Stability in nonlinear evolution problems by means of fixed point theorems

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Abstract. The stabilization of solutions to an abstract differential equation is investigated. The initial value problem is considered in the form of an integral equation. The equation is solved by means of the Banach contraction mapping theorem or the Schauder fixed point theorem in the space of functions decreasing to zero at an appropriate rate. Stable manifolds for singular perturbation problems are compared with each other. A possible application is illustrated on an initial-boundary-value problem for a parabolic equation in several space variables.

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1. Introduction

In this work we study the stability and stabilization of solutions to nonlinear evolution problems by application of fixed point theorems in appropriate Banach spaces of functions with specific behaviour as time tends to infinity. To this purpose we interpret the evolution problems as differential equations in Banach spaces and take advantage of the theory of C_0-semigroups of operators, and other relevant tools as applied for example in Pazy [10]. We investigate a problem of type (2.1) stated below. There is a body of results in this area for which we refer the reader to Hale [4], Krein and Dalecki [8], Pazy [10]. Our approach is classical in the sense that we split the nonlinear problem into a linearized part and a nonlinear perturbation assuming the existence of a stable equilibrium by normalization placed at zero. This is close to the approach applied for example in Rauch [11]. Next, we invert the linear part and search for a stabilizing solution as a fixed point of the corresponding integral operator.

We start in Section 2 by a simple application of the Banach contraction principle in the space $L^\infty_w(0,\infty;X)$ of functions $u:(0,\infty)\to X$ (X a Banach space) which are measurable, essentially bounded and tending to zero at an appropriate rate $w(t)^{-1}$ as $t\to\infty$. This technique allows a local result only. The same technique is applied in Section 3 to a singularly perturbed Problem (3.1) making it

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possible to establish at least locally the inclusion $\Omega_0 \subset \Omega_\varepsilon$ between stable manifolds (3.2) for $\varepsilon = 0$ and $\varepsilon > 0$ small. A large data result is proved in Section 4 with the help of the Schauder fixed point theorem in the space $L^\infty_w(0,\infty;X)$. Some growth restriction at infinity is necessary for the nonlinear perturbation. In Section 5 we deal briefly with the stabilization of the solution to a singularly perturbed Problem (5.1); the investigation is brought to the point when the results of Sections 2 and 4 can be applied. Finally, in Section 6 the stabilization of the solution to a parabolic initial-boundary value problem for large data is investigated via method developed in Section 4.

We adopt the usual notation $L^p(M;X)$ for the L^p-spaces of functions from a set $M \subset R^N$ into a Banach space X, $W^{k,p}(M;X)$ for the Sobolev spaces of kth order, $C^k(M;X)$ for the spaces of functions with continuous derivatives up to the order k, $L(X,Y)$ for the space of the continuous linear operators from X into Y with $L(X) = L(X,X)$, and so on. By $B_r(0;X)$ we denote the ball centered at 0 with radius r in the Banach space X.

2. Stability by the Banach contraction principle

In this section we investigate the stability of the stationary solution corresponding to the evolutionary problem

$$
\begin{aligned}
u'(t) + (A+B)u(t) &= 0, \quad t > 0, \\
u(0) &= x.
\end{aligned} \tag{2.1}
$$

Here $A : D(A) \subset X \to X$ is a linear operator in a Banach space X, $B : X \to X$ is an operator (in general nonlinear) and x a given element of X. In order to establish the stability of the stationary point we shall make use of the Banach contraction principle in the space of functions $u : [0,\infty) \to X$ which decrease in an appropriate rate as $t \to \infty$.

We make the following assumptions:

(i) $-A$ is a generator of a C_0-semigroup $\{T(t)\}_{t \ge 0}$ of bounded linear operators in X;

(ii) $B = D - F$, where $D \in L(X)$ and $F : X \to X$, $F(0) = 0$;

(iii) the semigroup $\widetilde{T}(t)$ generated by $-(A+D)$ satisfies the estimate $|\widetilde{T}(t)| \le \widetilde{\omega}(t)$, $t \ge 0$, with some $\widetilde{\omega} \in L^\infty(0,\infty)$;

(iv) there exists $r_0 > 0$ and a continuous function $\lambda : [0, r_0) \to R^+$ with $\lambda(0) = 0$ such that for any $r \in (0, r_0)$ we have $|F(u) - F(v)| \le \lambda(r)|u - v|$ for $u, v \in B_r(0;X)$;

(v) $\mu(r) := \sup_{t \in R^+} w(t) \int_0^t \widetilde{\omega}(t-s) w(s)^{-1} \lambda(w(s)^{-1} r)\, ds < \infty$ for $r \in (0, r_0]$, $\limsup_{r \to 0+} \mu(r) < 1$ for some function $w \in L^\infty_{\mathrm{loc}}(0,\infty)$ such that $w(t) \ge 1$ a.e. in $(0,\infty)$ and $\lim_{t \to \infty} w(t) = \infty$.

Let us note that in concrete situations typically $\widetilde{\omega}(t) \leq Me^{-\alpha t}, w(t) = e^{\beta t}$, $\lambda(r) = cr^{\mu}$ with some constants $M > 0$, $0 < \beta < \alpha$, $c > 0$ and $\mu \in (0, 1]$.

We shall work in the space

$$(2.2)\quad L^{\infty}_{w}(0,\infty;X) = \{u \in L^{\infty}(0,\infty;X) : \|u\|_w := \operatorname{ess\,sup}_{t\geq 0} w(t)\,|u(t)| < \infty\}.$$

It is a standard result that the space $L^{\infty}_{w}(0,\infty;X)$ is a Banach space under the norm $\|\cdot\|_w$.

In the following lemma we prove that the operator

$$(2.3)\qquad G(u)(t) = \int_0^t \widetilde{T}(t-s)Fu(s)\,ds, \quad u \in B_r(0; L^{\infty}_{w}(0,\infty;X))$$

is well defined and maps $B_r(0; L^{\infty}_{w}(0,\infty;X))$ into itself if $r > 0$ is sufficiently small.

Lemma 2.1. *Let assumptions* (i)–(v) *be satisfied. Then there exists $r_1 \in (0, r_0]$ such that for any $r \in (0, r_1)$ the operator G defined by* (2.3) *maps the ball $B_r(0; L^{\infty}_{w}(0,\infty;X))$ into itself and*

$$(2.4)\qquad \|G(u) - G(v)\|_w \leq \mu(r)\|u - v\|_w \quad \textit{for} \quad u, v \in B_r(0; L^{\infty}_{w}(0,\infty;X)),$$

where $\mu(r) < 1$.

PROOF: If $u, v \in B_r(0; L^{\infty}_{w}(0,\infty;X))$, then by (2.3), (iii) and (iv) we have

$$\begin{aligned} w(t)|G(u)(t)| &\leq w(t)\int_0^t |\widetilde{T}(t-s)|\,|Fu(s)|\,ds \\ &\leq w(t)\int_0^t \widetilde{\omega}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\|u\|_w\,ds \\ &\leq w(t)\int_0^t \widetilde{\omega}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\,ds\cdot r \leq \mu(r)r. \end{aligned}$$

Analogously we get

$$\begin{aligned} w(t)|G(u)(t) - G(v)(t)| &\leq \int_0^t \widetilde{\omega}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\|u-v\|_w\,ds \\ &\leq \mu(r)\|u-v\|_w. \end{aligned}$$

The result follows immediately. □

Define an operator H by

$$(2.5)\quad H(u)(t) = \widetilde{T}(t)x + G(u)(t); \quad u \in B_r(0; L^{\infty}_{w}(0,\infty;X)),\ t > 0,\ r \in (0, r_0).$$

If

$$(2.6)\qquad \limsup_{t\to\infty} w(t)\widetilde{\omega}(t) < \infty,$$

then H maps $B_r(0; L^{\infty}_{w}(0,\infty;X))$ into $L^{\infty}_{w}(0,\infty;X)$. We have the following result.

Theorem 2.2. *Let assumptions* (i)–(v) *be satisfied and let* (2.6) *hold. If* $|x|$ $(x \in X)$ *is sufficiently small, then the operator* H *defined by* (2.5) *has a unique fixed point in* $B_r(0; L^\infty_w(0,\infty;X))$ *for* $r > 0$ *small enough. This fixed point is a generalized solution of* (2.1), *and it satisfies* $|u(t)| \le r w(t)^{-1}$ *for* $t \ge 0$; *in particular we have* $\lim_{t\to\infty} u(t) = 0$ *in* X.

PROOF: By (2.6) there exists $\widetilde{c} > 0$ such that $w(t)\widetilde{\omega}(t) \le \widetilde{c}$, for $t \ge 0$. We choose $r > 0$ in Lemma 2.1 so that $\lambda(r) \le \frac{1}{2}$ and $\mu(r) < 1$. Then for $u, v \in B_r(0; L^\infty_w(0,\infty;X))$ and $\widetilde{c}|x| \le \frac{1}{2}r$ we have

$$\begin{aligned} w(t)|H(u)(t)| &\le w(t)|\widetilde{T}(t)x| + w(t)|G(u)(t)| \le w(t)\widetilde{\omega}(t)|x| + \lambda(r)r \\ &\le \widetilde{c}|x| + \tfrac{1}{2}r \le \tfrac{1}{2}r + \tfrac{1}{2}r = r, \end{aligned}$$

and

$$w(t)|H(u)(t) - H(v)(t)| = w(t)|G(u)(t) - G(v)(t)| \le \mu(r)\|u - v\|_w.$$

Thus H maps $B_r(0; L^\infty_w(0,\infty;X))$ into itself and is contractive therein. We apply the Banach contraction principle in $B_r(0; L^\infty_w(0,\infty;X))$ to obtain the result. $\square$

3. Comparison of stable manifolds for a singularly perturbed problem

In this section we study the parameter dependent problem

$$\text{(3.1)} \qquad \begin{aligned} u_\varepsilon'(t) + (\varepsilon A + B)u_\varepsilon(t) = 0, &\quad t > 0, \\ u_\varepsilon(0) = x, &\quad \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 > 0,\ x \in X. \end{aligned}$$

As in Section 2, $A : D(A) \subset X \to X$ is a linear operator in X whereas $B : X \to X$ may be nonlinear. We are going to consider stable manifolds associated with Problem (3.1), that is the sets

$$\Omega_\varepsilon = \{x \in X : u_\varepsilon(t) \to 0 \ \text{ in } \ X \ \text{ as } \ t \to \infty\},$$

where u_ε is the generalized solution of (3.1) (dependent on x).

