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On the WM property of Orlicz sequence spaces endowed with the Orlicz norm

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Abstract. We obtain the criterion of the WM property for Orlicz sequence spaces endowed with the Orlicz norm.

Keywords: Orlicz space, Orlicz norm, WM property

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B.B. Panda and O.P. Kapoor [1] introduced the concept of the WM property in 1975. The WM property is an important property in geometry of Banach spaces. Some criteria of WM properties for Orlicz function spaces endowed with the Luxemburg norm and Orlicz norm have been discussed in [2] and [4], respectively. Moreover, the criterion of the WM property in Orlicz sequence spaces endowed with the Luxemburg norm has also been discussed in [3]. A remained problem is the WM property of Orlicz sequence spaces endowed with the Orlicz norm. In this paper, we shall give the criterion of the WM property in Orlicz sequence spaces equipped with the Orlicz norm.

Let X be a Banach space, and let $B(X)$ and $S(X)$ denote the unit ball and unit sphere of X, respectively. X is said to have the *WM property* if for any $x \in S(X)$, $x_n \in B(X)$ $(n \in N)$, $\|x_n + x\| \to 2$ implies that there exists a support functional f at x, $f(x_n) \to 1$. It is known that f is said to be a *support functional* at $x \in S(X)$, if $f(x) = \|f\| = 1$.

Let $M(u)$ and $N(v)$ denote a pair of complementary N-functions, $P_-(u)$ and $P(u)$ denote the left and right derivates of $M(u)$, respectively. We say that $[a,b]$ $(a < b)$ is an *affine segment of* $M(u)$, if $M(u)$ is affine on $[a,b]$, but neither affine on $[a-\varepsilon, b]$ nor on $[a, b+\varepsilon]$ for all $\varepsilon > 0$. a and b are called the left and right end points of $[a,b]$, respectively. It is known that $M(u)$ has at most countable number of affine segments $[a_i, b_i]$ $(i = 1, 2, \cdots)$. For convenience, we denote $S_M^0 = R \setminus \bigcup_{i=1}^{\infty}[a_i, b_i]$. We call that an affine segment $[a,b]$ of $M(u)$ is *regular*, if both a and b are points of continuity of $P(u)$. If $[a,b]$ and $[b,c]$ $(a < b < c)$ are both affine segments of $M(u)$, then we call that they are *neighbour affine segments of* $M(u)$. $M(u)$ is said to satisfy the Δ_2-condition $(M \in \Delta_2)$, if there exist K, $u_0 > 0$ such that $M(2u) \le KM(u)$ for $0 \le u \le u_0$. We denote the modular of a sequence $x = \{x(i)\}_{i=1}^{\infty}$ by $\varrho_M(x) = \sum_{i=1}^{\infty} M(x(i))$. It is well known that the space

$$l_M = \{x = \{x(i)\}_{i=1}^{\infty} : \text{for some } \lambda > 0, \varrho_M(\lambda x) = \sum_{i=1}^{\infty} M(\lambda x(i))) < \infty\}$$

endowed with the Orlicz norm

$$\|x\|^0 = \inf_{k>0} (1 + \varrho_M(kx))/k = \sup\{\sum_{i=1}^{\infty} x(i)y(i) : \varrho_N(y) \le 1\},$$

or with the Luxemburg norm

$$\|x\| = \inf\{c > 0 : \varrho_M(x/c) \le 1\}$$

is a Banach sequence space which is denoted by l_M^0, l_M respectively. We know that (cf. [5]) for any $x \ne 0$, $\|x\|^0 = (1 + \varrho_M(kx))/k$ if $k \in K(x) = [k_x^*, k_x^{**}]$, where $k_x^* = \inf\{k > 0 : \varrho_N(P(kx)) \ge 1\}$, $k_x^{**} = \sup\{k > 0 : \varrho_N(P(kx)) \le 1\}$.

Lemma 1. *If $x \in S(l_M^0)$, then $v \in l_N$ is a support functional at x if and only if for any (or some) $k \in K(x)$*

(i) $\varrho_N(v) = 1$,
(ii) $x(i)y(i) \ge 0$ *and* $P_-(k|x(i)|) \le |v(i)| \le P(k|x(i)|)$.

