a. \tag{6}$$

Note that if $u > 0$ we have $\Phi \geq b - v + va/2 > a$. Hence $u = 0$. Thus $v = 1$, as wanted. □

Now we will consider the case of a nodal curve. We will use the following lemma.

Lemma 1.5. *Fix integers a, b, j with $j \geq 0$, $5 \leq a \leq b$ and $3j < (a+1)(b+1)$. Fix $J \subset S$ with $\operatorname{card}(J) = j$ and J general. Then for every $A \subset S$ with $z := \operatorname{card}(A) \leq a$ we have $h^1(\boldsymbol{I}_{J\cup A}(a-2, b-2)) \neq 0$ if and only if A is contained in a line of type $(0,1)$ or (if $b = a$) in a line of type $(1,0)$.*

Proof: By adjunction theory and 1.2 the case $j = 0$ is proved as Proposition 1.4. Hence we may assume $j > 0$ and take j and z minimal for which the thesis fails. Thus we may assume $h^1(\boldsymbol{I}_{J\cup A}(a-2, b-2)) = 1$ and $h^1(\boldsymbol{I}_{J'\cup A}(a-2, b-2)) = 0$ for every $J' \subset J$, $J' \neq J$. Thus again we may apply 1.2 and obtain the following inequality:

$$\Phi := u(b-v) + v(a-u) \leq j + z \leq j + a. \tag{7}$$

Since J is general and the effective divisor D given by 1.2 contains $J \cup A$, we have $uv + u + v \geq j$. Assume by contradiction $u > 0$ and if $v = 0$, $b > a$. The reader may check easily first the case $b \geq 2v + 2$, $v \geq 2$, then the case $b \geq 5$, $v = 1$ and then the case $b = 2v + 1$. Now assume $b = 2v$. By (7) we find a contradiction if $2j < a(b-2)$. Since $3j < (a+1)(b+1)$ and $b \geq a \geq 5$, we conclude. □

Proposition 1.6. *Fix integers a, b, t with $t \geq 0$, $5 \leq a \leq b$ and $3t < (a+1)(b+1)$. Let C' be an integral nodal curve with $\operatorname{Sing}(C')$ formed by t general points of S. Let C be the normalization of C'. Then C has no pencil of degree $< a$; if $a < b$, then the only pencil on C of degree a is induced by the family of lines of type $(0,1)$; if $a = b$, then the only pencils on C of degree a are induced by one of the family of lines of S. Hence $\operatorname{Cliff}(C) = a - 2$.*

Proof: Note that since $3t < (a+1)(b+1)$ and $b \geq a \geq 5$ there is such an integral curve C' ([1, Proposition 4.1]). Then apply Lemma 1.5. □

Now we will consider the case $n = 2m$ with $m \geq 2$ integer and with $S \cong F_e$ smooth. Since we are just interested in showing that the Clifford index is as small as possible for curves with that degree, arithmetic genus and number of nodes on any such smooth S, we will assume $e = 1$. Take as base of $\operatorname{Pic}(S)$ the effective divisors h, f with $h^2 = -1$, $h \cdot f = 1$, $f^2 = 0$. First we consider the case of a smooth C.

Proposition 1.7. *Let C be a smooth curve of type $ah + bf$ (or (a,b) for short) on S with $2 \leq a \leq b$ and $b \geq 3$. Then C has no pencil of degree $< a$ and the only pencil of degree a is induced by the ruling of S. In particular $\operatorname{Cliff}(C) = a - 2$.*

Proof: By 1.7 we have integers (u,v) with $D := (u,v)$, $L := (a,b)$ with $0 \leq 2u \leq a$, $0 \leq 2v \leq b$, $u \leq v$, either $b = 2v$ or $a - 2u \leq b - 2v$ and such that:

$$\Phi := u(b-v) + v(a-u) - u(a-u) \leq z \leq a. \tag{8}$$

Assume by contradiction $u > 0$. Since $v \geq u$ we have $\Phi \geq a - u + ub/2$. Hence $u = 0$. Thus $v = 1$, as wanted. □

Now we will consider the case of a nodal curve.