The aim of this section is to derive conditions under which there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ we have $\Omega_0 \subset \Omega_\varepsilon$ at least locally. To achieve this we shall establish convergence $u_\varepsilon(t) \to u_0(t)$ in X as $\varepsilon \to 0+$ pointwise on compact intervals, and universal stability of $u = 0$ for $\varepsilon \in (0, \varepsilon^*)$ (that means that there exists $r > 0$ such that $u_\varepsilon(t) \to 0$ in X as $t \to \infty$ whenever $|x| \le r$ and $\varepsilon \in (0, \varepsilon^*)$).

In what follows we shall make some assumptions.

(a) Let assumptions (i), (ii) and (iv) of Section 2 be satisfied. Further, denote by $\widetilde{\omega}_\varepsilon$ a $L^\infty_{\rm loc}(0,\infty)$ function such that the semigroup $\widetilde{T}_\varepsilon(t)$ generated by $-(\varepsilon A + D)$ satisfies the estimate

$$\text{(3.3)} \qquad |\widetilde{T}_\varepsilon(t)| \le \widetilde{\omega}_\varepsilon(t), \quad t \ge 0,\ \varepsilon \in (0, \varepsilon_0],$$

and introduce a family of weight functions $w_\varepsilon \in L^\infty_{\mathrm{loc}}(0,\infty)$ such that $w_\varepsilon(t) \geq 1$ a.e. in $(0,\infty)$ and $\lim_{t\to\infty} w_\varepsilon(t) = \infty$ $(\varepsilon \in (0,\varepsilon_0])$.

We make also the following assumption.

(b) There exist $r_1 \in (0, r_0]$ and $\kappa \in (0,1)$ such that we have
$$\mu_\varepsilon(r) := \sup_{t\in R^+} w_\varepsilon(t) \int_0^t \widetilde{\omega}_\varepsilon(t-s) w_\varepsilon(s)^{-1} \lambda(w_\varepsilon(s)^{-1} r)\, ds \leq \kappa$$
for $r \in (0, r_1]$, $\varepsilon \in (0, \varepsilon_0]$.

Lemma 3.1. *Let assumptions* (i), (ii) *of Section* 2 *be satisfied. In addition, let us assume that F is locally Lipschitz, that there exists a family of generalized solutions $u_\varepsilon(t)$ of Problem* (3.1) *which is uniformly bounded in $L^\infty(0,t_0)$ $(t_0 > 0)$ with respect to $\varepsilon \in (0,\varepsilon_0]$, and that there is a constant $\widetilde{c} > 0$ such that $\widetilde{\omega}_\varepsilon(t) \leq \widetilde{c}$ for $t \in [0,t_0]$, $\varepsilon \in (0,\varepsilon_0]$. Then $u_\varepsilon(t) \to u_0(t)$ in X as $\varepsilon \to 0+$ for all $t \in (0,t_0]$.*

Remark 3.2. Later on we shall provide some conditions which guarantee the existence and uniform boundedness of $u_\varepsilon(t)$ as assumed in Lemma 3.1.

Proof of Lemma 3.1: First, let us show that for any $x \in X$ and any $t > 0$ we have $\lim_{\varepsilon\to 0+} \widetilde{T}_\varepsilon(t)x = \widetilde{T}_0(t)x$. To prove this, consider the operator $S_\varepsilon(s) = T_\varepsilon(t-s)\widetilde{T}_\varepsilon(s)$, where $T_\varepsilon(\tau)$ is the semigroup generated by $(-\varepsilon A)$. Then, for $x \in D(A)$, the function $s \mapsto S_\varepsilon(s)x$ is differentiable and $S'_\varepsilon(s)x = -T_\varepsilon(t-s)D\widetilde{T}_\varepsilon(s)x$. Integrating $S'_\varepsilon(s)x$ from 0 to t yields

$$\widetilde{T}_\varepsilon(t)x = T_\varepsilon(t)x - \int_0^t T_\varepsilon(t-s)D\widetilde{T}_\varepsilon(s)x\, ds \quad \text{for } x \in D(A). \tag{3.4}$$

By continuity, the relation (3.4) can be extended to all $x \in X$. Further, $T_\varepsilon(t) = T(\varepsilon t)$ since $T(\tau)$ is generated by $-A$ and, consequently, $\lim_{\varepsilon\to 0+} |T_\varepsilon(t)x - x| = 0$ for any $x \in X$ locally uniformly with respect to t. In addition, there exist constants M, ω such that

$$|T_\varepsilon(t)| = |T(\varepsilon t)| \leq Me^{\varepsilon\omega t} \leq Me^{\varepsilon_0\omega t} \quad \text{for } t \geq 0. \tag{3.5}$$

By subtraction with $\varepsilon > 0$ and with $\varepsilon = 0$ we obtain

$$\begin{aligned}\widetilde{T}_\varepsilon(t)x - \widetilde{T}_0(t)x = (T(\varepsilon t)x - x) &- \int_0^t \big[T(\varepsilon(t-s)) - I\big] D\widetilde{T}_0(s)x\, ds \\ &- \int_0^t T(\varepsilon(t-s))D(\widetilde{T}_\varepsilon(s)x - \widetilde{T}_0(s)x)\, ds.\end{aligned} \tag{3.6}$$

By the continuity of T the first term tends to 0 as $\varepsilon \to 0+$ pointwise in t. The same is true for the second term by the Lebesgue dominated convergence theorem. Denoting $\varphi_\varepsilon(t) = |\widetilde{T}_\varepsilon(t)x - \widetilde{T}_0(t)x|$, from (3.6) we obtain with the aid of (3.5)

$$\varphi_\varepsilon(t) \leq g_\varepsilon(t) + C \int_0^t \varphi_\varepsilon(s)\, ds$$

with some function g_ε satisfying $\lim_{\varepsilon\to 0+} g_\varepsilon(t) = 0$ and with a constant C. The Gronwall lemma yields

$$\varphi_\varepsilon(t) \le g_\varepsilon(t) + Ce^{Ct}\int_0^t g_\varepsilon(s)e^{-Cs}\,ds$$

and the Lebesgue dominated convergence theorem implies $\lim_{\varepsilon\to 0+}\varphi_\varepsilon(t) = 0$.

The generalized solution $u_\varepsilon(t)$ of (3.1) satisfies the relation

$$u_\varepsilon(t) = \widetilde{T}_\varepsilon(t)x + \int_0^t \widetilde{T}_\varepsilon(t-s)Fu_\varepsilon(s)\,ds. \tag{3.7}$$

By subtraction we obtain

$$u_\varepsilon(t) - u_0(t) = \widetilde{T}_\varepsilon(t)x - \widetilde{T}_0(t)x + \int_0^t \big[\widetilde{T}_\varepsilon(t-s)Fu_\varepsilon(s) - \widetilde{T}_0(t-s)Fu_0(s)\big]\,ds.$$

This, the uniform boundedness of u_ε and the fact that F is locally Lipschitz yield

$$\begin{aligned}|u_\varepsilon(t) - u_0(t)| \le |\widetilde{T}_\varepsilon(t)x - \widetilde{T}_0(t)x| &+ \int_0^t |\widetilde{T}_\varepsilon(t-s)Fu_0(s) - \widetilde{T}_0(t-s)Fu_0(s)|\,ds\\ &+ \int_0^t |\widetilde{T}_\varepsilon(t-s)|\,|Fu_\varepsilon(s) - Fu_0(s)|\,ds\\ &\le a_\varepsilon(t) + C\int_0^t |u_\varepsilon(s) - u_0(s)|\,ds,\end{aligned}$$

where $\lim_{\varepsilon\to 0+} a_\varepsilon(t) = 0$ by what we have proved above, and C is a constant. The same Gronwall lemma argument as above yields $\lim_{\varepsilon\to 0+}|u_\varepsilon(t) - u_0(t)| = 0$.

Our next step is the universal stability of the stationary solution $u = 0$.

Theorem 3.3. *Let assumptions* (a) *and* (b) *be satisfied and let*

$$w_\varepsilon(t)\widetilde{\omega}_\varepsilon(t) \le \widetilde{C} < \infty \ \text{ for all } t \in [0,\infty) \text{ and } \varepsilon \in (0,\varepsilon_0] \tag{3.8}$$

with a constant $\widetilde{C}$ independent of t and ε. Then there exists $R > 0$ such that for any $x \in X$ with $|x| \le R$ the generalized solution $u_\varepsilon(t)$ of (3.1) *converges to* 0 *as $t \to \infty$ for all $\varepsilon \in (0,\varepsilon_0]$.*

Proof: As in Section 2 we intend to apply the Banach contraction principle to the operator H_ε given by

$$H_\varepsilon(u)(t) = \widetilde{T}_\varepsilon(t)x + \int_0^t \widetilde{T}_\varepsilon(t-s)Fu(s)\,ds. \tag{3.9}$$

It suffices to show that there is $R > 0$ such that if $x \in X$, $|x| \le R$ and $\varepsilon \in (0,\varepsilon_0]$, then there exists $r > 0$ such that H_ε maps the ball $B_r(0; L^\infty_{w_\varepsilon}(0,\infty;X))$ into itself and is a contraction in that ball.