Proof: See [6]. □

Lemma 2. *If $M(u)$ does not satisfy the Δ_2-condition ($M \notin \Delta_2$), then there exists $x \in S(l_M^0)$ having no support functional in l_N.*

Proof: Since $M \notin \Delta_2$ is equivalent to $P(u) \notin \Delta_2$ (cf. [5]), there exists $u_i \downarrow 0$ $(i \to \infty)$ such that

$$P((1 + 1/i)u_i) \ge 2^{i+1} P(u_i),\ u_i P(u_i) < 1/2^i.$$

Take natural number k_i satisfying

$$1/2^{i+1} \le k_i u_i P(u_i) < 1/2^i.$$

Let

$$x = (u_1, \cdots, u_1, \cdots, u_i, \cdots, u_i, \cdots),$$

where u_i is taken k_i times and let $x' = x/\|x\|^0$. We have

$$\varrho_N(P(\|x\|^0 x')) = \sum_{i=1}^{\infty} k_i N(P(u_i)) < \sum_{i=1}^{\infty} k_i u_i P(u_i) < 1$$

and for any $\varepsilon > 0$,

$$\begin{aligned}
\varrho_N(P(\|x\|^0(1+\varepsilon)x')) &= \varrho_N(P((1+\varepsilon)x)) \\
&\geq \sum_{i=1}^{\infty} k_i N(P((1+\varepsilon)u_i)) \\
&\geq \sum_{i>1/\varepsilon} k_i N(P((1+1/i)u_i)) \\
&> \sum_{i>1/\varepsilon}^{\infty} k_i \{u_i P((1+1/i)u_i) - M(u_i)\} \\
&= \sum_{i>1/\varepsilon}^{\infty} k_i u_i P((1+1/i)u_i) - \varrho_M(x) = \infty.
\end{aligned}$$

So, we know that $k_{x'} = \|x\|^0$, $\varrho_N(P(k_{x'}x')) < 1$. By Lemma 1, it follows that any support functional at x' is not in l_N. $\square$

Lemma 3. *Suppose that $M \in \Delta_2$. Let $\chi_x = \{x' : \|x+x'\|^0 = 2, \|x'\|^0 = 1\}$, then for any $x \in S(l_M^0)$, all of the elements in χ_x have a common support functional if and only if for any affine segment $[a,b]$ of $M(u)$, the following conditions hold:*

(i) *$N(P(a)) < 1/2$ implies b is a point of continuity of $P(u)$,*

(ii) *$N(P(a)) + N(P_-(a)) \leq 1$ implies a is a point of continuity of $P(u)$.*

(iii) *$N(P(a)) \geq 1/2$, $N(P(b)) < 1$ and b is a left end point of an affine segment of $M(u)$ implies that for any $\{u_i\} \subset S_M^0$, $N(P(b)) + \sum_{i=1}^{\infty} N(P_-(u_i)) \leq 1$, we have $\sum_{i=1}^{\infty}\{N(P(u_i)) - N(P_-(u_i))\} < N(P(b)) - N(P_-(b))$.*

Remark of Lemma 3. Despite the conditions (i)–(iii) are complicated, they are very weak and implied by the following alternative conditions: (i) $P(u)$ is continuous at all of the left and right end points of the affine segments of $M(u)$ contained in $[0, Q_-(N^{-1}(1))]$; (ii) $P(u)$ is strictly monotone on $[0, Q_-(N^{-1}(1))]$, where $Q_-(v)$ is the left derivate of $N(v)$.

Proof of Lemma 3: *Sufficiency.* By the definition of the Orlicz norm and the convexity of $M(u)$, we have

$$\begin{aligned}
0 &= \|x\|^0 + \|x'\|^0 - \|x+x'\|^0 \\
&\geq \frac{1+\varrho_M(kx)}{k} + \frac{1+\varrho_M(k'x')}{k'} \\
&\quad - \frac{kk'}{k+k'}(1+\varrho_M(\frac{kk'}{k+k'}(x+x'))) \geq 0.
\end{aligned}$$

Thus,

$$\frac{k}{k+k'}M(k'x'(i)) + \frac{k'}{k+k'}M(kx(i)) = M(\frac{kk'}{k+k'}(x(i)+x'(i)))$$

for all $i \in N$. Hence, for any $x' \in \chi_x$, $k'x'(i) = kx(i)$ if $kx(i) \in S_M^0$; $k'x'(i) \in [a,b]$ if $kx(i)$ belongs to an affine segment $[a,b]$ of $M(u)$, where $k \in K(x)$, $k' \in K(x')$. Without loss of generality, we can assume that $x(i) \geq 0$. Then $x'(i) \geq 0$ for any $x' \in \chi_x$. We consider the set of natural numbers
$J = \{i : kx(i) \in [a_i, b_i], [a_i, b_i]$ is an affine segment of $M(u)\}$.

If $J = \phi$, then for any $x' \in \chi_x$, $k'x'(i) = kx(i)$. Obviously, any support functional at x is a common support functional for all elements in χ_x.

If $J \neq \phi$, suppose without loss of generality, that $kx(1) = \max_{i \in J} kx(i)$, and $kx(1)$ belongs to an affine segment $[a,b]$ of $M(u)$.

Step 1. At first we will prove the following fact: If $kx(i) \in S_M^0$ or belongs to a regular affine segment of $M(u)$ for all $i \geq 2$, then all of the elements in χ_x have a common support functional $v \in l_N$. We will consider the following three cases.