Lemma 1.8. *Fix integers a, b, j with $j \geq 0$, $5 \leq a \leq b$ and $3j < (a+1)(2b+2-a)/2$. Fix $J \subset S$ with $\mathrm{card}(J) = j$ and J general. Then for every $A \subset S$ with $z := \mathrm{card}(A) \leq a$ we have $h^1(\_J \cup A(a-2, b-3)) \neq 0$ if and only if A is contained in a line of the ruling of S.*

PROOF: By 1.2 the case $j = 0$ is proved as Proposition 1.7. Hence we may assume $j > 0$ and take j and z minimal for which the thesis fails. Thus we may assume $h^1(\boldsymbol{I}_{J\cup A}(a-2, b-3)) = 1$ and $h^1 l(\boldsymbol{I}_{J'\cup A}(a-2, b-3)) = 0$ for every $J' \subset J$, $J' \neq J$. Thus again we may apply 1.2 and obtain the following inequality:

$$\Phi := u(b-v) + v(a-u) - u(a-u) \leq j + z \leq j + a. \tag{9}$$

Since J is general and the effective divisor D given by 1.2 contains $J \cup A$, we have $(u+1)(2v+2-u)/2 > j$. Thus it is sufficient to check that if $v \geq u > 0$ we have

$$2u(b-v) + 2v(a-u) - 2u(a-u) \geq (u+1)(2v+2-u) - 1 + a \tag{10}$$

with the constraint that $(a-2u, b-2v)$ is effective. To check (10) we may take b as small as possible with this constraint and the inequality $a \leq b$. First assume $b = a$. Hence (10) is

$$2va - 4uv + 3u^2 \geq 2v + u + 1 + a \tag{11}$$

and for $v \geq 2$ to check (11) with $a \geq 2u$ we reduce to the case $a = 2u$, $u = v$ which is true because $a \geq 4$. Now assume $b > a$. Hence by the minimality of b we have either $a - 2u = b - 2v$ or $a - 2u = b - 2v - 1$. First assume $b = a + 2v - 2u$ (hence $v > u$). Thus (10) is equivalent to

$$2av - u^2 - 2uv \geq 2v - u + 1 + a. \tag{12}$$

To check (12) we reduce to the case $v = u + 1$ and then conclude. Now assume $a - 2u = b - 2v - 1$. Now it is sufficient to check that

$$2av - u^2 - 2uv + 2u \geq 2v - u + 1 + a. \tag{13}$$

To check (13) we reduce to the case $u = v$ and then we conclude. □

Proposition 1.9. *Fix integers a, b, t with $t \geq 0$, $3t < (a+1)(2b+2-a)/2$. Let $T \subset S$ with $\mathrm{card}(T) = t$ and T general. Let C' be an integral nodal curve with $\mathrm{Sing}(C') = T$. Let C be the normalization of C'. Then C has no pencil of degree $< a$ and the only degree a pencil on C is induced by the ruling of S.*

PROOF: Note that since $3t < (a+1)(2b+2-a)/2$ and $b \geq a \geq 5$ there is such an integral curve C' ([1, Proposition 4.1]). Then apply Lemma 1.8.