So, let $R > 0$, $|x| \le R$, $r \in (0, r_1]$ and $u, v \in B_r(0; L^\infty_{w_\varepsilon}(0,\infty; X))$. Then, for $R \le (1-\kappa)r/\widetilde{C}$, we have

$$\begin{aligned} w_\varepsilon(t)|H_\varepsilon(u)(t)| &\le w_\varepsilon(t)\widetilde{\omega}_\varepsilon(t)\, R + w_\varepsilon(t)\int_0^t \widetilde{\omega}_\varepsilon(t-s)\lambda(w_\varepsilon(s)^{-1}r)w_\varepsilon(s)^{-1}\, ds\, \|u\|_{w_\varepsilon} \\ &\le \widetilde{C}R + \mu_\varepsilon(r)r \le \widetilde{C}R + \kappa r \le r, \end{aligned}$$

and consequently $\|H_\varepsilon(u)\|_{w_\varepsilon} \le r$. Similarly we have

$$\begin{aligned} &w_\varepsilon(t)|H_\varepsilon(u)(t) - H_\varepsilon(v)(t)| \\ &\qquad \le w_\varepsilon(t)\int_0^t \widetilde{\omega}_\varepsilon(t-s)\lambda(w_\varepsilon(s)^{-1}r)w_\varepsilon(s)^{-1}\, ds\, \|u-v\|_{w_\varepsilon} \le \kappa\|u-v\|_{w_\varepsilon}, \end{aligned}$$

and the assertion easily follows. □

The following main theorem of this section gives a local comparison result.

Theorem 3.4. *Let assumptions* (a), (b) *and* (3.8) *be satisfied. In addition, let F be locally Lipschitz and for any $x \in X$ and any $t_0 > 0$ let there exist a family of generalized solutions $u_\varepsilon(t)$ of Problem* (3.1) *which is uniformly bounded in $L^\infty(0, t_0)$ with respect to $\varepsilon \in (0, \varepsilon_0]$. Then for any $x \in \Omega_0$ there exists an $\varepsilon^* \in (0, \varepsilon_0]$ such that $\lim_{\varepsilon\to 0+} u_\varepsilon(t) = 0$, $\varepsilon \in (0, \varepsilon^*)$, $t \in [0, t_0]$ for corresponding solutions $u_\varepsilon(t)$ of* (3.1).

Proof: If $x \in \Omega_0$ then $\lim_{t\to\infty} u_0(t) = 0$ in X, where $u_0(t)$ is a generalized solution of (3.1) with $\varepsilon = 0$. Take $R > 0$ whose existence is guaranteed by Theorem 3.3 and find $t_0 > 0$ such that $|u_0(t)| \le \frac{1}{2}R$ for all $t \ge t_0$. Let $u_\varepsilon(t)$, $\varepsilon \in (0, \varepsilon_0]$ be the family of generalized solutions of (3.1). In view of our assumptions Lemma 3.1 may be applied on the interval $(0, t_0]$ to obtain that, in particular, there exists $\varepsilon^* \in (0, \varepsilon_0]$ such that $|u_\varepsilon(t_0) - u_0(t_0)| \le \frac{1}{2}R$ for $\varepsilon \in (0, \varepsilon^*)$. Thus we get

$$|u_\varepsilon(t_0)| \le |u_0(t_0)| + |u_\varepsilon(t_0) - u_0(t_0)| \le \tfrac{1}{2}R + \tfrac{1}{2}R = R.$$

But taking $x = u_\varepsilon(t_0)$, by Theorem 3.3 we can construct a solution $v_\varepsilon(t)$ of (3.1), which converges to 0 as $t \to \infty$ ($\varepsilon \in (0, \varepsilon^*)$). By uniqueness ($F$ is locally Lipschitz continuous) we get $v_\varepsilon(t) = u_\varepsilon(t_0 + t)$ so that $\lim_{t\to\infty} u_\varepsilon(t) = 0$ in X for $\varepsilon \in (0, \varepsilon^*)$ as well. □

Remark 3.5. If $x \in D(A)$ then it may be easily seen that there exists $t_1 > 0$ such that $|T(t)x - x| \le t(1 + |Ax|)$. Consequently in the proof of Lemma 3.1 we have $|T(\varepsilon t)x - x| \le \varepsilon t_0(1 + |Ax|)$ for $0 \le \varepsilon \le t_1/t_0 =: \varepsilon_1$ and by (3.6) and the Gronwall lemma it follows $|\widetilde{T}_\varepsilon(t)x - \widetilde{T}_0(t)x| \le \operatorname{const}\varepsilon(1 + |Ax|)$, $t \in [0, t_0]$. This yields $|u_\varepsilon(t) - u_0(t)| \le \operatorname{const}\varepsilon(1 + |Ax|)$, $t \in [0, t_0]$, $\varepsilon \in [0, \varepsilon_1]$. Now, from the proof of Theorem 3.4 for any $R > 0$ we get the existence of $\varepsilon \in (0, \varepsilon^*)$ such that $\Omega_0 \cap B_R(0; D(A)) \subset \Omega_\varepsilon \cap B_R(0; D(A))$ for all $\varepsilon \in (0, \varepsilon^*)$, where $D(A)$ is equipped with the graph norm $|x|_{D(A)} = |x| + |Ax|$ for $x \in D(A)$. For strongly positive operators (see Krasnosel'skij [7]) the requirement $x \in D(A)$ can be relaxed to $x \in D(A^\alpha)$ for some $\alpha > 0$ with the obvious modification in the assertion. The rate of convergence is then of order ε^α.

We give a reference to the question of the existence and the uniform boundedness of the solutions u_ε to (3.1). For Problem (2.1), a necessary and sufficient condition for the existence of a solution is given in Iwamiya, Takahashi and Oharu [5] in a quite general setting, and it may be modified for the singularly perturbed Problem (3.1). We present here a rather less general result of Crandall and Liggett [2] which is sufficient for our purpose.

Proposition 3.6. *Let assumption* (i) *of Section* 2 *be satisfied and let $B : X \to X$ be continuous. Denote by $\langle \cdot, \cdot \rangle$ the pairing between X and its dual space X^*, $\Phi(y) := \{\varphi \in X^* : \langle y, \varphi \rangle = |y|^2 = |\varphi|^2\}$ and $\langle z, y \rangle_i = \inf\{\langle z, \varphi \rangle : \varphi \in \Phi(y)\}$ for $z, y \in X$. If there exists $\alpha \in R$ such that*

$$\langle (\varepsilon A + B)y - (\varepsilon A + B)z, y - z \rangle_i \le \alpha |y - z|^2 \text{ for } y, z \in D(A), \varepsilon \in (0, \varepsilon_0],$$

and

$$R(I - \lambda(\varepsilon A + B)) = X \text{ for } \lambda \in (0, \infty) \text{ with } \lambda\alpha < 1 \text{ and } \varepsilon \in (0, \varepsilon_0],$$

then for each $\varepsilon \in (0, \varepsilon_0]$ there is a nonlinear continuous semigroup $S_\varepsilon := \{S_\varepsilon(t) : t \ge 0\}$ of continuous operators on X such that

$$S_\varepsilon(t)x = T(\varepsilon t)x + \int_0^t T(\varepsilon(t-s))BS_\varepsilon(s)x\,ds \text{ for } t \ge 0 \text{ and } x \in X, \tag{3.10}$$

$$|S_\varepsilon(t)x - S_\varepsilon(t)y| \le e^{\alpha t}|x - y| \text{ for } t \ge 0 \text{ and } x, y \in X. \tag{3.11}$$

Remark 3.7. Proposition 3.6 yields a generalized solution $u_\varepsilon(t) = S_\varepsilon(t)x$ of (3.1) satisfying (3.7) by (3.10). In our case the generalized solution is uniformly bounded since $S_\varepsilon(t)(0) \equiv 0$, and by (3.11) we have

$$|u_\varepsilon(t)| = |S_\varepsilon(t)x - S_\varepsilon(t)(0)| \le e^{\alpha t}|x| \le \text{const} \text{ for } t \in [0, t_0] \ (t_0 > 0).$$

The uniqueness follows from the local Lipschitz continuity of F by a standard Gronwall lemma argument.

4. Stability by the Schauder fixed point theorem

We consider the problem

$$u'(t) + (A + B)u(t) = 0, \quad t > 0, \tag{4.1}$$

$$u(0) = x, \tag{4.2}$$

where A is a linear (in general unbounded) operator in a Banach space X, which is a generator of a C_0-semigroup of bounded linear operators $T(t)$, $t \geq 0$, and $B : X \to X$ is a continuous (in general nonlinear) operator. We say that (4.1), (4.2) has a generalized solution if there is a function $u \in L^\infty_{\mathrm{loc}}(0,\infty;X)$ such that

$$u(t) = T(t)x - \int_0^t T(t-s)Bu(s)\,ds, \quad t \geq 0. \tag{4.3}$$

We assume that there exists $u = \overline{u} \in X$ such that

$$(A+B)\overline{u} = 0, \quad \overline{u} \in D(A), \tag{4.4}$$

(where $D(A)$ is the definition domain of A) and intend to establish conditions under which $u(t) \to \overline{u}$ as $t \to \infty$ in X.

