Mazur–like topological linear spaces and their products

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Abstract. Topological linear spaces having the property that some sequentially continuous linear maps on them are continuous, are investigated. It is shown that such properties (and close ones, e.g., bornological-like properties) are closed under large products.

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J. Isbell in his talk at the 1st Prague Topological Symposium in 1961 said that a topological linear space X satisfies Mazur's theorem if every sequentially continuous linear functional on X is continuous. Spaces having the just described property are called Mazur spaces in [15]. We shall study some variations of such a property and will call the resulting spaces Mazur-like spaces. The terms are justified by the fact that S. Mazur was probably the first one who started to investigate sequentially continuous linear functionals from a general point of view. In 1946 he gave a lecture in Warsaw (see [9]) where he talked about the following result:

Every sequentially continuous linear functional on the power $\mathbb{R}^X$ is of the form $f(\phi) = \alpha_1\phi(p_1) + \cdots + \alpha_k\phi(p_k)$ for some integer k, points $p_1, \cdots, p_k$ of X, and reals $\alpha_1, \cdots, \alpha_k$, if and only if every two-valued measure on X being zero on points, is zero on X.

S. Mazur adds that the same result is valid if instead of a discrete space X one takes a metric space X, and instead of the power $\mathbb{R}^X$ one takes the space $C_p(X)$ of continuous functions on X endowed with the pointwise convergence. He promised to published details in Fund. Math. but that never happened.

J. Isbell says in [6] that the Mazur's result was completed by V. Pták to the form given below in 1956 (unpublished). At about the same time, S. Mrówka published his generalization of the Mazur's result (see [11]) that we call, together with Isbell, Mazur theorem:

For a completely regular space X, the space $C_p(X)$ is a Mazur space iff the space X is realcompact.

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J. Isbell then announced generalizations of Mazur theorem to other classes of continuous functions like differentiable ones; these results were published in [7]; see also [12] for a general characterization of linear subspaces of $C_p(X)$ that are Mazur.

A stronger version of Mazur spaces was defined in [3] for locally convex spaces under the name C-sequential spaces (C stands for convex); R.M. Dudley required all sequentially continuous linear mappings (not only linear functionals) on a given space to be continuous; he proved that every bornological space is C-sequential. R.F. Snipes in [13] shows that the class of C-sequential spaces is coreflective in all topological linear spaces (or locally convex spaces), and that the class is closed under inductive limits. For specialists in category theory, the first fact follows directly from the definition of C-sequential spaces, the other is a direct consequence of coreflectivity. Probably the best survey of properties of Mazur spaces can be found in A. Wilansky's book [15].

We recall that certain weak topologies on Mazur spaces are complete, completion of a Mazur space is a Mazur space, and that coarser linear topologies compatible with a given Mazur space are again Mazur.

A cardinal κ is called (Ulam) nonmeasurable if there is no nontrivial two-valued measure on κ being zero on points; equivalently, every ultrafilter on κ that is closed under countable intersections, has a nonempty intersection; in other words, a discrete space of cardinality κ is realcompact, i.e., can be embedded as a closed subspace into a power of reals. See, e.g., [4] for those and other topological notions and terms we shall use in this paper.

We shall now recall several concepts from category theory (see [1] for details and for other notions). We shall work in an epireflective subcategory $\mathcal{K}$ of the category `TLS` of real topological linear spaces; epireflectivity of $\mathcal{K}$ means that $\mathcal{K}$ is closed under products and subspaces in `TLS`; **we always assume that** $\mathbb{R} \in \mathcal{K}$ to avoid the two trivial cases, namely the subclasses consisting of the zero space, or of all indiscrete topological linear spaces. We shall consider nontrivial coreflective classes $\mathcal{C}$ in $\mathcal{K}$. By 'nontrivial' we understand that the class contains $\mathbb{R}$, to avoid almost exotic spaces, i.e., those having zero continuous linear functionals only. The coreflectivity for $\mathcal{C}$ in our case means that the class $\mathcal{C}$ is closed under quotients and under inductive limits in $\mathcal{K}$. Equivalently, for every space $X \in \mathcal{K}$ there exists a finer space cX belonging to $\mathcal{C}$ such that every continuous linear mapping from a space in $\mathcal{C}$ to X is continuous already into the finer space cX. Every class of spaces from $\mathcal{K}$ has a coreflective hull in $\mathcal{K}$. Since finite products coincide with finite free sums, every coreflective class is finitely productive (i.e., closed under finite products). By `LCS` we denote the epireflective subclass of `TLS` consisting of locally convex spaces.