Now we consider the case of a smooth curve C' on a minimal degree cone surface $S' \subset \mathbf{P}^n$, $n \geq 3$. Hence S' is a cone over a smooth rational normal curve of $\mathbf{P}^{n-1}$

and it has as minimal resolution $S \to S'$ with $S \cong F_{n-1}$, with h as exceptional divisor and with $|h + (n-1)f|$ inducing the morphism $S \to \mathbf{P}^n$. Let $C \cong C'$ be the strict transform in S of C'. Since C' is smooth and not a line, there is an integer $a > 0$ and an integer $q = 0$ or 1 with $C' \in |ah + (a(n-1)+q)f|$. We want to prove that Cliff $(C) = a-2$ and that C has a unique pencil of degree a, the one induced by the ruling of S and S'. Now the function Φ is $\Phi = u(a-u)(n-1)+u$ if $q = 0$, $\Phi = u(a-u)(n-1)+a-u$ if $q = 1$, with $0 \leq 2u \leq a$. Hence we conclude. □

2. Proof of Theorem 0.1

In this section we will prove Theorem 0.1, and Propositions 2.2, 2.3 and 2.4 below. We stress that [9] was the key paper in which the authors introduced the ideas and methods for the construction of curves with invariants (d, g) used in [12], [14] and [6]. In this section we will fix the following notations. Let $S = S(h)$ be a smooth rational surface which is the blowing up of $\mathbf{P}^2$ at $6-h$ general points $P(i)$, $1 \leq i \leq 6-h$. We will take as basis of Pic$(S) \cong \mathbf{Z}^{7-h}$ the total transform of a line of the plane and the opposite of the exceptional divisors $E(i)$. Thus every curve or line bundle will be given as $(b; b(1), \ldots, b(6-h))$ with b and $b(i)$ integers. We may even assume $b(i) \geq b(j)$ if $i \geq j$. We know ([11, V.4.12]) the ample cone of S. We know the class of the rigid divisors corresponding to the 27 lines of the cubic $S(0)$ and that if $h \leq 1$, $(u; u(1), \ldots, u(6-h))$ is effective but not in this class with $u(i) \geq u(j)$ for $i \geq j$, then $u(6-h) \geq 0$, $u \geq u(1)$ and $2u \geq u(1)+u(2)+u(3)+u(4)+u(5)$ (see also [11, Example V.4.8]). However, in [6] and [14] for the existence of smooth integral curves in the class $(b; b(1), \ldots, b(6-h))$ with $b(i) \geq b(j)$ for $i \geq j$ it was used the stronger sufficient condition

$$b \geq b(1) + b(2) + b(3) \tag{14}$$

(see [12, p. 303], [6, eq. (1.4), i.e. the inequality $\alpha_3 \geq \alpha_2$ in the first line of eq. (1.8)]). S is embedded as a degree $4k-1+h$ surface S_k of $\mathbf{P}^{3k+h}$ by the complete linear system associated to $(k+2; k, 1, \ldots, 1)$. As in [B] we will use Reider-Lazarsfeld method explained in 1.2. Fix $(b; b(1), \ldots, b(6-h))$ normalized (i.e. ordered with $b(i) \geq b(j)$ for $i \geq j$) and corresponding to the complete family V of smooth integral curves of degree d and genus g (hence satisfying also (14)). Fix a general pairs (C, Z) with $C \in V$, $Z \subset C$, $y := \mathrm{card}(Z) \leq b - b(1)$, Z moving in a g^1_y on C. By 1.2 there are integers x, $x(j)$, $1 \leq j \leq 6-h$, with $x > 0$, $x(j) \geq 0$ for all j, $x \geq x(i) + x(j)$ for all pairs $\{i, j\}$ and such that both $(x; x(1), \ldots, x(6-h))$ and $(b-2x; b(1)-2x(1), \ldots, b(6-h)-2x(6-h))$ are effective, with $(x; x(1), \ldots, x(6-h))$ corresponding to an effective divisor D containing Z. Now the inequality (4) is

$$\Phi := x(b-x) - \sum_{1 \leq j \leq 6-h} x(j)(b(j) - x(j)) \leq y \leq b - b(1) \tag{15}$$

$$\text{with } \ 0 \leq 2x \leq b \ \text{ and } \ x \geq x(i) + x(j) \ \text{ for all } \ i \neq j. \tag{16}$$