We suppose that $\overline{u} = 0$ since the general case can be easily reduced to this case, so that (4.4) can be written as

$$B(0) = 0. \tag{4.5}$$

To show that under appropriate assumptions $u(t) \to 0$ as $t \to \infty$ in X we shall make use of the Schauder fixed point theorem to find the solution of (4.3) in an appropriate Banach space of functions $u : R^+ \to X$ for which $u(t) \to 0$ in X as $t \to \infty$.

Let $w \in L^\infty_{\mathrm{loc}}(0,\infty)$ be such that $w(t) \geq 1$ a.e. in $(0,\infty)$ and $\lim_{t\to\infty} w(t) = \infty$. As in Section 2 we define

$$L^\infty_w(0,\infty;X) = \{u \in L^\infty(0,\infty;X) : \|u\|_w := \operatorname{ess\,sup}_{t\geq 0} w(t)\,|u(t)| < \infty\}. \tag{4.6}$$

In the sequel we shall need the following compactness criterion which we prove for reader's convenience.

Proposition 4.1. *A set $K \subset L^\infty(0,\infty;X)$ is relatively compact in $L^\infty_w(0,\infty;X)$ if the following conditions hold:*

(i) *there is a set $M \subset (0,\infty)$ of measure zero such that for any $t \in (0,\infty)\backslash M$ the orbit $X_t = \{f(t) : f \in K\}$ of t under K is relatively compact in X;*
(ii) *for any $\varepsilon > 0$ there is a finite partition of $(0,\infty)$ into measurable sets $A_1,\dots,A_n$ such that*

$$\operatorname{ess\,sup}_{s,t\in A_j} |w(s)f(s) - w(t)f(t)| < \varepsilon$$

for all $j \in \{1,\dots,n\}$ and all $f \in K$.

Proof: Let $\varepsilon > 0$. There is a measurable partition $A_1,\dots,A_n$ of $(0,\infty)$ and a set $B \subset (0,\infty)$ of measure zero such that $|w(s)f(s) - w(t)f(t)| < \frac{1}{3}\varepsilon$ whenever $t,s \in A_j\backslash B$ for some j and $f \in K$. We may assume that $A_j\backslash(M \cup B)$ is nonempty for all j. Choose points $t_j \in A_j \backslash (M \cup B)$ for $j \in \{1, \dots, n\}$ and define a map $P : K \to X^n$ by

$$P(f) = (w(t_1)f(t_1), \dots, w(t_n)f(t_n)) \text{ for all } f \in K.$$

The set $P(K)$ is relatively compact in X being a subset of a relatively compact set $w(t_1)X_{t_1} \times \cdots \times w(t_n)X_{t_n}$. Let $\{P(f_1), \dots, P(f_p)\}$ be a $\frac{1}{3}\varepsilon$-net for $P(K)$ with respect to the norm $|(x_1, \dots, x_n)| = \max(|x_1|, \dots, |x_n|)$. We show that $\{f_1, \dots, f_p\}$ is an ε-net for K: Let $f \in K$. There is $k \in \{1, \dots, p\}$ such that $|P(f) - P(f_k)| < \frac{1}{3}\varepsilon$. Given $t \in (0, \infty) \backslash (M \cup B)$, there is $j \in \{1, \dots, n\}$ such that $t \in A_j \backslash (M \cup B)$. So

$$\begin{aligned} |w(t)f(t) - w(t)f_k(t)| &\le |w(t)f(t) - w(t_j)f(t_j)| + |w(t_j)f(t_j) - w(t_j)f_k(t_j)| \\ &\quad + |w(t_j)f_k(t_j) - w(t)f_k(t)| \\ &< \tfrac{1}{3}\varepsilon + |P(f) - P(f_k)| + \tfrac{1}{3}\varepsilon < \varepsilon, \end{aligned}$$

which shows that $\|f - f_k\|_w < \varepsilon$. □

Assume that

$$B = D - F \text{ where } D \in L(X) \text{ and } F \text{ is locally Lipschitz.} \tag{4.7}$$

Then the operator $-(A + D)$ generates a C_0-semigroup $\widetilde{T}(t)$, $t \ge 0$, of continuous linear operators in X (see Pazy [10, Section 3.1, Theorem 1.1]). It can be shown that the operator F is locally bounded and that it transforms each strongly measurable function from $(0, \infty)$ into X onto a strongly measurable function from $(0, \infty)$ into X. Let $u \in L^\infty_{\text{loc}}(0, \infty; X)$. Then the function

$$v(t) = \int_0^t \widetilde{T}(t - s)Fu(s)\, ds$$

belongs to $L^\infty_{\text{loc}}(0, \infty; X)$ as well. Indeed, given $t_0 > 0$ we have $|Fu(s)| \le \text{const}$ a.e. in $(0, t_0)$ and there exist $M > 0$ and $\omega \in R$ such that $|\widetilde{T}(\tau)| \le Me^{\omega\tau}$, $\tau \ge 0$, which yields the desired estimate. Define

$$G(u)(t) = \int_0^t \widetilde{T}(t - s)Fu(s)\, ds, \quad t \ge 0, \text{ for } u \in L^\infty_{\text{loc}}(0, \infty; X). \tag{4.8}$$

Then G maps $L^\infty_{\text{loc}}(0, \infty; X)$ into itself. Now, more generally, let there exist a function $\widetilde{\omega} \in L^1_{\text{loc}}([0, \infty))$ such that

$$|\widetilde{T}(\tau)| \le \widetilde{\omega}(\tau), \quad \tau > 0. \tag{4.9}$$

We have the following.

Lemma 4.2. *Let* (4.7) *hold and let the weight function* w *be such that*

$$\limsup_{t\to\infty} w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,ds < \infty. \tag{4.10}$$

Then G *maps* $L_w^\infty(0,\infty;X)$ *into itself and is locally Lipschitz.*

PROOF: By (4.10) we have $w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,ds \le \text{const} < \infty$ for $t \ge 0$. Since for $u \in L_w^\infty(0,\infty;X)$ we have $|u(s)| \le \text{const} < \infty$, a.e. and $F(0)=0$, by (4.7) we have $|Fu(s)| \le k|u(s)|$ a.e. with some $k>0$. Hence

$$|\widetilde{T}(t-s)Fu(s)| \le \widetilde{\omega}(t-s)k|u(s)| \le k\|u\|_w\,\widetilde{\omega}(t-s)w(s)^{-1},$$

and consequently

$$|w(t)G(u)(t)| \le kw(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,ds\,\|u\|_w \le \text{const}\,\|u\|_w$$

which yields the first result.

Now, having $u,v \in L_w^\infty(0,\infty;X)$, $\|u\|_w$, $\|v\|_w \le R$ $(R>0)$, we have u,v bounded, and by (4.7) there is $k=k(R)>0$ such that $|Fu(s)-Fv(s)| \le k|u(s)-v(s)|$ a.e. Thus we obtain

$$w(t)|G(u)(t)-G(v)(t)| \le kw(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,ds\,\|u-v\|_w,$$

which yields the Lipschitz continuity of G in the ball $B_R(0;L_w^\infty(0,\infty;X))$. □

In the following lemma we give a sufficient condition for G to map some ball in $L_w^\infty(0,\infty;X)$ into itself.

Lemma 4.3. *Let there exist a nondecreasing function* φ *such that*

$$|F(u)| \le \varphi(|u|) \quad \text{for all} \quad u \in X, \tag{4.11}$$

$$\kappa := \sup_{\sigma>0}(\sigma^{-1}\varphi(\sigma))\sup_{t>0} w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,ds < 1, \tag{4.12}$$

$$\widetilde{S} := \sup_{s>0} w(s)\widetilde{\omega}(s) < \infty. \tag{4.13}$$

Then for any $R>0$ *we have* $\|G(u)\|_w \le \kappa R$ *for all* $u \in B_R(0;L_w^\infty(0,\infty;X))$*; for any* $x \in X$ *there exists* $R>0$ *such that the mapping* H *defined by* (2.5) *maps* $B_R(0;L_w^\infty(0,\infty;X))$ *into itself. The radius* R *can be chosen independently of* $x \in B_r(0;X)$ *for any fixed* $r>0$.

Proof: Let $R > 0$ and $u \in B_R(0; L^\infty_w(0,\infty;X))$. Then we have

$$\begin{aligned}
w(t)|G(u)(t)| &\le w(t)\int_0^t \widetilde{\omega}(t-s)|F(u(s))|\,ds \le w(t)\int_0^t \widetilde{\omega}(t-s)\varphi(|u(s)|)\,ds\\
&= w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}w(s)R^{-1}\varphi(w(s)^{-1}w(s)|u(s)|)\,ds\,R\\
&\le w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}w(s)R^{-1}\varphi(w(s)^{-1}R)\,ds\,R\\
&\le \sup_{\sigma>0}\,(\sigma^{-1}\varphi(\sigma))\,w(t)\int_0^t \widetilde{\omega}(t-s)w(s)^{-1}\,dsR \le \kappa R.
\end{aligned}$$

Hence, if $r > 0$ and $x \in B_r(0;X)$ then

$$w(t)|H(u)(t)| \le \sup_{s\ge 0}\,(w(s)\widetilde{\omega}(s))\,|x| + \kappa R \le \widetilde{S}r + \kappa R.$$

Now it suffices to take R so large that $\widetilde{S}r + \kappa R \le R$. □

For a typical example, let us note that, if $|F(u)| \le c_0|u|^{\mu_0}$ for $|u| \le \eta_0$, $|F(u)| \le c_1|u|^{\mu_1}$ for $|u| > \eta_0$ with some constants $c_0, c_1 > 0$, $\mu_0 \ge 1$, $\mu_1 \in [0,1)$, $\eta_0 > 0$ and $\widetilde{\omega}(t) = e^{-\alpha t}$, $w(t) = e^{\beta t}$ with $0 < \beta < \alpha$ then κ in (4.12) is estimated from above by $(\alpha-\beta)^{-1}\max\{c_0\eta_0, c_1/\eta_0\}$.