Case I. If $kx(1) \in (a,b)$, then for any support functional v at x, $v(1) = P(a)$. So, v is the common support functional of χ_x.

Case IIa. If $kx(1) = a$ and $\sum_{i=2}^{\infty} N(P_-(kx(i))) + N(P(a)) \leq 1$. Since $1 \leq \varrho_N(P(kx)) = \sum_{i=2}^{\infty} N(P(kx(i))) + N(P(a))$, then there exists $v \in l_N$, $v(1) = P(a)$, $P_-(kx(i)) \leq v(i) \leq P(kx(i))$ as $i \geq 2$, $\varrho_N(v) = 1$. By Lemma 1, v is the common support functional of χ_x.

Case IIb. If $kx(1) = a$ and $\sum_{i=2}^{\infty} N(P_-(kx(i))) + N(P(a)) > 1$. In this case we have $k'x'(1) = a$ for all $x' \in \chi_x$. If not, there exists $x' \in \chi_x$, $k'x'(1) > a$. Then

$$1 \geq \varrho_N(P_-(k'x')) = \sum_{i=2}^{\infty} N(P_-(kx(i))) + N(P(a)) > 1.$$

A contradiction. Hence, by Lemma 1, any support functional at x is a common support functional of χ_x.

Case IIIa. $kx(1) = b$ and b is not the left end point of any affine segments of $M(u)$. If b is a point of continuity of $P(u)$ or $\sum_{i=2}^{\infty} N(P(kx(i))) + N(P_-(b)) \geq 1$, then there exists $v \in l_N$, $v(1) = P_-(b)$, $P_-(kx(i)) \leq v(i) \leq P(kx(i))$ as $i \geq 2$. $\varrho_N(v) = 1$. Hence, by Lemma 1, v is the common support functional of χ_x. If b is not a point of continuity of $P(u)$ and $\sum_{i=2}^{\infty} N(P(kx(i))) + N(P_-(b)) < 1$, then there exists $v \in l_N$, $P(a) < v(1) \leq P(b)$, $v(i) = P(kx(i))$ as $i \geq 2$, $\varrho_N(v) = 1$. We prove that v is the common support functional of χ_x. In fact, if $x' \in \chi_x$, then $k'x'(1) = b$. If not, $k'x'(1) \neq b$. Since b is not a left end point of any affine segment of $M(u)$, we have $k'x'(1) < b$. Thus

$$\begin{aligned} 1 \leq \varrho_N(P(k'x')) &= \sum_{i=2}^{\infty} N(P(kx(i))) + N(P(a)) \\ &= \varrho_N(v) - (N(v(1)) - N(P(a))) < 1. \end{aligned}$$

A contradiction. Hence, by Lemma 1, v is the common support functional of χ_x.

Case IIIb. $kx(1) = b$ and b is a left end point of an affine segment $[b, c]$ $(b < c)$. In this case we have $N(P_-(b)) \geq 1/2$. If not, $N(P_-(b)) < 1/2$, i.e. $N(P(a)) < 1/2$, this implies b is a point of continuity of $P(u)$ by the condition (i). A contradiction. Now we prove that $k'x'(1) \geq b$ for all $x' \in \chi_x$; or $k'x'(1) \leq b$ for all $x' \in \chi_x$. If not, there exist $x', x'' \in \chi_x$, $k'x'(1) < b$, $k''x''(1) > b$. Then,

$$1 \geq \varrho_N(P_-(k''x'')) = \sum_{i=2}^{\infty} N(P_-(kx(i))) + N(P(b)).$$

Note $k'x'(i)$ belongs to S_M^0 or to a regular affine segment of $M(u)$ for all $i \geq 2$. By the condition (iii), we have

$$\begin{aligned}\varrho_N(P(k'x')) &= \sum_{i=2}^{\infty} N(P(kx(i))) + N(P_-(b)) \\ &< \sum_{i=2}^{\infty} N(P_-(kx(i))) + N(P(b)) \leq 1.\end{aligned}$$

So x' has no support functional in l_N, this contradicts $M \in \Delta_2$.

If $k'x'(1) \leq b$ for all $x' \in \chi_x$. Similarly to the case IIIa, one can easily verify that there exists a common support functional of χ_x.

If $k'x'(1) \geq b$ for all $x' \in \chi_x$. Similarly to the cases IIa and IIb, one can easily verify that χ_x has a common support functional.

Step 2. We prove the sufficiency. We will consider the following three cases.

I. $N(P(a)) < 1/2$. In this case, we have $N(P(a)) + N(P_-(a)) < 1$. By the conditions (i) and (ii), we obtain that $[a, b]$ is a regular affine segment.

