Smooth approximation in weighted Sobolev spaces

T. Kilpeläinen

Abstract. We give necessary and sufficient conditions for the equality $H = W$ in weighted Sobolev spaces. We also establish a Rellich-Kondrachov compactness theorem as well as a Lusin type approximation by Lipschitz functions in weighted Sobolev spaces.

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Suppose that w is a p-admissible weight, $p > 1$, in the sense of [7] (see **I**–**IV** below) and μ the measure with density w, i.e. $d\mu(x) = w(x)dx$. As in [7] we let $H^{1,p}(\Omega;\mu)$ be the closure of $C^\infty(\Omega)$ in $L^p(\Omega;\mu) \times L^p(\Omega;\mu)$ in the norm

$$\|\varphi\|_{1,p,\mu} = \Big(\int_\Omega |\varphi|^p\,d\mu\Big)^{1/p} + \Big(\int_\Omega |\nabla\varphi|^p\,d\mu\Big)^{1/p}.$$

Define

$$W^{1,p}(\Omega;\mu) = \{u \in L^1_{\mathrm{loc}}(\Omega;dx)\colon\, u, Du \in L^p(\Omega;\mu)\}\,,$$

where $Du \in L^1_{\mathrm{loc}}(\Omega;dx)$ is the distributional gradient of u, i.e.

$$\int_\Omega u\nabla\varphi\,dx = -\int_\Omega Du\varphi\,dx$$

whenever $\varphi \in C_0^\infty(\Omega)$. We equip $W^{1,p}(\Omega;\mu)$ with the norm $\|\cdot\|_{1,p,\mu}$.

The space $H^{1,p}(\Omega;\mu)$ has appeared to be useful in studying partial differential equations; see [4] and [7]. The spaces $W^{1,p}(\Omega;\mu)$ have been studied for example by Kufner [10]. It is easy to give examples of p-admissible weights w so that $H^{1,p}(\Omega;\mu) \not\subset W^{1,p}(\Omega;\mu)$; cf. [7, p. 13]. In the case when w belongs to Muckenhoupt's A_p-class (with the same p as is the integrability exponent in $H^{1,p}(\Omega;\mu)$) it is known that the two definitions result in the same space of functions, i.e. $H^{1,p}(\Omega;\mu) = W^{1,p}(\Omega;\mu)$; cf. e.g. [9]. In the unweighted case this equality is often credited to Meyers and Serrin [13], but it appeared already in the work [2] of Deny and Lions. Nowadays it is well known and can be found in textbooks, e.g. in [3].

In this note our aim is to give necessary and sufficient conditions for weights guaranteeing that $H = W$. As byproducts we obtain a Lusin type approximation of $H^{1,p}(\mathbf{R}^n;\mu)$ functions by Lipschitz functions and a weighted Rellich-Kondrachov theorem.

1. Theorem. *If $W^{1,p}(\Omega;\mu)$ is a Banach space, then $H^{1,p}(\Omega;\mu) \subset W^{1,p}(\Omega;\mu)$.*

PROOF: This is immediate, for smooth functions in $H^{1,p}(\Omega;\mu)$ are dense in $H^{1,p}(\Omega;\mu)$ and they also belong to $W^{1,p}(\Omega;\mu)$. □

Remarks. If $w^{1/(1-p)}$ is locally integrable in $\mathbf{R}^n$, we infer from Hölder's inequality that $W^{1,p}(\Omega;\mu)$ is a Banach space, cf. [7, p. 14] or [11, 1.5].

Clearly, it is necessary for the equality $H^{1,p}(\Omega;\mu) = W^{1,p}(\Omega;\mu)$ that $W^{1,p}(\Omega;\mu)$ is a Banach space, for $H^{1,p}(\Omega;\mu)$ is.

For the reader's convenience we recall here that a locally integrable function w is termed *p-admissible* if the following four conditions are fulfilled:

I $0 < w < \infty$ almost everywhere in $\mathbf{R}^n$ and the measure μ with $d\mu(x) = w(x)dx$ is *doubling*, i.e. there is a constant $C > 0$ such that

$$\mu(B(x_0, 2r)) \le C\mu(B(x_0, r))$$

for each ball $B(x_0, r)$.

II If D is an open set and $\varphi_i \in C^\infty(D)$ is a sequence of functions such that $\int_D |\varphi_i|^p \, d\mu \to 0$ and $\int_D |\nabla\varphi_i - v|^p \, d\mu \to 0$ as $i \to \infty$, where v is a vector-valued measurable function in $L^p(D;\mu)$, then $v = 0$.