Lemma 4.4. *Let $K \subset L^\infty_w(0,\infty;X)$ be bounded and let $Y \hookrightarrow\hookrightarrow X$ be a Banach space such that either*

(i) *$\widetilde{T} \in L^1_{\mathrm{loc}}(0,\infty;L(X,Y))$ and $F : X \to X$ is locally bounded*
or
(ii) *$\widetilde{T} \in L^1_{\mathrm{loc}}(0,\infty;L(Y))$ and $F : X \to Y$ is locally bounded from X to Y.*

Then for any $t \ge 0$ the set $\{G(u)(t) : u \in K\}$ is relatively compact in X.

Proof: Let (i) hold. For any $u \in K$ we have

$$\begin{aligned}
\Big|\int_0^t \widetilde{T}(t-s)Fu(s)\,ds\Big|_Y &\le \int_0^t |\widetilde{T}(t-s)|_{L(X,Y)}\,|Fu(s)|\,ds\\
&\le \text{const}\int_0^t |\widetilde{T}(s)|_{L(X,Y)}\,ds
\end{aligned}$$

since $|u(s)| \le \text{const}\,w(s)^{-1} \le \text{const}$ for almost all $s \in (0,t)$ and all $u \in K$. So, for all $t \ge 0$, $\{G(u)(t) : u \in K\}$ is bounded in Y and, by the compactness of the imbedding, relatively compact in X.

If (ii) holds, then similarly we have

$$\Big|\int_0^t \widetilde{T}(t-s)Fu(s)\,ds\Big|_Y \le \int_0^t |\widetilde{T}(t-s)|_{L(Y)}\,|Fu(s)|_Y\,ds \le \text{const}\int_0^t |\widetilde{T}(s)|_{L(Y)}\,ds$$

with the same conclusion. □

The next lemma provides a sufficient condition for (ii) of Proposition 4.1.

Lemma 4.5. *Let $Y \hookrightarrow X$ be a Banach space with the norm $|\cdot|_Y$, $w \in C([0,\infty))$ and let there exist $r_0 > 0$, $\delta_0 > 0$ and functions $\eta \geq 0$, $\psi \in L^1_{\mathrm{loc}}([0,\infty); R^+)$ such that the following relations are satisfied:*

$$\left\{\begin{array}{l} |F(u)| \leq \mathrm{const}\, |u|^{1+\delta_0} \ \ \textit{for} \ \ u \in B_{r_0}(0;X); \\ |\widetilde{T}(\tau)y - y| \leq \eta(\tau)|y|_Y \ \ \textit{for} \ \ y \in Y \ \ \textit{and} \ \ \tau \geq 0; \\ |\widetilde{T}(\tau)x|_Y \leq \psi(\tau)|x| \ \ \textit{for} \ \ x \in X \ \ \textit{and} \ \ \tau > 0; \\ \limsup\limits_{\tau\to 0+} \eta(\tau) = 0; \\ \limsup\limits_{t\to\infty} w(t)\displaystyle\int_0^t \widetilde{\omega}(t-\tau)w(\tau)^{-1-\delta_0}\,d\tau = 0. \end{array}\right. \tag{4.14}$$

If $M \subset L^\infty_w(0,\infty;X)$ is bounded, then the set $K = G(M) \subset L^\infty_w(0,\infty;X)$ satisfies condition (ii) *of Proposition* (4.1)*.*

Proof: First, let $t^* > 0$ be arbitrary and assume that $t^* < s < t < \infty$. If $u \in M$ and $v = G(u)$, we can write

$$\begin{aligned} w(t)v(t) - w(s)v(s) &= \int_0^t w(t)\widetilde{T}(t-\tau)Fu(\tau)\,d\tau \\ &\quad - \int_0^s w(s)\widetilde{T}(s-\tau)Fu(\tau)\,d\tau \\ &= \int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)Fu(\tau)\,d\tau \\ &\quad + \int_s^t w(t)\widetilde{T}(t-\tau)Fu(\tau)\,d\tau. \end{aligned} \tag{4.15}$$

Since $|u(\tau)| \leq \mathrm{const}\, w(\tau)^{-1}$ and since F is locally Lipschitz continuous, then $|Fu(\tau)| \leq \mathrm{const}\,|u(\tau)|^{1+\delta_0}$ even if $|u(\tau)| \geq r_0$, and consequently

$$\begin{aligned} &\Big|\int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)Fu(\tau)\,d\tau\Big| \\ &\quad \leq \mathrm{const} \int_0^s (w(t)\widetilde{\omega}(t-\tau) + w(s)\widetilde{\omega}(s-\tau))w(\tau)^{-1-\delta_0}\,d\tau\, \|u\|_w^{1+\delta_0} \\ &\quad \leq \mathrm{const}\left(w(s)\int_0^s \widetilde{\omega}(s-\tau)w(\tau)^{-1-\delta_0}\,d\tau\right. \\ &\qquad \left. + w(t)\int_0^t \widetilde{\omega}(t-\tau)w(\tau)^{-1-\delta_0}\,d\tau\right)\|u\|_w^{1+\delta_0}. \end{aligned} \tag{4.16}$$

Similarly we get

$$w(t)\Big|\int_s^t \widetilde{T}(t-\tau)Fu(\tau)\,d\tau\Big| \leq \mathrm{const}\, w(t)\int_{t^*}^t \widetilde{\omega}(t-\tau)w(\tau)^{-1-\delta_0}\,d\tau\, \|u\|_w^{1+\delta_0}. \tag{4.17}$$

So by (4.14), for any $\varepsilon > 0$, t^* can be chosen so that

$$|w(t)v(t) - w(s)v(s)| < \tfrac{1}{2}\varepsilon \ \text{ for } \ t > s > t^*.$$

Let $k \in N$ and put $t_j = jt^*/(k-1)$, $j = 0, 1, \ldots, k-1$. Choose a particular $j \in \{1, \ldots, k-1\}$ and estimate $|w(t)v(t) - w(s)v(s)|$ for $t_{j-1} \le s \le t \le t_j$. Denote

$$\delta_k = \sup\{|w(\tau_1) - w(\tau_2)| : |\tau_1 - \tau_2| \le t^*/(k-1),\ \tau_1, \tau_2 \in [0, t^*]\}.$$

Then using (4.14) we can estimate the integral (4.16) as follows:

$$\begin{aligned}
&\Big|\int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)Fu(\tau)\,d\tau\Big| \\
&\quad \le w(t)\Big|\int_0^s (\widetilde{T}(t-s) - I)\widetilde{T}(s-\tau)Fu(\tau)\,d\tau\Big| \\
&\qquad + |w(t) - w(s)|\Big|\int_0^s \widetilde{T}(s-\tau)Fu(\tau)\,d\tau\Big| \\
&\quad \le w(t)\int_0^s |\widetilde{T}(t-s) - I|_{L(Y,X)}\,|\widetilde{T}(s-\tau)|_{L(X,Y)}\,\mathrm{const}\,w(\tau)^{-1}\,d\tau \\
&\qquad + \delta_k\,\mathrm{const}\int_0^s |\widetilde{T}(s-\tau)|w(\tau)^{-1}\,d\tau \\
&\quad \le \ \mathrm{const}\,\Big[w(t)\eta(t-s)\int_0^s \psi(s-\tau)w(\tau)^{-1}\,d\tau + \delta_k\int_0^s \widetilde{\omega}(s-\tau)w(\tau)^{-1}\,d\tau\Big].
\end{aligned}$$

By (4.14) and the uniform continuity of w on $[0, t^*]$ the last expression is less than $\frac{1}{4}\varepsilon$ for a sufficiently large $k \in N$.

Finally, (4.17) is estimated in the following way:

$$\begin{aligned}
w(t)\Big|\int_s^t \widetilde{T}(t-\tau)Fu(\tau)\,d\tau\Big| &\le \mathrm{const}\,w(t)\int_{t_{j-1}}^t \widetilde{\omega}(t-\tau)w(\tau)^{-1}\,d\tau \\
&\le \mathrm{const}\sup_{\tau\in[0,t^*]} w(\tau)\int_0^{t^*/(k-1)} \widetilde{\omega}(\sigma)\,d\sigma;
\end{aligned}$$

the last expression may be made less than $\frac{1}{4}\varepsilon$ when k is chosen appropriately large. So the system of intervals $[t_{j-1}, t_j)$, $j = 1, \ldots, k-1$, $[t_{k-1}, \infty)$ is the desired measurable partition of R^+ corresponding to the given $\varepsilon > 0$ as required in condition (ii) of Proposition 4.1. The proof is complete. □

Remark 4.6 (Analytic semigroups)**.** The assumptions concerning the semigroup $\widetilde{T}(t)$ in the preceding lemmas can be easily met when assuming that $\widetilde{T}(t)$ is a (compact) analytic semigroup. It suffices to choose $w(t) = e^{\alpha t}$ with an appropriate $\alpha > 0$. In particular, (4.14) is then satisfied with $Y = D((A+D)^\alpha)$, $\eta(\tau) = \mathrm{const}\,\tau^\alpha$, $\psi(\tau) = \mathrm{const}\,\tau^{-\alpha}$. The corresponding results on analytic semigroups can be found for instance in Pazy [10, Sections 2.5, 2.6].

We are now in position to state the main result of this section.