If there exists some i such that $kx(i) \in [a_i, b_i]$. Note that $kx(1) = \max_{i \in J} kx(i)$ and $kx(1) \in [a, b]$, it is easy to see that $a_i \leq a$, so $N(P(a_i)) < 1/2$. This implies $[a_i, b_i]$ is a regular affine segment of $M(u)$ too. So, χ_x has a common support functional.

II. $N(P(a)) \geq 1/2 \geq \{N(P(a)) + N(P_-(a))\}/2$. By the condition (ii), $P_-(a) = P(a)$, so $N(P(a)) = 1/2$. If $kx(2) \in [a, b]$, we have $x(3) = x(4) = \cdots = 0$. In fact, if $x(3) > 0$, then

$$\begin{aligned}1 \geq \varrho_N(P_-(kx)) &\geq N(P_-(kx(1))) + N(P_-(kx(2))) + N(P_-(kx(3))) \\ &= 2N(P(a)) + N(P_-(kx(3))) = 1 + N(P_-(kx(3))) > 1.\end{aligned}$$

This is a contradiction. Thus we have one of the following two cases for x.

II1. $x = (x(1), x(2), 0, 0, \cdots)$, $kx(1), kx(2) \in [a, b]$. By Lemma 1,
$y = (P(a), P(a), 0, 0, \cdots)$ is a common support functional of χ_x.

II2. $x = (x(1), x(2), x(3), \cdots)$, $kx(1) \in [a, b]$. $kx(i)$ $(i \geq 2)$ belongs to S_M^0 or to a regular affine segment. However, this case has been discussed.

III. $N(P(a)) + N(P_-(a)) > 1$. Clearly, we have $kx(i) \leq a$ for all $i \geq 2$ by $\varrho_N(P_-(kx)) \leq 1$. We will consider the following two cases: $kx(1) \in (a, b]$ and $kx(1) = a$.

III1. $kx(1) \in (a, b]$. We will first prove $kx(i) < a$ $(i > 1)$. Otherwise $kx(2) \geq a$, whence,

$$1 \geq \varrho_N(P_-(kx)) \geq N(P(a)) + N(P_-(a)).$$

This contradicts the hypothesis of III.

If there exists some $i \geq 2$, such that $kx(i)$ belongs to an affine segment $[a', b']$, then from

$$1 \geq \varrho_N(P_-(kx)) \geq N(P(a)) + N(P_-(a')) \geq N(P(a')) + N(P_-(a'))$$

and the condition (ii), we have $P_-(a') = P(a)$. From $1 \geq N(P(a)) + N(P(a'))$ and $N(P(a)) > 1/2$, we immediately obtain $N(P(a')) < 1/2$. Thus $[a', b']$ is a regular affine segments of $M(u)$. This implies that $kx(i)$ belongs to S_M^0 or to some regular affine segment of $M(u)$ for any $i \geq 2$. χ_x has a common support functional.

III2. $kx(1) = a$. We shall consider the problem in the following three cases.

III2.1. $N(P_-(a)) > 1/2$. Since $\varrho_N(P_-(kx)) \leq 1$, we have $kx(i) < a$ as $i > 1$. If $kx(i)$ belongs to an affine segment $[a', b']$ for some $i > 1$, similarly to the case of III1, $[a', b']$ is a regular affine segment of $M(u)$. Hence $kx(i) \in S_M^0$ or $kx(i)$ belongs to a regular affine segment of $M(u)$ for every $i > 1$. So, χ_x has a common support functional of χ_x.

III2.2. $N(P_-(a)) = 1/2$. Note that $kx(i) \leq a$ for all $i \geq 2$. If $kx(i)$ belongs to some affine segment $[a', b']$ of $M(u)$ and $b' = a$ for some $i \in N$, then $kx(j) = 0$ if $j \neq i, 1$. In fact, if $kx(j) > 0$, then $\varrho_N(P_-(kx)) \geq 2N(P_-(a)) + N(P_-(kx(j))) > 1$. A contradiction. Let $v(1) = v(i) = P_-(a), v(j) = 0$ as $j \neq i, 1$. Note $x' \in \chi_x$ implies $k'x'(1) \leq a$. Then $v = \{v(i)\}$ is a common support functional of χ_x. If $kx(i)$ belongs to some affine segment $[a', b']$ of $M(u)$ and $b' < a$ for some $i \in N$, note that $N(P(b')) < 1/2$, by the condition (i) and (ii), $[a', b']$ is a regular affine segment. $kx(i)$ $(i > 1)$ belongs to S_M^0 or to some regular affine segments. χ_x has a common support functional.

III2.3. $N(P_-(a)) < 1/2$. Note that $kx(i) \leq a$ for all $i \geq 2$. If $kx(i) < a$ for all $i \geq 2$, then for every $i \geq 2$, $kx(i)$ belongs to S_M^0 or to some regular affine segment of $M(u)$. The result is deduced.