It is known that a product of nontrivial bornological spaces is bornological iff the index set of the product is nonmeasurable (the so called Mackey-Ulam theorem); the same assertion is true for Mazur spaces (by the Mazur theorem) The last mentioned result is reproved in [5] as a part of a more general result for topological groups. Since its proof uses some previous results about topological groups that are not needed for topological linear spaces, we shall give here a simplified proof of the result and, at the same time, prove a more general theorem containing both results mentioned at the beginning of this paragraph (and other), as special cases.

In the first part we shall introduce some definitions and basic properties of spaces we are interested in. The second part contains our main result, namely that certain classes of topological linear spaces containing a class of Mazur-like spaces, are closed under large products. In the third part, examples are given showing that the previous results are not valid in TLS, and subclasses of TLS are suggested, where the results hold.

1. Mazur-like classes

In this section, all spaces are supposed to be locally convex ($\mathcal{K} = \mathtt{LCS}$). We shall now give a rather general definition of spaces similar to Mazur spaces for which the analog of Mackey-Ulam theorem holds. In modifications of Mazur spaces one can use various ranges of linear mappings (instead of $\mathbb{R}$) and various classes of mappings that are requested to be continuous; in this last variation we restrict ourself to some classes only:

Definition 1. Let $\mathcal{R}$ be a class of topological linear spaces and $\mathcal{S}$ be a class of linear mappings. A locally convex space X is called an *$\mathcal{R}$-S-space* if every linear mapping from $\mathcal{S}$ defined on X into a topological linear space from $\mathcal{R}$ is continuous.

Thus Mazur spaces are $\mathbb{R}$-$\mathcal{S}$-spaces, where $\mathcal{S}$ is the class of all sequentially continuous linear mappings (we shall write $\mathbb{R}$-$\mathcal{S}$ instead of more precise $\{\mathbb{R}\}$-$\mathcal{S}$) We shall mostly consider cases where $\mathcal{S}$ is related to sequentially continuous linear mappings.

Every $\mathcal{R}$-S-space is an $\mathcal{R}'$-S'-space for $\mathcal{R} \supset R'$ and $\mathcal{S} \supset S'$. For given $\mathcal{R}, S$, the class of $\mathcal{R}$-S-spaces is coreflective.

Two extreme cases will be used in the sequel. We shall say that a linear mapping $f : X \to Y$ is *strongly sequentially continuous* if it preserves convergence of well-ordered nets of nonmeasurable length (i.e., if a well-ordered net, or a chain in another terminology, $\{x_\alpha\}_{\alpha<\kappa}$ converges to 0 in X, κ is a nonmeasurable cardinal, then $\{fx_\alpha\}_{\alpha<\kappa}$ converges to 0 in Y). Clearly, every strongly sequentially continuous linear mapping is sequentially continuous.

We shall say that a linear mapping $f : X \to Y$ is *weakly sequentially continuous* if $\{fx_n\}$ converges to 0 in Y whenever $\{x_n\}$ is such a sequence in X that $\{\alpha_n x_n\}$ converges to 0 for every sequence $\{\alpha_n\}$ of reals (such a sequence $\{x_n\}$ will be called strongly convergent to 0). Clearly, every sequentially continuous linear mapping is weakly sequentially continuous.

Definition 2. The $\mathcal{R}$-S-spaces are called *strong Mazur spaces* for $\mathcal{R}$ equal to all locally convex spaces and $\mathcal{S}$ equal to all weakly sequentially continuous linear mappings.

The $\mathbb{R}$-S-spaces are called *weak Mazur spaces* for $\mathcal{S}$ equal to all strongly sequentially continuous linear mappings.