III There are constants $\varkappa > 1$ and $C > 0$ such that

$$\Big(\frac{1}{\mu(B)} \int_B |\varphi|^{\varkappa p} \, d\mu\Big)^{1/\varkappa p} \le C\, r \Big(\frac{1}{\mu(B)} \int_B |\nabla\varphi|^p \, d\mu\Big)^{1/p}$$

whenever $B = B(x_0, r)$ is a ball in $\mathbf{R}^n$ and $\varphi \in C_0^\infty(B)$.

IV There is a constant $C > 0$ such that

$$\int_B |\varphi - \varphi_B|^p \, d\mu \le C\, r^p \int_B |\nabla\varphi|^p \, d\mu \tag{2}$$

whenever $B = B(x_0, r)$ is a ball in $\mathbf{R}^n$ and $\varphi \in C^\infty(B)$ is bounded. Here

$$\varphi_B = \frac{1}{\mu(B)} \int_B \varphi \, d\mu \,.$$

It has turned out that the assumptions **I** and **IV** will imply both **II** and **III**. The uniqueness of the gradient **II** was observed by Semmes; see [8, Lemma 5.6]. Hajłasz and Koskela proved in [6] that the Poincaré inequality **IV** and the doubling property **I** imply the Sobolev inequality **III**.

In this note we show that the Poincaré inequality has an essential role also in smooth or Lipschitz approximation of functions in weighted Sobolev spaces.

First observe that it readily follows from the definition of $H^{1,p}(\Omega;\mu)$ that the Poincaré inequality (2) holds for all $\varphi \in H^{1,p}(\Omega;\mu)$. In what follows we say that the Poincaré inequality (2) holds for a function φ if it holds with a constant $C > 0$ independent of the ball B.

3. Theorem. *Suppose that the Poincaré inequality* (2) *holds for all functions from* $W^{1,p}(\mathbf{R}^n;\mu)$*. Then* $W^{1,p}(\Omega;\mu) \subset H^{1,p}(\Omega;\mu)$*.*

As the main ingredient of the proof we establish the following result which may be of independent interest.

4. Theorem. *Suppose that the Poincaré inequality* (2) *holds for* u *in* $W^{1,p}(\mathbf{R}^n;\mu)$*. Then* $u \in H^{1,p}(\mathbf{R}^n;\mu)$ *and for each* $\varepsilon > 0$*, there is a Lipschitz function* f *in* $\mathbf{R}^n$ *such that*

$$\mu(\{x\colon f(x) \neq u(x)\}) < \varepsilon$$

and

$$\|u - f\|_{1,p} < \varepsilon\,.$$

Proof: By approximating u by its truncations we are free to assume that $0 \le u \le 1$.

Let x and y be Lebesgue points of u with $|x-y| \le 1$. Let k_0 be the integer with $2^{-k_0} < |x-y| \le 2^{1-k_0}$. For $k = 1, 2, \dots$ write

$$B_k = B(x, 2^{-k})$$

and

$$a_k = \fint_{B_k} u\, d\mu\,.$$

Since x is a Lebesgue point for u, we have

$$\begin{aligned} |u(x) - a_{k_0}| &= \lim_{j\to\infty} |a_j - a_{k_0}| \le \sum_{k=k_0}^{\infty} |a_{k+1} - a_k| \\ &\le \sum_{k=k_0}^{\infty} \fint_{B_{k+1}} |u - a_k|\, d\mu \le c \sum_{k=k_0}^{\infty} \big(\fint_{B_k} |u - a_k|^p\, d\mu\big)^{1/p} \\ &\le C \sum_{k=k_0}^{\infty} 2^{-k} \big(\fint_{B_k} |\nabla u|^p\, d\mu\big)^{1/p}, \end{aligned}$$

where Hölder's and Poincaré's inequalities and the doubling property of μ have also been used. Thanks to our choice of k_0, the last sum does not exceed

$$c|x-y|\big(M|\nabla u|^p(x)\big)^{1/p},$$

where M stands for the weighted maximal function

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} f\, d\mu\,.$$

Similarly, we find that

$$|u(y) - \fint_{B(y,2^{k_0+2})} u\,d\mu| \le c|x-y|\big(M|\nabla u|^p(y)\big)^{1/p},$$

and further in a similar manner

$$\begin{aligned}|a_{k_0} - \fint_{B(y,2^{k_0+2})} u\,d\mu| &\le c\fint_{B(y,2^{k_0+2})} |u - u_{B(y,2^{k_0+2})}|\,d\mu \\ &\le c|x-y|\big(M|\nabla u|^p(y)\big)^{1/p}.\end{aligned}$$