Set $\Phi'' : \Phi - (b - b(1))$. If $x = 0$, we find that D is rigid, while $Z \subset D$ moves covering C and C moves in S. If $x = 1$ we conclude easily. Assume by contradiction $x \geq 2$. Φ and Φ'' do not change if we substitute $x(i)$ with $b(i) - x(i)$. However, making this substitution all other constraints are satisfied if $x(i) > b(i) - x(i)$. Hence we will assume $2x(i) \leq b(i)$ for all i. Now note that for fixed x the functions Φ and Φ'' may only decrease if we permute the integers $x(u)$ in such a way that $x(i) \geq x(j)$ if $i \geq j$. Note that when $x(2) > 0$ if instead of $x(1)$, $x(2)$ we take $x(1)+1$, $x(2)-1$ and then normalize the new $6-h$ integers, Φ and Φ'' may only decrease. Hence we may assume $x(1)$ as large as possible with the restriction $2x(1) \leq b(1)$. Then note that after this normalization Φ and Φ'' may only decrease if we decrease x; hence we may assume $x = x(1)+x(2)$. Thus we reduce to the following 3 cases.

First case: $x = x(1)$ and $x(j) = 0$ for $j > 1$. Hence $\Phi = x(b - b(1)) \geq 2(b - b(1))$, a contradiction.

Second case: $x(j) = 0$ for $j > 2$, $x(1) = [b(1)/2]$ and $x = x(1) + x(2)$. Hence $\Phi = [b(1)/2](b - b(1)) + x(2)(b - b(2)) > b - b(1)$, a contradiction.

Third case: $x(i) = [b(i)/2]$, $x(2) = [b(2)/2]$, $x(j) = \min\{x(3), [b(j)/2]\}$ if $j > 3$ and $x = x(1) + x(2)$. Taking the derivatives, we see that Φ'' may only decrease if we decrease b taking fixed the other integers. Hence by (14) we reduce to the case $b = b(1) + b(2) + b(3)$ and find $\Phi'' > 0$ (hence a contradiction) if $b(1) \geq 5$ or $b(2) \geq 4$. Assume $b(1) = 4$ and $b(2) = 3$. We may assume $x(1) = 2$, $x(i) = 1$ for $i > 1$, $x = x(1) + x(2) = 3$, $b = b(1) + b(2) + b(3)$ and $b(j) = b(3)$ for $j > 3$. If $b(3) = 3$, we have $\Phi'' = 1 + 2h$, a contradiction. If $b(3) = 2$ we have $\Phi'' = 3 + h$. Now assume $b(1) = 4$ and $b(2) \leq 2$. We reduce to the case $b(j) = 2$ for all $j \geq 2$, $x(1) = 2$, $x(j) = 1$ for $j > 1$, $x = 3$, $b = 8$ and find $\Phi'' = 6 + h$, a contradiction. Now assume $b(1) \leq 3$ (hence $b \leq 9$). Note that $d < (k+2)b - kb(1)$. Hence if we are forced to reduce to a case with $b \leq 9$ to find a positive lower bound for Φ'', then $d < 8k + 17$, a contradiction. □

Remark 2.1. The same proof for $k = 1$ and $h = 0, 1, 2$ (using respectively [12, Theorem 0.2] (or [9]), [14, Theorem 1.0.1], and [14, Theorem 2.1.2] instead of [7]) gives respectively the following results 2.2, 2.3 and 2.4.