It is easy to show that $\mathbb{R}$ is strong Mazur, so that every coreflective class containing all strong Mazur spaces is a nontrivial coreflective class.

Every strong Mazur space is bornological (if $\{x_n\}$ strongly converges to 0 then for every sequence $\{\alpha_n\}$ in $\mathbb{R}$ the set $\{\alpha_n x_n\}$ is bounded and thus $\{f x_n\}$ must converge to 0 provided that f preserves bounded sets).

We shall consider coreflective classes $\mathcal{C}$ containing all strong Mazur spaces. For examples of such $\mathcal{C}$ we may take all bornological spaces, all ultrabornological spaces, all semibornological spaces (see, e.g., [15] and [8] for these concepts), all (weak) Mazur spaces, all locally convex spaces defined by the property that every their linear functional continuous on a class of sets containing all null sequences (together with the point zero) is continuous. In the last example we may take the class of compact sets (then we get that k-continuous linear functionals are continuous) or of countable sets, or of bounded sets. The first examples of this paragraph up to Mazur spaces are examples of classes contained in the class of weak Mazur spaces.

The coreflective classes containing strong Mazur spaces and composed of weak Mazur spaces will be called Mazur-like classes in this paper.

2. Productivity

As in the previous section we assume that $\mathcal{K} = \mathtt{LCS}$. The original Mazur's result from 1946 asserts that a power $\mathbb{R}^\kappa$ is a Mazur space iff κ is nonmeasurable. We shall now generalize that result not in the direction used by Pták and Mrówka — they replaced κ by a topological space; we shall replace the copies of $\mathbb{R}$ in the power by arbitrary spaces from a Mazur-like class.

The method of the main proof is a modification of the Mazur's method used in [10]. First, we need the following easy assertion:

Lemma 1. *Every linear mapping on X into a locally convex space that is continuous on the strong Mazur coreflection of X, is weakly sequentially continuous on X.*

Proof: Suppose that X has a topology τ and its strong Mazur coreflection has a topology $c\tau$. It suffices to show that every sequence $\{x_n\}$ strongly converging to zero in (X,τ), converges to zero in $(X, c\tau)$. Take the finest topology t on the set X such that every $\{x_n\}$ converging strongly to zero in τ converges to zero in t. Then (X,t) has the same sequences strongly converging to zero as (X,τ) has, and the finest locally convex space (X,t') coarser than (X,t) is strongly Mazur (indeed, continuous linear maps on (X,t') coincide with continuous linear maps on (X,t), and the last ones coincide with those linear maps on X preserving convergence of the generating sequences, i.e., convergence of sequences strongly converging to zero in (X,τ)). Thus (X,t) is finer than $(X,c\tau)$. $\square$

Theorem 1. *Every coreflective class containing the class of strong Mazur spaces is closed under products of nonmeasurably many spaces.*

Proof: Take a coreflective class $\mathcal{C}$ containing all strong Mazur spaces. Suppose that every $X_i, i \in I$, belongs to $\mathcal{C}$ and $|I|$ is nonmeasurable. We want to show that $X = \Pi_I X_i$ belongs to $\mathcal{C}$. Since every locally convex space can be embedded into a product of pseudonormable linear spaces, it suffices to show that every continuous linear mapping f from the coreflection (in $\mathcal{C}$) of X into a normable space M is continuous on X. Our condition that $\mathcal{C}$ contains all strong Mazur spaces implies that f is weakly sequentially continuous on X (use Lemma 1). We show that f depends on finitely many coordinates and our theorem then follows from the fact that $\mathcal{C}$ is finitely productive.

Define $J = \{i \in I : fx \text{ is nonzero for some } x_i \in X \text{ with } \mathrm{pr}_{I\setminus\{i\}} x_i = 0\}$. We may choose such x_i that $||fx_i|| = 1$. The set J is finite because otherwise every one-to-one sequence $\{x_{i_n}\}$ composed of just constructed points strongly converges to 0 but their f-images are far from 0. Now we prove that f depends on J. Suppose that it is not the case. Thus there is some $x \in X$ with $fx \neq 0$ and $\mathrm{pr}_J x = 0$. We shall get a contradiction by showing that $|I|$ must then be measurable.