If there exist $i_1, \cdots, i_m$ $(m \geq 2)$ such that $kx(i) = a$, $i \in \{i_1, \cdots, i_m\}$. By the condition (i), one can easily verify that a is not the right end point of any affine segment of $M(u)$. Note $N(P(a)) + N(P_-(a)) > 1$, so we have $k'x'(i) = a$ $(i = i_1, \cdots, i_m)$ for any $x' \in \chi_x$. Therefore, if there exist finite $i \in N$ with $kx(i) = a$, we can also conclude that χ_x has a common support functional.

Necessity. If the condition (i) is not true, then there exists an affine segment $[a, b]$ of $M(u)$ such that, $N(P(a)) < 1/2$ and b is not a point of continuity of $P(u)$.

Take $w > 0$ satisfying

$$N(P(a)) + N(w) \leq 1, \ \ P(a) < w < P(b).$$

Select a nonnegative sequence (u_i), u_i $(i = 1, 2, \cdots)$ is a point of continuity of $P(u)$ satisfying

$$N(P(a)) + N(w) + \sum_{i=1}^{\infty} N(P(u_i)) = 1.$$

Take c with $a < c < b$, and put

$$x' = (b, b, u_1, u_2, \cdots), \ \ x'' = (b, c, u_1, u_2, \cdots), \ \ x''' = (c, b, u_1, u_2, \cdots),$$

$$x = \frac{x'}{\|x'\|^0}, \ \ x_1 = \frac{x''}{\|x''\|^0}, \ \ x_2 = \frac{x'''}{\|x'''\|^0},$$

and

$$y_1 = (w, P(a), P(u_1), P(u_2), \cdots), \ \ y_2 = (P(a), w, P(u_1), P(u_2), \cdots).$$

It is easy to see that $\varrho_N(y_i) = 1$ $(i = 1, 2)$ and

$$\begin{aligned}
1 \geq \langle y_i, x\rangle &= \langle y_i, x'\rangle / \|x'\|^0 \\
&= (P(a)b + wb + \sum_{i=1}^{\infty} u_i P(u_i)) / \|x'\|^0 \\
&= (\varrho_N(y_i) + \varrho_M(x')) / \|x'\|^0 = (1 + \varrho_M(x')) / \|x'\|^0 \\
&= (1 + \varrho_M(\|x'\|^0 x)) / \|x'\|^0 \geq \|x\|^0 = 1.
\end{aligned}$$

Hence $\langle y_i, x\rangle = 1$. Similarly, we can prove that $\langle y_i, x_i\rangle = 1$ $(i = 1, 2)$.

By Lemma 1, we deduce that y_i $(i = 1, 2)$ is a unique support functional at x_i. But $y_1 \neq y_2$. On the other hand, from $\langle y_i, x_i + x\rangle = 2$ $(i = 1, 2)$, we know that $\|x_i + x\|^0 = 2$. i.e. $x_1, x_2 \in \chi_x$. But, there is no common support functional at x_1 and x_2. It is a contradiction.

If the condition (ii) is not true, then there exists a structural affine segment $[a, b]$ of $M(u)$ with $N(P(a)) + N(P_-(a)) \leq 1$, where a is not a point of continuity of $P(u)$. Select a nonnegative sequence (u_i), where u_i $(i = 1, 2, \cdots)$ is a point of continuity of $P(u)$ satisfying

$$N(P(a)) + N(p_-(a)) + \sum_{i=1}^{\infty} N(P(u_i)) = 1.$$

Take c with $a < c < b$, and put

$$x' = (a, a, u_1, u_2, \cdots), \ \ x'' = (c, a, u_1, u_2, \cdots), \ \ x''' = (a, c, u_1, u_2, \cdots),$$

$$x = \frac{x'}{\|x'\|^0}, \ \ x_1 = \frac{x''}{\|x''\|^0}, \ \ x_2 = \frac{x'''}{\|x'''\|^0},$$

and

$$y_1 = (P(a), P_-(a), P(u_1), P(u_2), \cdots),\ y_2 = (P_-(a), P(a), P(u_1), P(u_2), \cdots).$$

Similarly to the cases in (i), we can complete the proof.