Proposition 2.2. *For all integers d, g with $d \geq 20$ and*

$$(1/3)^{1/2} \cdot d^{3/2} - d + 1 \leq g \leq d(d-3)/6 + 1,$$

there are integers b, $b(j)$, $1 \leq j \leq 6$, with $b(i) \geq b(j)$ if $i \geq j$, $b(6) \geq 0$, $b \geq b(1) + b(2) + b(3)$, and a smooth connected curve C on a smooth cubic surface $S \subset \mathbf{P}^3$, S blowing up of $\mathbf{P}^2$ at 6 general points $P(j)$ $(1 \leq j \leq 5)$ with degree $d := 3b - \sum_{1 \leq j \leq 6} b(j)$, genus $g := 1 + [b2 - \sum_{1 \leq j \leq 6} b(j)2 - d]/2$ and such that C has no pencil of degree $< b - b(1)$ and the only degree $b - b(1)$ pencils on C are induced by the pencil of lines of $\mathbf{P}^2$ through one of the points $P(j)$ with $b(j) = b(1)$. Hence $\mathrm{Cliff}(C) = b - b(1) - 2$.

Proposition 2.3. *For all integers d, g with $d \geq 25$ and*

$$(d+12)(d+9)^{1/2} - (11/2)d - 35 \leq g \leq (1/8)d^2 - d/2 + 1,$$

there are integers b, $b(j)$, $1 \leq j \leq 5$, with $b(i) \geq b(j)$ if $i \geq j$, $b(5) \geq 0$, $b \geq b(1) + b(2) + b(3)$, and a smooth connected curve C on a smooth Del Pezzo surface $S \subset \mathbf{P}^4$, S blowing up of $\mathbf{P}^2$ at 5 general points $P(j)$ $(1 \leq j \leq 6)$ with degree $d := 3b - \sum_{1 \leq j \leq 5} b(j)$, genus $g := 1 + [b^2 - \sum_{1 \leq j \leq 5} b(j)^2 - d]/2$ and such that C has no pencil of degree $< b - b(1)$ and the only degree $b - b(1)$ pencils on C are induced by the pencil of lines of $\mathbf{P}^2$ through one of the points $P(j)$ with $b(j) = b(1)$. Hence Cliff $(C) = b - b(1) - 2$.

Proposition 2.4. *For all integers d, g with $d \geq 35$ and*

$$(d+30)(2d+40)^{1/2} - (23/2)d - 189 \leq g \leq (1/10)d^2 - d/2 + 1,$$

there are integers b, $b(j)$, $1 \leq j \leq 5$, with $b(i) \geq b(j)$ if $i \geq j$, $b(5) \geq 0$, $b \geq b(1) + b(2) + b(3)$, and a smooth connected curve C on a smooth Del Pezzo surface $S \subset \mathbf{P}^5$, S blowing up of $\mathbf{P}^2$ at 4 general points $P(j)$, $(1 \leq j \leq 4)$ with $\deg = d := 3b - \sum_{1 \leq j \leq 4} b(j)$, *genus $g := 1 + [b^2 - \sum_{1 \leq j \leq 4} b(j)^2 - d]/2$ and such that C has no pencil of degree $< b - b(1)$ and the only degree $b - b(1)$ pencils on C are induced by the pencil of lines of $\mathbf{P}^2$ through one of the points $P(j)$ with $b(j) = b(1)$. Hence* Cliff $(C) = b - b(1) - 2$.

Remark 2.5. Assume that the base field has characteristic $p > 0$. Since we used Reider-Bogomolov technique 1.2 on a smooth rational surface, this part works by [15, Theorem 7]. In [12, § 5] it was proved that a certain construction due to Gruson and Peskine of smooth curves on a smooth cubic surface or on a non normal quartic surface works also in positive characteristic. The same proof (see in particular [12, 5.1], to check that $(k+2; k, 1, \dots 1)$ is very ample) works for the construction given in [7, § 1], of a smooth curve with invariants (d, g) on the surface S considered in Theorem 0.1. The results in [14] were explicitly stated and proved in positive characteristic. Hence Theorem 0.1 and Propositions 2.2, 2.3 and 2.4 are true also in positive characteristic.

3. Space curves on a smooth quartic surface

In this section (using heavily [13]) we will prove the following result.