We say in this proof that a subset $K \subset I \setminus J$ has the property (p) if there exists $x_K \in X$ with $fx_K \neq 0$ and $\mathrm{pr}_{I\setminus K}(x_K) = 0$ (thus $I \setminus J$ has (p)). We assert that there exists $K \subset I \setminus J$ having the property (p) and such that there are no two disjoint subsets of K both having (p). If not, then there is a disjoint sequence $\{K_n\}$ of subsets of $I \setminus J$ such that every K_n has (p) and there is a sequence $\{\alpha_n\}$ of reals such that $\{\alpha_n f(x_{K_n})\}$ does not converge to zero in M. This is a contradiction with the weak sequential continuity of f because $\{x_{K_n}\}$ strongly converges to zero in X.

Now it suffices to define μ on all subsets of K to be 0 or 1 depending whether the subset has not or has the property (p). Clearly, μ is a nontrivial two-valued measure being zero on all finite sets. That is a contradiction with our assumption that I is nonmeasurable.

Thus, f depends on the finite set J in the sense that there is a linear mapping g on $\Pi_J X_i$ such that $f = g\,\mathrm{pr}_J$. Clearly, g is continuous on $c(\Pi_J X_i)$ (since $\Pi_J X_i$ is a retract of $\Pi_I X_i$ and, thus, $c(\Pi_J X_i) = \Pi_J X_i$ is a retract of $c(\Pi_I X_i)$). Consequently, g is continuous on $\Pi_J X_i$ and, therefore, f is continuous on X. $\square$

In the preceding proof, we could use the known result by S. and P. Dierolf from [2] (see also [14, Theorems 5.17, 5.21], or [8, Lemma 8.8.3]): *In a coreflective subcategory $\mathcal{C}$ of spaces, a product of κ its nontrivial members belongs to $\mathcal{C}$ iff $\mathbb{R}^\kappa$ belongs to $\mathcal{C}$.* For us, it would be no simplification and so, our proof deals with general products instead of powers of reals. We shall use the mentioned result in the proof of Theorem 2.

A 'converse' of preceding Theorem (i.e., *products of measurably many members of a coreflective class $\mathcal{C}$ does not belong to $\mathcal{C}$*) is not true without adding some conditions on $\mathcal{C}$ or on the spaces under consideration: one of the coreflective classes fulfilling the condition of Theorem 1 is the class of all spaces that is certainly productive. So, to get a converse, our $\mathcal{C}$ cannot be too large. Also, every product of indiscrete spaces is indiscrete and may belong to $\mathcal{C}$ for any index set of the product.

Theorem 2. *Let $\mathcal{C}$ be a coreflective class contained in the class of weak Mazur spaces. No product of measurably many nontrivial Hausdorff spaces from $\mathcal{C}$ belongs to $\mathcal{C}$.*

Proof: Using the P. and S. Dierolf's result mentioned after the proof of Theorem 1, it suffices to show that a power $\mathbb{R}^I$ does not belong to $\mathcal{C}$ whenever $|I|$ is measurable. We shall show that $\mathbb{R}^I$ is not a weak Mazur space. There is a two-valued measure μ on I that is zero on countable sets and has the value 1 on I. As a mapping $\{0,1\}^I \to \{0,1\}$, μ is strongly sequentially continuous but not continuous. Now it suffices to extend μ to a strongly sequentially continuous linear functional on $\mathbb{R}^I$; the extension will not be continuous since its restriction to $\{0,1\}^I$ is not continuous. For every $x = \{r_i\}_I \in \mathbb{R}^I$ define fx to be that unique real number r such that $\mu\{i \in I : r_i = r\} = 1$ (recall that μ is $(2^\omega)^+$-additive). Clearly, $f(\alpha x) = \alpha f x$ for every real number α. To show additivity $f(x+y) = fx + fy$, where x is as before and $y = \{s_i\}_I$, it suffices to realize that if $\mu\{i \in I : r_i = r\} = 1, \mu\{i \in I : s_i = s\} = 1$, then $\mu\{i \in I : r_i + s_i = r + s\} = 1$. Similarly one can show that f is strongly sequentially continuous: take κ nonmeasurable and a long sequence $x_\alpha = \{r_{\alpha,i}\}_I, \alpha < \kappa$, converging to zero in $\mathbb{R}^I$ and suppose that for each α, $\mu\{i \in I : r_{\alpha,i} = r_\alpha\} = 1$; then $\mu\{i \in I : r_{\alpha,i} = r_\alpha$ for each $\alpha\} = 1$, thus $f(x_\alpha)$ converge to 0 because r_α converge to 0. $\square$