Combining these estimates we arrive at

$$|u(x)-u(y)| \le c|x-y|\big((M|\nabla u|^p(x))^{1/p} + (M|\nabla u|^p(y))^{1/p}\big),$$

where $c \ge 1$ is a constant, independent of x and y. So, if L is the set of the Lebesgue points of u and $\lambda \ge 1$, then the restriction on u to the set

$$K_\lambda = \{x \in L\colon M|\nabla u|^p(x) \le \lambda^p\}$$

is Lipschitz with constant $c\lambda$. By a well known extension theorem for Lipschitz functions (cf. [3]), there is a Lipschitz-function f_λ on $\mathbf{R}^n$ such that $\operatorname{Lip}(f_\lambda) = c\lambda$ and the restriction to K_λ of f_λ coincides with u. By truncating, if necessary, we may assume that $0 \le f_\lambda \le 1$.

We claim that f_λ is the function we are looking for. By a well known weak type inequality for maximal functions [14, p. 13] we have

$$\mu(\complement K_\lambda) \le \frac{c}{\lambda^p}\int_{\mathbf{R}^n} |\nabla u|^p\,d\mu.$$

Thus

$$\int_{\mathbf{R}^n} |u-f_\lambda|^p\,d\mu = \int_{\complement K_\lambda} |u-f_\lambda|^p\,d\mu \le c\mu(\complement K_\lambda) \to 0$$

as $\lambda \to \infty$ and

$$\int_{\mathbf{R}^n} |\nabla u - \nabla f_\lambda|^p\,d\mu = \int_{\complement K_\lambda} |\nabla u - \nabla f_\lambda|^p\,d\mu \le c\int_{\complement K_\lambda} |\nabla u|^p\,d\mu + c\lambda^p\mu(\complement K_\lambda) \le c.$$

Hence $\{f_\lambda\}_\lambda$ is bounded in $H^{1,p}(\mathbf{R}^n;\mu)$ (see [7, Lemmas 1.11 and 1.15]). Since $f_\lambda \to u$ in $L^p(\mathbf{R}^n;\mu)$, we find that $u \in H^{1,p}(\mathbf{R}^n;\mu)$ and that $\nabla f_\lambda \to \nabla u$ weakly in $L^p(\mathbf{R}^n;\mu)$; see [7, 1.32]. Appealing to Mazur's theorem we find a sequence of convex combinations of f_λ's that converges to u in $H^{1,p}(\mathbf{R}^n;\mu)$. These functions are Lipschitz and coincide with u in K_λ. The theorem follows. □

Observe that in the proof of Theorem 4 we never really employed the fact that the gradient of u is distributional. Hence we have:

5. Theorem. *Suppose that* $u \in H^{1,p}(\mathbf{R}^n;\mu)$. *Then for each* $\varepsilon > 0$, *there is a Lipschitz function* f *in* $\mathbf{R}^n$ *such that*

$$\mu(\{x\colon f(x) \neq u(x)\}) < \varepsilon$$

and

$$\|u - f\|_{1,p} < \varepsilon\,.$$

Theorem 5 in the unweighted case is due to Liu [12]; see also [3, Section 6.6.3]. See [5] for an interesting study of Sobolev spaces, where Hajłasz defines Sobolev functions by using their Lipschitz property.

6. Remark. Taking a look at the proof of Theorem 4, we observe the following fact that is used later: *If* $u \in H^{1,p}(\mathbf{R}^n;\mu)$, $0 \le u \le 1$ *and* $M > 0$, *there exists a Lipschitz function* $v \in H^{1,p}(\mathbf{R}^n;\mu)$ *such that* $\operatorname{Lip}(v) \le M$,

$$\|u - v\|_p \le \frac{c_1}{M}\|\nabla u\|_p\,,$$

and

$$\|\nabla v\|_p \le c_2\|\nabla u\|_p + c_3\,,$$

where $c_j = c_j(n,p,\mu) > 0$.

Proof of Theorem 3: Fix $u \in W^{1,p}(\Omega;\mu)$. Let $D \subset\subset \Omega$ and choose a cut-off function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ on D. Then $u\eta \in W^{1,p}(\mathbf{R}^n;\mu)$ and hence $u\eta \in H^{1,p}(\mathbf{R}^n;\mu)$ by Theorem 4. Consequently, $u \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$. Because both u and ∇u are in $L^p(\Omega;\mu)$, u belongs to $H^{1,p}(\Omega;\mu)$ by [7, Lemma 1.15]. □

Combining Theorems 1 and 3 we obtain:

7. Theorem. *The space* $H^{1,p}(\Omega;\mu)$ *coincides with* $W^{1,p}(\Omega;\mu)$ *if and only if* $W^{1,p}(\Omega;\mu)$ *is a Banach space and the Poincaré inequality* (2) *holds for functions from* $W^{1,p}(\mathbf{R}^n;\mu)$.