Theorem 4.7. *Let the following assumptions be satisfied:*

(i) *A is a generator of a C_0-semigroup in X;*

(ii) *$B = D + F$, $D \in L(X)$ and $\widetilde{T}(t)$ satisfies*
$|\widetilde{T}(t)| \le \widetilde{\omega}(t)$ for $t > 0$, where $\widetilde{\omega} \in L^1_{\mathrm{loc}}([0,\infty))$;

(iii) *$F : X \to X$ is locally Lipschitz continuous and there exist constants $\delta_0 > 0$, $r_0 > 0$, $k_0 > 0$, and a nondecreasing function φ such that*
$|F(u)| \le k_0|u|^{1+\delta_0}$ for $u \in B_{r_0}(0;X)$,
and (4.11), (4.12) *hold;*

(iv) *there exists a Banach space $Y \hookrightarrow\hookrightarrow X$ such that either*

(a) *$\widetilde{T} \in L^1_{\mathrm{loc}}(0,\infty; L(X,Y))$*

or

(b) *$\widetilde{T} \in L^1_{\mathrm{loc}}(0,\infty; L(Y))$, $F : X \to Y$ is locally bounded;*

in both cases we assume that
$|\widetilde{T}(\tau)y - \widetilde{T}(0)y| \le \eta(\tau)|y|_Y$, $y \in Y$, $\tau \ge 0$,
$|\widetilde{T}(\tau)\xi|_Y \le \psi(\tau)|\xi|$, $\xi \in X$, $\tau > 0$,
where $\eta \ge 0$, $\limsup\limits_{\tau\to 0+} \eta(\tau) = 0$, $\psi \in L^1_{\mathrm{loc}}([0,\infty); R^+)$;

(v) *there is a positive function $w \in C([0,\infty))$ with $\lim\limits_{t\to\infty} w(t) = \infty$ and such that*

$$\limsup_{t\to\infty} w(t)\widetilde{\omega}(t) = 0,$$

$$\limsup_{t\to\infty} w(t)\int_0^t \widetilde{\omega}(t-\tau)w(\tau)^{-1-\delta_0}\,d\tau = 0.$$

Then for any $x \in X$ there exists a (*unique*) *generalized solution $u \in L^\infty_w(0,\infty;X)$ of Problem* (4.1), (4.2).

Proof: We shall make use of the Schauder fixed point theorem in the ball $B_R(0; L^\infty_w(0,\infty;X))$ with $R > 0$ sufficiently large. To this end (as in (2.5)) define

$$H(u)(t) = \widetilde{T}(t)x + G(u)(t), \quad t \ge 0,\ u \in L^\infty_w(0,\infty;X),$$

where G is defined by (4.8). By Lemma 4.2 the operator H maps $L^\infty_w(0,\infty;X)$ continuously into itself. By assumption (v) and Lemma 4.3, for any u in the ball $B_R(0; L^\infty_w(0,\infty;X))$ with $R > 0$ sufficiently large we find

$$\|H(u)\|_w \le \|\widetilde{T}(\cdot)x\|_w + \|G(u)\|_w \le \widetilde{S}|x| + \kappa R \le R.$$

Finally, by Lemmas 4.4, 4.5 and Proposition 4.1, the mapping H is compact in $L^\infty_w(0,\infty;X)$. So H satisfies the assumptions of the Schauder fixed point theorem in $B_R(0; L^\infty_w(0,\infty;X))$ for a sufficiently large $R > 0$ and hence it has a fixed point $u \in L^\infty_w(0,\infty;X)$, that is $u = H(u)$. Thus u is a solution of (4.3). □

5. Stabilization for a singularly perturbed problem

Let us consider the problem

$$\begin{aligned} &\varepsilon u_\varepsilon'(t) + (A+B)u_\varepsilon(t) = 0, \quad t > 0, \\ &u_\varepsilon(0) = x, \quad \varepsilon \in [0,\varepsilon_0],\ \varepsilon_0 > 0,\ x \in X, \end{aligned} \tag{5.1}$$

where A and B satisfy assumptions (i)–(iii) of Section 2. Then the generalized solution of Problem (5.1) is $u_0(t) \equiv 0$ for $\varepsilon = 0$ and a function $u_\varepsilon \in L^\infty_{\mathrm{loc}}(0,\infty;X)$ such that

$$u_\varepsilon(t) = \widetilde{T}(t/\varepsilon)x + \frac{1}{\varepsilon}\int_0^t \widetilde{T}\Big(\frac{t-s}{\varepsilon}\Big)Fu_\varepsilon(s)\,ds, \quad t \ge 0, \tag{5.2}$$

if $\varepsilon \in (0,\varepsilon_0]$. Introducing new variables

$$\tau = t/\varepsilon, \quad v(\tau) = u_\varepsilon(\varepsilon\tau), \tag{5.3}$$

we transform Problem (5.1) into the problem

$$\begin{cases} v'(\tau) + (A+B)v(\tau) = 0, \quad \tau > 0, \\ \qquad\qquad v(0) = x. \end{cases} \tag{5.4}$$

Now we can apply the results of Section 2 and 4 to Problem (5.4). If we succeed in finding a weight function $w(t)$ such that the hypotheses of Theorem 2.2 or 4.7, respectively, are satisfied, then we get

$$|v(\tau)| \le Cw(\tau)^{-1}, \quad \tau \ge 0 \tag{5.5}$$

with a constant C depending only on the radius r of the ball $B_r(0;X)$ the initial datum $x \in X$ is taken from. The relation (5.5) translated to (t,u)-setting reads

$$|u_\varepsilon(t)| \le Cw(t/\varepsilon)^{-1}, \quad t \ge 0,\ \varepsilon \in (0,\varepsilon_0].$$

This yields not only the stabilization of the solutions for $x \in B(0;X)$ to the zero stationary solution as $t \to \infty$ but also the pointwise convergence of $u_\varepsilon(t)$ to 0 as $\varepsilon \to 0+$ for $t > 0$, and the rate of convergence in terms of t/ε.

The reader can easily formulate the corresponding theorems for Problem (5.1) by just modifying Theorems 2.2 and 4.7, respectively.

6. Application to a parabolic equation

As an illustration of application of the results of Section 4 let us consider a semilinear parabolic equation and formulate an explicit condition that guarantees the assumptions of Theorem 4.7. So, let $\Omega \subset R^n$ be a bounded domain with uniformly C^2-boundary and let

$$\begin{aligned} &a_{ij} \in C^2(\overline{\Omega}), \quad a_{ji} = a_{ij}, \quad i,j = 1,\dots,n \\ &\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2 \ \text{ for } \xi \in R^n, \quad \xi \ne 0 \ \text{ with } c_0 > 0 \end{aligned} \tag{6.1}$$

and

$$g \in C^1(R), \qquad g(0) = 0. \tag{6.2}$$

Consider the initial-boundary-value problem

$$\begin{aligned} u_t - \sum_{i,j=1}^{n} (a_{ij} u_{x_j})_{x_i} &= g(u), \quad && x \in R^n, \quad t > 0 \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t > 0, \\ u(x,0) &= u^0(x), && x \in R^n. \end{aligned} \tag{6.3}$$

Let $s \in (0,1)$, $p > n/s$, and let $X := \mathring{W}_p^s(\Omega)$, where W_q^r $(0 < r < \infty$, $1 \le q \le \infty)$ stands for the usual Sobolev space, $\mathring{W}_q^r(\Omega)$ being the closure in $W_q^r(\Omega)$ of $C_0^\infty(\Omega)$. Also, denote by $\|\cdot\|_q$ the norm in $L^q(\Omega)$ and by $\|\cdot\|_{r,q}$ in W_q^r. Let us note that if $r \in (0,1)$ then for $v \in W_q^r(\Omega)$ we use the norm

$$\|v\|_{r,q} := \left\{ \int_\Omega |v(x)|^q \, dx + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x-y|^{n+rq}} \, dx dy \right\}^{1/q}$$

(see for instance [1]).

Proposition 6.1. *Let assumption* (6.1) *be satisfied. If A is defined by*

$$\begin{aligned} D(A) &:= W_p^2(\Omega) \cap \mathring{W}_p^1(\Omega), \\ (Av)(x) &:= -\sum_{i,j=1}^{n} (a_{ij}(x) v_{x_j}(x))_{x_i} \ \ \textit{for } v \in D(A), \end{aligned} \tag{6.4}$$

then the operator $(-A)$ is a generator of an analytic exponentially decreasing semigroup $\{T(t)\}_{t \ge 0}$ of continuous linear operators in $L^p(\Omega)$ which is an exponentially decreasing C_0−semigroup of contractions in X.