Theorem 3.1. *Assume characteristic* 0. *Fix integers d, g with $d \geq 6$ and $d - 2 \leq g < (d^2/8)$. There is an integer $s \geq 1$ and an integer $g(0)$ with $0 \leq g(0) \leq d - 4s - 3$ and such that $g = g(0) + sd + s(2s+3)$, a smooth quartic surface $S \subset \mathbf{P}^3$ and a smooth connected curve $C \subset S$ with* $\deg(C) = d$, $p_a(C) = g$ *and* Cliff $(C) \geq 3s$.

Proof: The proof is divided into 5 steps.

Step 1. For this range of integers (d, g) S. Mori in [13] proved the existence of a smooth quartic surface X and a smooth connected curve $Y \subset X$ with $\deg(Y) =$

d and $p_a(Y) = g$. Recall that the proof in [13] was by induction on d from the case of a curve $C' \subset S$, $\deg(S) = 4$, with $\deg(C') := d' := d - 4$ to a curve of degree d which is a flat specialization of a reducible nodal curve $C'' = C' \cup A$ with $A := S \cap M$, M plane, and $C' \cap A = C' \cap M$. Hence $\mathrm{card}(C' \cap A) = d'$ and $g(C'') = g(C') + d' - 1$. To prove 3.1 we have to check that for a general smoothing, C, of C'' we have $\mathrm{Cliff}(C) \geq \mathrm{Cliff}(C') + 3$. We fix a general 1-parameter smoothing of C'' inside a quartic surface and take a line bundle on the complement of the special fiber C'' and which computes the Clifford index of the general smooth fiber. Since the total space of the deformation is smooth, this line bundle extends (not uniquely) to a line bundle on the total space. Since $\mathrm{card}(C' \cap A) = d'$, this extension is uniquely determined if we impose that its restriction, R, to C'' has $-d' + 2 \leq \deg(R\,|\,A) \leq 3$. By semicontinuity we may assume $h^0(C'', R) \geq 2$.

Step 2. First assume $h^0(A, R\,|\,A) \leq 1$ and $h^0(C', R\,|\,C') \geq 2$. Thus we may find a non trivial section of $R\,|\,C'$ vanishing on d' points of C'. Varying the plane M (or just using that d' is at least the lower bound we want for the Clifford index for (d, g)), we find a contradiction.

Step 3. By Step 2 we may assume $h^0(A, R\,|\,A) \geq 2$. Hence by the adjunction formula we have $\deg(R\,|\,A) = 3$ and there is $P \in A$ with $R\,|\,A \cong \boldsymbol{O}_A(1)(-P)$, $R\,|\,A$ induced by the pencil of lines in M through P.

Step 4. However (and this is a key point) to prove 3.1 we may assume the existence of a line bundle R' on C' of degree $\deg(R\,|\,C')$ on C' with $h^0(C', R') = 2$ and with a non trivial section s vanishing at 3 non collinear points $P(1)$, $P(2)$ and $P(3)$. If we take as M the plane spanned by $P(1)$, $P(2)$ and $P(3)$, we find easily that in general $M \cap S'$ will be smooth and intersecting transversally C'. Thus we may assume $R' = R\,|\,C'$. But now the condition that $P(1)$, $P(2)$, $P(3)$ and P are not collinear (which follows from the non collinearity of $P(1)$, $P(2)$ and $P(3)$) implies that s does not satisfy the gluing condition at $P(1)$, $P(2)$ and $P(3)$ needed to lift it to a section of $R\,|\,C''$. This implies $h^0(C'', R\,|\,C'') \leq 1$, a contradiction.

Step 5. The starting point of the induction in [13] was an integral curve of degree $d(0)$ and genus $g(0)$ with $0 \leq g(0) \leq d(0) - 3$. Hence if we arrive inductively at the pair of invariants (d, s) in s steps, we have $d = d(0) + 4s$ and by Step 4 we have $\mathrm{Cliff}(C) \geq 3s$. □

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