If the condition (iii) is not satisfied, then there exist two neighbour affine segments $[a, b]$ and $[b, c]$ of $M(u)$ such that $N(P(a)) > 1/2, N(P(b)) < 1$, and there is a sequence $\{u_k\} \subset S_M^0$ satisfying

$$N(P(b)) + \sum_{i=1}^{\infty} N(P_-(u_i)) \le 1,$$

and

$$\sum_{i=1}^{\infty}\{N(P(u_i)) - N(P_-(u_i))\} \ge N(P(b)) - N(P_-(b)).$$

Take w_k such that $P_-(u_k) \le w_k \le P(u_k)$ and

$$\sum_{i=1}^{\infty}\{N(w_i)) - N(P_-(u_i))\} = N(P(b)) - N(P_-(b)).$$

Let $v_k \downarrow 0$ satisfy

$$N(P(b)) + \sum_{i=1}^{\infty} N(P_-(u_i)) + \sum_{i=1}^{\infty} N(P(v_i)) = 1.$$

Then we have

$$N(P_-(b)) + \sum_{i=1}^{\infty} N(w_i) + \sum_{i=1}^{\infty} N(P(v_i)) = 1.$$

Put

$$y_1 = (P(b), P_-(u_1), P(v_1), P_-(u_2), P(v_2), \cdots),$$
$$y_2 = (P_-(b), w_1, P(v_1), w_2, P(v_2), \cdots).$$

Then $\varrho_N(y_i) = 1 (i = 1, 2)$. Let

$$z = (b, u_1, v_1, u_2, v_2 \cdots),\ z_1 = (c, u_1, v_1, u_2, v_2 \cdots),\ z_2 = (a, u_1, v_1, u_2, v_2, \cdots)$$

and

$$x = z/\|z\|^0,\ x_1 = z_1/\|z_1\|^0,\ x_2 = z_2/\|z_2\|^0.$$

Note

$$1 \ge \langle x, y_2\rangle = \{bP_-(b) + \sum_{i=1}^{\infty} u_i w_i + \sum_{i=1}^{\infty} v_i P(v_i)\}/\|z\|^0$$
$$= (\varrho_M(z) + \varrho_N(y_2))/\|z\|^0 \ge \|x\|^0 = 1,$$

which implies y_2 is a support functional at x. Similarly, y_2 is also a support functional at x_2. So $\|x + x_2\| = 2$, $x_2 \in \chi_x$. In exactly the same manner as above, we can prove that $x_1 \in \chi_x$. But x_1 and x_2 do not have any common support functional. In fact, if v_1 is a support functional at x_1, then $v_1(1) \ge P(b)$; if v_2 is a support functional at x_2, then $v_2(1) \le P(a)$. The proof is finished. $\square$

Lemma 4. *Let $M \in \Delta_2$, $N \in \Delta_2$, $x, x_n \in S(l_M^0)$ and $\|x_n + x\|^0 \to 2$ $(n \to \infty)$, then* $\lim_{j\to\infty} \sup_n \|x_n - [x_n]_j\|^0 = 0$, *where* $[y]_j(i) = y(i)$, *if* $i \le j$; $[y]_j(i) = 0$, *if* $i > j$.

Proof: Let

$$\|x\|^0 = (1 + \varrho_M(kx))/k, \|x_n\|^0 = (1 + \varrho_M(k_n x_n))/k_n.$$

Since $M \in \Delta_2$, we only need to prove that

$$\lim_{j\to\infty} \sup_n \varrho_M(k_n x_n - [k_n x_n]_j) = \lim_{j\to\infty} \sup_n \sum_{i>j} M(k_n x_n(i)) = 0.$$

If not, then there exist $i_n \uparrow \infty$ and $\varepsilon_0 > 0$ satisfying $\sum_{i>i_n} M(k_n x_n(i)) \ge \varepsilon_0$. Since $N \in \Delta_2$, $\sup k_n < \infty$ (cf. [5]), we may assume without loss of generality, that $k_n \to k'$. Take $\tau \in (0,1)$ such that $k\tau/k' \in (0,1)$. Choose τ_n satisfying

$$\frac{k(1+\tau)}{k+k_n} + \frac{k_n(1-\tau_n)}{k+k_n} = 1.$$

It is easy to see that $\tau_n \to k\tau/k'$. Moreover, $N \in \Delta_2$ implies that there exists $\delta > 0$ such that

$$M(u/(1+\tau)) \le (1-\delta)M(u)/(1+\tau)$$

for all u, $0 \le u \le 2k'M^{-1}(1)$. Take $\tau' \in (k\tau/k', 1)$. Since $M \in \Delta_2$, we have $\varrho_M(kx(i)/(1-\tau')) < \infty$. Hence, $\sum_{i>i_n} M(kx(i)/(1-\tau_n)) \to 0$ $(n \to \infty)$. From

$$\begin{aligned}
0 \leftarrow & \|x_n\|^0 + \|x\|^0 - \|x + x_n\|^0 \\
\ge & \frac{1+\varrho_M(k_n x_n)}{k_n} + \frac{1+\varrho_M(kx)}{k} \\
& - \frac{k+k_n}{kk_n}(1 + \varrho_M(\frac{kk_n}{k+k_n}(x + x_n))) \ge 0,
\end{aligned}$$