Proof: The proof of the analyticity of the semigroup generated by A in $L^p(\Omega)$ can be found for example in [12]. Also, the fractional powers of A are well defined. Taking $v \in D(A)$, $v^* := v|v|^{p-2}$, by Green's lemma, (6.1) and Poincaré's inequality in $\mathring{W}_2^1(\Omega)$ we find

$$\begin{aligned} \|Av\|_p \Big(\int_\Omega |v|^p \, dx \Big)^{(p-1)/p} &\ge \langle Av, v^* \rangle \ge c_0 (p-1) \int_\Omega \sum_{j=1}^{n} \Big(\frac{\partial}{\partial x_j} |v|^{p/2} \Big)^2 dx \\ &\ge \frac{4 c_0 m (p-1)}{p^2} \int_\Omega |v|^p \, dx, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the duality between L^p and $L^{(p-1)/p}$, $c_0>0$ is the constant from (6.1) and

$$m := \inf\Big\{ \int_\Omega |\nabla v|^2\,dx \Big/ \int_\Omega v^2\,dx :\ v\in \mathring{W}_p^1(\Omega), v\neq 0\Big\}. \tag{6.5}$$

Now taking $\varphi\in D(A)$ and setting $u(t):=T(t)\varphi$ we get

$$u_t + Au = 0. \tag{6.6}$$

Put $u^* := u|u|^{p-2}$. Then

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p = \big\langle u_t, u^*\big\rangle = -\big\langle Au, u^*\big\rangle \le -\frac{4c_0 m(p-1)}{p^2}\|u\|_p^p$$

from where $\|u(t)\|_p \le e^{-ct}\|u(0)\|_p$ with $c:=4c_0m(p-1)/p^2$. By continuity the above estimate can be extended to $\varphi\in L^p(\Omega)$. Thus we have

$$\|T(t)\|_{L(L^p(\Omega))} \le e^{-ct}. \tag{6.7}$$

Further, having defined the space

$$D_A(\theta,q) := \{v\in L^p(\Omega) : t\to z(t) := \|t^{1-\theta-1/q}AT(t)v\|_p \in L^q(0,\infty)\},$$
$$(0<\theta<1, 1\le q\le\infty)$$

equipped with the norm $\|v\|_{D_A(\theta,q)} := \|v\|_p + \|z\|_{L^q(0,\infty)}$, it may be shown [9, Theorem 3.2.3] that

$$D_A\Big(\frac{s}{p},p\Big) = \mathring{W}_p^s(\Omega) \tag{6.8}$$

algebraically and topologically. It is immediate that $\{T(t)\}_{t\ge 0}$ is a C_0-semigroup on X and by (6.7) we have

$$\|T(t)\|_{L(X)} \le e^{-ct}, \quad t\ge 0. \tag{6.9}$$

□

Assume now that

$$g(u) = du + f(u), \quad u\in R \tag{6.10}$$

with $d<c$. In what follows we shall make use of the equivalent form of Problem (6.3) which can be written as follows:

$$u(t) = A^\nu \exp[-(d+A)t]A^{-\nu}u^0 + \int_0^t A^\nu \exp[-(d+A)(t-s)]A^{-\nu} f(u(s))\,ds, \tag{6.11}$$

where ν is for convenience chosen from the interval $(s/p, 1)$. Now we intend to apply Theorem 4.7 with $\tilde{T}(t) := A^\nu exp[-(d+A)t]$, $x := A^{-\nu}u^0$, $Dv := dv$, $Fv := A^{-\nu}f(v)$ for $v \in X$. This requires a slight modification of our considerations in Section 4. First, in place of (4.13) we use the estimate $|\exp[-(d+A)t]\,u_0| \leq e^{-(c-d)t}\,|u^0|$. Further, in (4.15) the semigroup property of $T(t)$ is to be used, the operator A^ν being once applied only which makes it possible to require the third estimate in (4.14) with $T(\tau)$ instead of $\widetilde{T}(\tau)$ with obvious corrections in the on going estimates of the individual terms in (4.15). The modification in (4.14) has to be considered in the condition (iv) of Theorem 4.7. Now, consider the operator A restricted to X. Then $(-A)$ is a generator of the C_0-semigroup $\{T(t)/X\}_{t\geq 0}$ on X. Indeed, $T(t)X \subset X$ for $t \geq 0$ is elementary and by the Lebesgue theorem, for $v \in X$ we have $\lim_{\tau\to 0}\int_0^\infty t^{p-s-1}\|AT(t)(T(\tau)v - v)\|_p^p\,dt = 0$. So $\{T(t)/X\}_{t\geq 0}$ is a C_0-semigroup in X and by [10, Theorem 5.5] the assertion follows. So the condition (i) of Theorem 4.7 is satisfied. Let us verify (ii). Let $v \in X$ and $t > 0$. Then we have (cf. [10, Theorem 6.13])

$$\begin{aligned}
&\int_0^\infty \tau^{p-s-1}\|A^{\nu+1}\exp[-(d+A)(t+\tau)]v\|_p^p\,d\tau \\
&\qquad\leq \int_0^\infty \tau^{p-s-1}\|A\exp[-(d+A)\tau]v\|_p^p\,d\tau\,\|A^\nu\exp[-(d+A)t]\|_{L(L^p(\Omega))}^p \\
&\qquad\leq C(\nu,\varepsilon)t^{-p\nu}e^{-(c-d-\varepsilon)pt}\|v\|_X^p,
\end{aligned}$$

where $C(\nu,\varepsilon)$ is a constant and $0 < \varepsilon < c-d$. This yields

$$\|\tilde{T}(t)\|_{L(X)} \leq c(\nu)t^{-\nu}e^{-\gamma t} \quad \text{for} \quad t > 0, \quad \text{where} \quad \gamma = (c-d)/2. \tag{6.12}$$

So (ii) is satisfied with $\widetilde{\omega}(t) := c(\nu)t^{-\nu}e^{-\gamma t}$.

Lemma 6.2. *Let $f \in C^1(R)$, $f(0) = 0$, with f' locally Lipschitz continuous, s, p as above. Then f maps $\mathring{W}_p^s(\Omega)$ into itself and is locally Lipschitz continuous.*

Proof: We shall outline the proof of the Lipschitz continuity only since the rest is standard. Let $r > 0$ and $v, w \in B_r(0; \mathring{W}_p^s(\Omega))$. Then for $x, y \in \Omega$ we can write

$$\begin{aligned}
&f(v(x)) - f(w(x)) - \big(f(v(y)) - f(w(y))\big) \\
&= \int_0^1 \big[f'(\alpha v(x) + (1-\alpha)w(x)) - f'(\alpha v(y) + (1-\alpha)w(y))\big]\,d\alpha\,\big(v(x) - w(x)\big) \\
&\quad + \int_0^1 f'(\alpha v(y) + (1-\alpha)w(y))\,d\alpha\,\big(v(x) - w(x) - (v(y) - w(y))\big) \\
&=: A_1 + A_2.
\end{aligned}$$

Since $\|z\|_\infty \leq C\,\|z\|_{s,p}$ for all $z \in W_p^s(\Omega)$ we have $|\alpha v(\xi) + (1-\alpha)w(\xi)| \leq \max\{\|v\|_\infty, \|w\|_\infty\} \leq Cr$ for all $\alpha \in [0,1]$, $\xi \in \Omega$. Hence by assumption there exists $L(r) > 0$ such that $|f'(z_1(x)) - f'(z_2(x))| \le L(r)|z_1(x) - z_2(x)|$ for all $z \in B_r(0; W_p^s(\Omega))$ and $x \in \Omega$. So we have

$$\begin{aligned}|A_1| &\le L(r)\int_0^1 |\alpha(v(x)-v(y)) + (1-\alpha)(w(x)-w(y))|\,d\alpha\,|v(x)-w(x)| \\ &\le L(r)\max\left\{|v(x)-v(y)|,\,|w(x)-w(y)|\right\}\|v-w\|_\infty\end{aligned}$$

which yields

$$\left(\int_\Omega\int_\Omega \frac{|A_1|^p}{|x-y|^{n+sp}}\,dxdy\right)^{1/p} \le L(r)Cr\,\|v-w\|_{s,p}.$$

Similarly we find

$$|A_2| \le \sup\left\{|f'(\rho)|; |\rho| \le Cr\right\}|v(x)-w(x)-(v(y)-w(y))|$$

from where

$$\left(\int_\Omega\int_\Omega \frac{|A_2|^p}{|x-y|^{n+sp}}\,dxdy\right)^{1/p} \le \text{const}\,\|v-w\|_{s,p}.$$

□

Assume now that there exist positive constants δ_0, k, r_1 such that

$$\begin{aligned}&|f(v)| \le k|v|^{1+\delta_0}, \quad |f'(v)| \le k|v|^{\delta_0} \text{ for } |v| \le r_1, \\ &S := \sup_{v\neq 0}\left|\frac{f(v)}{v}\right| < \infty.\end{aligned} \tag{6.13}$$

Let C be a constant of the imbedding $W_p^s(\Omega) \hookrightarrow L^\infty(\Omega)$, i.e. $\|v\|_\infty \le C\,\|v\|_{s,p}$ for $v \in W_p^s(\Omega)$, and take $r_0 > 0$ such that $Cr_0 \le r_1$. Then for $v \in B_{r_0}(0; X)$ by (6.13) we have $\|f(v)\|_p \le Ck\|v\|_{s,p}^{1+\delta_0}$. Similarly as in the proof of Lemma 6.2 we can show that for $v \in \mathring{W}_p^s(\Omega)$ we have $|f(v(x)) - f(v(y))| \le k\|v\|_\infty^{\delta_0}|v(x)-v(y)|$, $x, y \in \Omega$. Hence

$$\int_\Omega\int_\Omega \frac{|f(v(x)) - f(v(y))|^p}{|x-y|^{n+sp}}\,dxdy \le k^pC^{p\delta_0}\|A^{-\nu}\|_{L(L^p(\Omega))}^p\|v\|_{s,p}^{p(1+\delta_0)}$$

since, clearly, $A^{-\nu} \in L(L^p(\Omega))$ and $\|A^{-\nu}z\|_{s,p} \le \|A^{-\nu}\|_{L(L^p(\Omega))}\|z\|_{s,p}$ for $z \in \mathring{W}_p^s(\Omega)$. Thus we have proved $\|F(v)\| \le \text{const}\,\|v\|_{s,p}^{1+\delta_0}$. Further, for $z \in D(A^\nu)$ we have

$$\begin{aligned}\int_0^\infty t^{p-s-1}\|AT(t)z\|_p^p\,dt &\le \int_0^\infty t^{p-s-1}\|A^{1-\nu}T(t)\|_p^p\,dt\,\|A^\nu z\|_p^p \\ &\le C(\nu)^p\int_0^\infty t^{p\nu-s-1}e^{-cpt}dt\,\|z\|_{D(A^\nu)}^p \\ &= C(\nu)^p(cp)^{s-p\nu}\Gamma(p\nu - s)\|z\|_{D(A^\nu)}^p = k(\nu,p)^p\|z\|_{D(A^\nu)}^p,\end{aligned}$$