Corollary. *A product of nontrivial Hausdorff spaces from a Mazur-like class $\mathcal{C}$ belongs to $\mathcal{C}$ iff the number of spaces in the product is nonmeasurable.*

Instead of nontrivial Hausdorff spaces one can speak about spaces having nontrivial Hausdorff modifications.

Corollary. *Let $\mathcal{P}$ be one of the following properties of Hausdorff spaces: bornological, ultrabornological, semibornological, Mazur, C-space. Then a product of κ many nontrivial spaces having $\mathcal{P}$ has $\mathcal{P}$, iff κ is nonmeasurable.*

3. Non-locally convex spaces

When we look carefully at the proof of Theorem 1 and find places where local convexity was used, we see that there are three such places. In the first paragraph we used that locally convex spaces have weak topologies with respect to normable spaces. This was needed both in the second and third paragraphs, where we made use of the fact that no sequence of nonzero elements in a normed space strongly converges to zero. This last property is crucial in our proof and we shall show that Theorem 1 is not true in the realm of all topological linear spaces, i.e., if $\mathcal{K}$ $=$ TLS.

It was proved by Noble that any product of less than $\mathfrak{s}$ of sequential topological groups is sequential (here $\mathfrak{s}$ is the sequential cardinal that is a large cardinal but is never bigger than the real-measurable cardinal — see e.g. [5] for details). Since every sequentially continuous group-homomorphism between topological linear spaces is a linear mapping, we see that any product of less than $\mathfrak{s}$ of Mazur spaces is Mazur again. So, our example works only in the case when the real-measurable cardinal is (Ulam) nonmeasurable — if our cardinals exist, then it is consistent that the requested situation occurs.

Example. Let κ be a cardinal number that is real-measurable and non-measurable, and μ be a real-measure on κ that is 0 on points of κ and 1 on the whole set. We shall show that $X = \mathbb{R}^\kappa$ is not an $\mathcal{R}$-S-space whenever $\mathcal{R}$ contains the following space Y and $\mathcal{S}$ consists of weakly sequentially continuous linear maps. Denote by Y the topological linear space having as the underlying set $\mathbb{R}^\kappa$ and its topology defined by means of the paranorm

$$||f|| = \int_\kappa \frac{|f(x)|}{1+|f(x)|}\, d\mu$$

(we regard points of Y as mappings f from κ into $\mathbb{R}$). Clearly, the topologies of X and Y are not comparable.

Claim: *Every sequence converging strongly to zero in X converges strongly to zero in Y.*

It is easy to see that if a sequence $\{f_n\}$ strongly converges to zero in X, then the sets $\mathrm{Coz}\,(f_n)$ of points with non-zero f_n-values, form a point-finite family, i.e., every point of κ belongs to at most finitely many of the sets $\mathrm{Coz}\,(f_n)$. This implies $\mu(\mathrm{Coz}\,(f_n)) \to 0$ (otherwise $\mu(\limsup \mathrm{Coz}\,(f_n)) > 0$ and our family would not be point-finite) and, consequently, $||f_n|| \to 0$ strongly in Y since $||f_n|| \le \mu(\mathrm{C}oz(f_n))$. □

Now, the identity map $X \to Y$ is weakly sequentially continuous and is not continuous, which implies that $(\mathbb{R})^\kappa$ is not an $\mathcal{R}$-S-space.