we have

$$2 \leftarrow \frac{k+k_n}{kk_n}(1 + \varrho_M(\frac{kk_n}{k+k_n}(x + x_n))).$$

Hence

$$\begin{aligned}
\frac{2kk'}{k+k'}-1 &\leftarrow \varrho_M\Big(\frac{kk_n}{k+k_n}(x+x_n)\Big)\\
&= \Big(\sum_{i=1}^{i_n}+\sum_{i>i_n}\Big)M\Big(\frac{kk_n}{k+k_n}(x(i)+x_n(i))\Big)\\
&\le \sum_{i=1}^{i_n}\Big\{\frac{k}{k+k_n}M(k_nx_n(i))+\frac{k_n}{k+k_n}M(kx(i))\Big\}\\
&+\sum_{i>i_n}M\Big(\frac{k(1+\tau)}{k+k_n}\times\frac{k_nx_n(i)}{1+\tau}+\frac{k_n(1-\tau_n)}{k+k_n}\times\frac{kx(i)}{1-\tau_n}\Big)\\
&\le \sum_{i=1}^{i_n}\Big\{\frac{k}{k+k_n}M(k_nx_n(i))+\frac{k_n}{k+k_n}M(kx(i))\Big\}\\
&+\sum_{i>i_n}\Big(\frac{k(1+\tau)}{k+k_n}M\Big(\frac{k_nx_n(i)}{1+\tau}\Big)+\frac{k_n(1-\tau_n)}{k+k_n}M\Big(\frac{kx(i)}{1-\tau_n}\Big)\Big)\\
&\le \sum_{i=1}^{i_n}\Big\{\frac{k}{k+k_n}M(k_nx_n(i))+\frac{k_n}{k+k_n}M(kx(i))\Big\}\\
&+\sum_{i>i_n}\Big\{\frac{k(1+\tau)}{k+k_n}(1-\delta)\frac{M(k_nx_n(i))}{1+\tau}+\frac{k_n}{k+k_n}M\Big(\frac{kx(i)}{1-\tau_n}\Big)\Big\}\\
&\le \frac{k}{k+k_n}\varrho_M(k_nx_n)-\sum_{i>i_n}\frac{k}{k+k_n}\delta M(k_nx_n(i))\\
&+\frac{k_n}{k+k_n}\varrho_M(kx)+\frac{k_n}{k+k_n}\sum_{i>i_n}\Big\{M\Big(\frac{kx(i)}{1-\tau}\Big)\Big)-M(kx(i))\Big\}\\
&\le \frac{k}{k+k_n}\varrho_M(k_nx_n)+\frac{k_n}{k+k_n}\varrho_M(kx)-\frac{k\delta\varepsilon_0}{k+k_n}+o(1/n)\\
&\to \frac{2kk'}{k+k'}-1-\frac{k\delta\varepsilon_0}{k+k'}.
\end{aligned}$$

This is a contradiction. □

Theorem. *l_M^0 has the WM property if and only if $M\in\Delta_2$, $N\in\Delta_2$ and $M(u)$ satisfies the conditions* (i)–(iii) *in Lemma* 3.

PROOF: *Necessity.* If $M(u)$ does not satisfy any of the conditions in Lemma 3, we know that there exist $x\in S(l_M^0)$, and $x_1,x_2\in\chi_x$ such that x_1,x_2 have no common support functional. Let $x_n=\{x_1,x_2,x_1,x_2,\cdots\}$, then there is no support functional f at x such that $f(x_n)\to 1$. This contradicts the WM property of l_M^0.

If $M \notin \Delta_2$, by Lemma 2, there exists $x \in S(l_M^0)$ such that any support functional at x does not belong to l_N. Let $x_n = (x(1), x(2), \cdots, x(n), 0, 0, \cdots)$. Obviously, $\|x_n\|^0 \to \|x\|^0 = 1$. $\|x_n + x\|^0 \to 2$. But for any support functional $y + \phi$ at x, where $y \in l_N$ and ϕ is a singular functional, $\phi(x) \neq 0$, we have $\langle y + \phi, x_n \rangle = \langle y, x_n \rangle \to 1 - \phi(x) > 0$. This contradicts the WM property of l_M^0.