Our argument yields a Rellich-Kondrachov theorem in weighted Sobolev spaces:

8. Theorem. *Suppose that* u_j *is a bounded sequence in* $H^{1,p}(\mathbf{R}^n;\mu)$. *Then there is a subsequence of* u_j *that converges pointwise a.e. and in* $L^q(\Omega;\mu)$ (*to a function* $u \in H^{1,p}(\mathbf{R}^n;\mu)$) *whenever* Ω *is bounded and* $1 \le q < \varkappa p$; *here* $\varkappa > 1$ *is the number in the Sobolev inequality* **III**.

Before we start the proof a few remarks are due: Usually one employs the Rellich-Kondrachov theorem when it is required to have stronger convergence than just weak. Therefore to establish the a.e. convergence is the only hard part of the theorem; then the Sobolev inequality takes care of the convergence in L^q.

Normally the Rellich-Kondrachov theorem is formulated for bounded sequences of Sobolev spaces on a nice, e.g. Lipschitz, domain. What is important is that the functions can be extended to the Sobolev space on $\mathbf{R}^n$ without loosing the boundedness in the norm. We pre-assume the boundedness in $H^{1,p}(\mathbf{R}^n;\mu)$ because extension properties in weighted spaces are not yet well understood (see, however [1], where Chua proves that if μ arises from an A_p-weight, then there is a bounded extension operator from $H^{1,p}(\Omega;\mu)$ to $H^{1,p}(\mathbf{R}^n;\mu)$, if Ω is for instance a Lipschitz domain; see also [15]).

9. Corollary. *Suppose that Ω is bounded and u_j is a bounded sequence in $H_0^{1,p}(\Omega;\mu)$. Then there is a subsequence of u_j that converges pointwise a.e. and in $L^q(\Omega;\mu)$ (to a function $u\in H_0^{1,p}(\Omega;\mu)$) whenever $1\le q<\varkappa p$; here $\varkappa>1$ is the number in* **III**.

Proof of Theorem 8: Let B be a fixed ball. By appealing to standard approximations by truncations and a diagonal subsequence argument we may assume that $0\le u_j\le 1$; also it suffices to select a subsequence that depends on B.

We claim that there is a subsequence of u_j which is a Cauchy sequence in $L^p(\Omega;\mu)$. To this end, fix $\varepsilon>0$ and choose for each j a c/ε-Lipschitz function v_j such that

$$\|v_j-u_j\|_p<\varepsilon$$

(see Remark 6). Because $0\le u_j\le 1$, we may (for ε small enough) assume that the sequence v_j is uniformly bounded in $\overline{B}$. Since the functions v_j are Lipschitz with a fixed constant, Ascoli's theorem gives us a uniformly on $\overline{B}$ convergent subsequence v_{j_i}. Thus

$$\|u_{j_k}-u_{j_i}\|_{L^p(\overline{B};\mu)}\le\|u_{j_k}-v_{j_k}\|_p+\|v_{j_k}-v_{j_i}\|_{L^p(\overline{B};\mu)}+\|v_{j_i}-u_{j_i}\|_p<3\varepsilon$$

when i and k are large enough. In the other words, u_{j_i} is a Cauchy sequence in $L^p(\overline{B};\mu)$ (and hence it has an a.e. convergent subsequence).

Now the theorem follows by interpolating: the cases where $q\le p$ are trivial and if $p<q<\varkappa p$, then we choose λ such that

$$\frac{1}{q}=\frac{\lambda}{p}+\frac{1-\lambda}{\varkappa p}$$

and obtain

$$\begin{aligned}\|u_{j_k}-u_{j_i}\|_{L^q(\overline{B};\mu)}&\le\|u_{j_k}-u_{j_i}\|^{\lambda}_{L^p(\overline{B};\mu)}\|u_{j_k}-u_{j_i}\|^{(1-\lambda)}_{L^{\varkappa p}(\overline{B};\mu)}\\&\le c\varepsilon^{\lambda},\end{aligned}$$

for the sequence

$$\|u_{j_k}-u_{j_i}\|_{L^{\varkappa p}(\overline{B};\mu)}$$

remains bounded by the Sobolev inequality **III**. The theorem follows. □

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Department of Mathematics, University of Jyväskylä, P.O. Box 35,
40351 Jyväskylä, Finland

E-mail: TeroK@math.jyu.fi

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