where $C(\nu) := \sup_{t>0}\{t^{\nu-1}e^{-ct}\|A^{1-\nu}T(t)\|\}$, $k(\nu,p) = C(\nu)(cp)^{s-p\nu}\Gamma(p\nu - s)$. Hence $\|z\|_{s,p} \le k(\nu,p)\|A^\nu z\|_p$. Taking $v \in X$ and putting $z = Fv$, we find

$$\|Fv\|_{s,p} \le k(\nu,p)\|A^\nu A^{-\nu} f(v)\|_p = k(\nu,p)\|f(v)\|_p \le k(\nu,p)S\|v\|_p,$$

so that (4.11) is satisfied with $\varphi(r) = Sk(\nu,p)r$, $r \ge 0$. Choose $\alpha \in (0,\gamma)$ and put $w(t) := e^{\alpha t}$ for $t \ge 0$. Elementary calculations with (4.12) yield $\kappa = \kappa_0 k(\nu,p)S$, where $\kappa_0 = \int_0^\infty \tau^{-\nu}e^{(\alpha-\gamma)\tau}d\tau$. So, to fulfil (4.12) we require

$$S < \kappa_0^{-1}k(\nu,p)^{-1}. \tag{6.14}$$

Thus we have satisfied all conditions listed in (iii). Let us choose in (iv), $Y := D(A) = W_p^2(\Omega)\cap \mathring{W}_p^1(\Omega)$. Then by the Sobolev imbedding theorem Y is compactly imbedded in X. By [6, Proposition 5.2], for $0 < \theta < s/p$ we have $D_A(s/p,p) \subset D(A^\theta)$ and for $v \in D_A(s/p,p)$,

$$A^\theta v = \frac{1}{\Gamma(1-\theta)}\int_0^\infty t^{-\theta}AT(t)v\,dt.$$

For $\varepsilon \in (0,c)$ there is $M(\varepsilon) > 0$ such that $\|AT(t)v\|_p \le M(\varepsilon)t^{-1}e^{-(c-\varepsilon)t}\|v\|_p$ for $t > 0$ (see [10]); hence we have

$$\int_1^\infty t^{-\theta}\|AT(t)v\|_p\,dt \le \text{const}\,\|v\|_p.$$

In addition, by Hölder's inequality we have

$$\begin{aligned}\int_0^1 t^{-\theta}\|AT(t)v\|_p\,dt &= \int_0^1 t^{(s+1)/p-1-\theta}t^{1-(s+1)/p}\|AT(t)v\|_p\,dt\\ &\le \Big(\int_0^1 t^{(s+1-p-\theta p)/(p-1)}\,dt\Big)^{(p-1)/p}\Big(\int_0^1 t^{p-s-1}\|AT(t)v\|_p^p\,dt\Big)^{1/p}\\ &\le \text{const}\,\|v\|_X,\end{aligned}$$

whenever $(1+s-p-\theta p)/(p-1) > -1$, i.e. $\theta < s/p$. So we have proved that $\|A^\theta v\|_p \le \text{const}\,\|v\|_X$ for $v \in X$ and $\theta \in (0,s/p)$ arbitrary but fixed. Now, (iv)(a) is satisfied since for $v \in \mathring{W}_p^s(\Omega)$, ν as above and $\nu_0 \in (0,s/p)$, $\nu_1 = \nu - \nu_0$ we have

$$\begin{aligned}\|A^\nu T(t)v\|_Y &\le \text{const}\,\|A^{1+\nu}T(t)v\|_p \le \text{const}\,\|A^{\nu_1}T(t)\|_{L(L^p(\Omega))}\|A^{\nu_0}v\|_p\\ &\le \text{const}\,t^{-\nu_1}e^{-\gamma t}\|v\|_{s,p}.\end{aligned}$$

Assume now $2s < p$ if $n = 1$ and $\nu \in (s/p, 1-s/p)$ in any case. Besides, let $0 < \theta < 1-\nu-s/p$. Then, in (iv) the nontrivial term in the norm $\|\widetilde{T}(\tau)v - \widetilde{T}(0)v\|_X^p$

can be estimated as follows (cf. [10, Theorem 6.13]):

$$\begin{aligned}\int_0^\infty & t^{p-s-1}\|AT(t)A^\nu(\exp[-(d+A)\tau]v-v)\|_p^p\,dt\\ &\le \text{const}\int_0^\infty t^{p-s-1}\|A^{\nu+\theta}T(t)\|^p_{L(L^p(\Omega))}\|(\exp[-(d+A)\tau]-I)A^{1-\theta}v\|_p^p\,dt\\ &\le \text{const}\int_0^\infty t^{p-s-1-p(\nu+\theta)}e^{p(c-d-\varepsilon)t}\,dt\,\tau^{p\theta}\|Av\|_p^p\\ &\le \text{const}\,\tau^{p\theta}\|v\|_Y^p \quad\text{for } v\in D(A),\end{aligned}$$

and

$$\|AT(\tau)v\|_p\le\|A^{1-\theta}T(\tau)\|_p\|A^\theta v\|_p\le\text{const}\,\tau^{\theta-1}\|v\|_X \quad\text{for } v\in X.$$

This yields (iv) with $\eta(\tau):=\text{const}\,\tau^\theta$, $\psi(\tau):=\text{const}\,\tau^{\theta-1}$. Finally, as far as (v) is concerned, since $\alpha<\gamma$ it remains to satisfy the last condition in (v). By the choice of w and $\widetilde{\omega}$ we have

$$\begin{aligned}w(t)\int_0^t\widetilde{\omega}(t-\tau)w(\tau)^{-1-\delta_0}\,d\tau &= e^{\alpha t}\int_0^t(t-\tau)^{-\nu}e^{-\gamma(t-\tau)}e^{-\alpha(1+\delta_0)\tau}\,d\tau\\ &= e^{-\alpha\delta_0 t}\int_0^t\sigma^{-\nu}\exp[\alpha(1+\delta_0)-\gamma]\sigma\,d\sigma.\end{aligned}$$

The last expression tends to 0 as $t\to\infty$ whenever $\alpha<\gamma(1+\delta_0)^{-1}$. So the following theorem holds.

Theorem 6.3. *Let $\Omega\subset R^n$ be a bounded domain with a C^2-boundary and let the functions a_{ij},g be such that (6.1), (6.2) hold. In addition, assume that there are given numbers $s\in(0,1)$, $p>n/s$, where $2s<p$ if $n=1$. Define m by (6.5), $c:=4c_0m(p-1)/p^2$, and assume that (6.10) holds with $d<c$ and that (6.13), (6.14) are satisfied. Choose $\alpha:0<\alpha<\frac{1}{2}(c-d)(1+\delta_0)^{-1}$ and put $w(t):=e^{\alpha t}$, $t\ge 0$. Then for any $u^0\in\mathring{W}^s_p(\Omega)$ there exists a unique solution $u\in L^\infty_w(0,\infty;\mathring{W}^s_p(\Omega))$ of Problem (6.3); in particular we have*

$$\|u(t)\|_{s,p}\le\text{const}\,e^{-\alpha t},\quad \textit{for almost all } t>0. \tag{6.15}$$

Proof: The theorem is a consequence of Theorem 4.1 and our previous considerations.

□

Remark 6.4. The constant in (6.15) depends only on u_0,p,s,Ω and the constants from conditions (6.1). (6.10) and (6.13). The constant m is the least eigenvalue of the operator $(-\Delta)$ if considered as an operator in $L^2(\Omega)$ with the domain of definition $W^2_2(\Omega)\cap\mathring{W}^1_2(\Omega)$. Let us note that using the results contained for example in [9] we could consider much more general elliptic operator and boundary conditions than those in (6.3) to obtain an analogous result but we prefer to avoid extra technical complications intending to make the presentation of the method more lucid. □

References

[1] Appel J., Zabrejko P.P., *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.

[2] Crandall M., Liggett T., *Generation of semigroups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1977), 265–298.

[3] Hale J.K., *Theory of Functional Differential Equations*, Springer, New York, 1977.

[4] Hale J.K., *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs No. 25, Amer. Math. Soc., Providence, 1988.

[5] Iwamiya T., Takahashi T., Oharu T., *Characterization of nonlinearly perturbed semigroups*, in: Functional Analysis and Related Topics, Proceedings, Kyoto 1991, H. Komatsu (ed.), Lecture Notes in Math. No. 1540, Springer, New York, 1993.

[6] Komatsu H., *Fractional powers of generators, II Interpolation spaces*, Pacific J. Math. **21** (1967), 89–111.

[7] Krasnosel'skij M.A., Zabrejko P.P., Pustyl'nik E.I., Sobolevskij P.E., *Integral Operators in the Spaces of Integrable Functions* (in Russian), Nauka, Moscow, 1966.

[8] Krein M., Dalecki J., *Stability of the Solutions of Differential Equations in Banach Spaces*, Amer. Math. Soc., Providence, 1974.

[9] Lunardi A., *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.

[10] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[11] Rauch J., *Stability of motion for semilinear equations*, in: Boundary Value Problems for Linear Evolution Partial Differential Equations, Proceedings of the NATO Advanced Study Institute held in Liège, Belgium, September 6–17, 1976, H. G. Garnir (ed.), NATO Advanced Study Institute Series C, Mathematical and Physical Sciences, vol. 29, Reidel Publishing Co., Dordrecht, 1977.

[12] Tanabe H., *Equations of Evolution*, Pitman, London, 1979.

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