If $N \notin \Delta_2$, there is a positive sequence (u_i), where u_i satisfies $u_i \downarrow 0$ and

$$N((1 + 1/i)u_i) \geq 2^{i+1} N(u_i), N(u_i) < 1/2^i.$$

Take natural number k_i such that

$$1/2^{i+1} \leq k_i N(u_i) < 1/2^i.$$

Select $a_n > 0$, $N(a_n) + k_n N(u_n) = 1$. Clearly $a_n \uparrow a$, $a > 0$ and $N(a) = 1$. Putting

$$z_n = (a_n, u_n, \cdots, u_n, 0, \cdots),$$
$$y = (a, 0, 0, \cdots),\ y_n = (0, u_n, \cdots, u_n, 0, \cdots),$$

where u_n is taken k_n times, we have $\varrho_N(z_n) = 1$. We can easily check that

$$\varrho_N(y) = 1,\ \|y\|_N = 1;\ \varrho_N((1 + 1/n)y_n) \geq 1,\ \|y_n\|_N \geq 1/(1 + 1/n).$$

By the Hahn-Banach Theorem, there exist $x_n, x \in S(l_M^0)$ such that

$$\langle x, y \rangle = \|y\|_N = 1, \langle x_n, y_n \rangle = \|y_n\|_N \geq 1/(1 + 1/n)$$
$$2 \geq \|x + x_n\|^0 > \langle x, z_n \rangle + \langle x_n, z_n \rangle$$
$$= a_n \langle x, y \rangle / a + \langle x_n, y_n \rangle \geq a_n/a + 1/(1 + 1/n) \to 2.$$

Therefore, $\|x + x_n\|^0 \to 2$. But for any support functional z at x, $\langle z, x_n \rangle = 0$, which contradicts the WM property of l_M^0.

Sufficiency. Let $1 = \|x_n\|^0 = (1 + \varrho_M(k_n x_n))/k_n = \|x\|^0 = (1 + \varrho_M(kx))/k$, $\|x + x_n\|^0 \to 2$. Since $N \in \Delta_2$, $\{k_n\}_{n=1}^\infty$ is bounded. However, the sequence $\{x_n(i)\}_{i=1}^\infty$ $(i = 1, 2, \cdots)$ are bounded. For any subsequence of $\{x_n\}$, using the diagonal method, we can select a subsequence, still denoted by $\{x_n\}$, satisfying $k_n \to k'$, $x_n(i) \to x_i'$ $(i = 1, 2, \cdots)$. Denote $x' = \{x_i'\}_{n=1}^\infty$. By the Fatou Theorem, $\|x'\|^0 \leq \sup_n \|x\|^0 = 1$. So $x' \in B(l_M^0)$. By Lemma 4,

$$\lim_{j \to \infty} \sup_n \|x_n - [x_n]_j\|^0 = \lim_{j \to \infty} \sup_n \sum_{i > j} M(k_n x_n(i)) = 0.$$

It is easy to see that

$$\|x_n - x'\|^0 \leq \|x_n - [x_n]_j\|^0 + \|[x_n]_j - [x']_j\|^0 + \|x' - [x']_j\|^0 \to 0 (n \to \infty).$$

Therefore, we deduce from $\|x_n + x\|^0 \to 2$, that $\|x + x'\|^0 = 2$. This implies that $\|x'\|^0 = 1$, i.e. $x' \in \chi_x$. It follows from

$$1 = (1 + \varrho_M(k_n x_n))/k_n \to (1 + \varrho_M(k'x'))/k' \geq \|x'\|^0 = 1,$$

that $k' \in K(x')$. Take a common support functional y at χ_x. For arbitrary $\varepsilon > 0$, we take a i_0 such that

$$\sum_{i>i_0} (M(k_n x_n(i)) + N(y(i))) < \varepsilon.$$

Since $k_n x_n(i) \to k' x'_i (i = 1, 2 \cdots)$, then for sufficiently large n, we get

$$|\sum_{i=1}^{i_n} k_n x_n(i) y(i) - \sum_{i=1}^{i_n} k' x'_i y(i)| < \varepsilon,$$
$$|\sum_{i=1}^{i_n} M(k_n x_n(i)) - \sum_{i=1}^{i_n} M(k' x'_i)| < \varepsilon.$$

Therefore, for n large enough, we have

$$\begin{aligned} k_n &\geq \sum_{i=1}^{\infty} k_n x_n(i) y(i) \geq \sum_{i=1}^{i_n} k_n x_n(i) y(i) - \varepsilon \\ &\geq \sum_{i=1}^{i_n} k' x'_i y(i) - 2\varepsilon = \sum_{i=1}^{i_n} (M(k' x'_i) + N(y(i)) - 2\varepsilon \\ &\geq \sum_{i=1}^{i_n} M(k_n x_n(i)) + \sum_{i=1}^{i_n} N(y(i)) - 3\varepsilon \geq \varrho_M(k_n x_n) + \varrho_N(y) - 4\varepsilon \\ &= 1 + \varrho_M(k_n x_n) - 4\varepsilon = k_n - 4\varepsilon. \end{aligned}$$

It is easy to see that $\langle x_n, y\rangle \to 1$.

Since $\{x_n\}$ is an arbitrary subsequence of $\{x_n\}$ and y does not depend on the subsequence, so for the sequence $\{x_n\}$, we still have $\langle x_n, y\rangle \to 1$. $\square$

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