The testing range of a weakly sequentially continuous mapping on X in the previous example cannot have a weak topology with respect to spaces having the property that no sequence of nonzero elements strongly converges to zero. It is not difficult to see that every continuous linear mapping F on our space Y into a space having the just described property, is zero.

Every topological linear space can be embedded into a product of pseudometrizable linear spaces but, as we have just seen, not every pseudometrizable linear space can be embedded into a product of spaces having no nontrivial sequences strongly converging to zero. Denote by $\mathcal{K}_0$ the epireflective hull in `TLS` of spaces having no nontrivial sequences strongly converging to zero; thus $\mathcal{K}_0$ consists of spaces X having the property that for every $x \neq 0$ there exists a continuous linear map from X into a space without nontrivial sequences strongly converging to 0, with $fx \neq 0$. If $\mathcal{K}$ is a bireflective subcategory of $\mathcal{K}_0$ containing $\mathbb{R}$, then we can repeat Sections 1 and 2 replacing everywhere locally convex spaces by spaces from

$\mathcal{K}$ and normable spaces by spaces having no nontrivial sequences strongly converging to 0. We have the following result (where $\mathcal{K}_0$-strong Mazur space means $\mathcal{K}_0$-{weakly sequentially continuous linear maps}-space):

Theorem 3. *Let $\mathcal{C}$ be a coreflective class in $\mathcal{K}_0$ containing the class of $\mathcal{K}_0$-strong Mazur spaces and contained in the class of weak Mazur spaces. A product of Hausdorff spaces from $\mathcal{C}$ belongs to $\mathcal{C}$ iff the number of nontrivial coordinate spaces is nonmeasurable.*

It is clear that locally bounded spaces have no nontrivial sequences strongly converging to zero. I do not know whether metrizable spaces belonging to $\mathcal{K}_0$ are locally bounded, i.e., whether $\mathcal{K}_0$ is generated by locally bounded spaces.

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References

[1] Adámek J., Herrlich H., Strecker G., *Abstract and Concrete Categories*, Wiley Interscience, New York, 1990.

[2] Dierolf P., Dierolf S., *On linear topologies determined by a family of subsets of a topological vector spaces*, Gen. Top. and its Appl. **8** (1978), 127–140.

[3] Dudley R.M., *On sequential convergence*, Trans. Amer. Math. Soc. **112** (1964), 483–507.

[4] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[5] Hušek M., *Sequentially continuous homomorphisms on products of topological groups*, Top. & Appl. **70** (1996), 155–165.

[6] Isbell J.R., *Mazur's theorem*, Proc. Top. Symp. Prague 1961 (Academia Prague 1962), pp. 221–225.

[7] Isbell J.R., Thomas E.S., Jr., *Mazur's theorem on sequentially continuous linear functionals*, Proc. Amer. Math. Soc. **14** (1963), 644–647.

[8] Jarchow H., *Locally Convex Spaces*, Teubner, Stuttgart, 1981.

[9] Mazur S., *Sur la structure des fonctionelles linéaires dans certains espaces* ($\mathcal{L}$), Ann. Soc. Polon. Math. **19** (1946), 241.

[10] Mazur S., *On continuous mappings on cartesian products*, Fund. Math. **39** (1952), 229–238.

[11] Mrówka S., *On the form of certain functionals*, Bull. Acad. Polon. Sci. Ser. Math. **5** (1957), 1061–1067.

[12] Mrówka S., *On the form of pointwise continuous linear positive functionals and isomorphisms of function spaces*, Studia Math. **21** (1961), 1–14.

[13] Snipes R.F., *C-sequential and S-bornological spaces*, Math. Ann. **202** (1973), 273–283.

[14] Sydow W., *Über die Kategorie der topologischen Vektorräume*, Doktor-Dissertation (Fernuniversität Hagen, 1980).

[15] Wilansky A., *Modern Methods in Topological Vector Spaces*, McGraw-Hill, 1978.

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[1] Added in proof: A kind of a generalization of Theorem 1 valid for all coreflective classes of topological linear spaces is described in the author's paper *Productivity of coreflective classes of topological linear spaces* (to appear in Top. & Appl.).