On a nonstationary discrete time infinite horizon growth model with uncertainty

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Abstract. In this paper we examine a nonstationary discrete time, infinite horizon growth model with uncertainty. Under very general hypotheses on the data of the model, we establish the existence of an optimal program and we show that the values of the finite horizon problems tend to that of the infinite horizon as the end of the planning period approaches infinity. Finally we derive a transversality condition for optimality which does not involve dual variables (prices).

Keywords: growth model, discrete time, infinite horizon, finite horizon, uncertainty, utility function, technology multifunction, optimal program, transversality condition

Classification: 90A20

1. Introduction

In a recent paper [12], the first author examined a nonstationary, infinite horizon multisector growth model with uncertainty and discounted future utilities. For the general model, first he proved the existence of a strongly optimal program and then he characterized it using a system of supporting prices. One of the necessary conditions for optimality, was a transversality condition, that says that the expected value of the input (and of the output) goes to zero as the time $k \to \infty$.

The purpose of this paper is to generalize some of the results in [12]. For the existence part we weaken the hypotheses on the data of the model to include some economically important cases and for the necessary condition, we obtain a transversality condition that does not involve dual variables (i.e. price systems)

Furthermore our existence result also proves the convergence of the value of the finite horizon problems to the value of the infinite horizon one (approximation result). The open-endedness of the future is very important from an economic viewpoint, because it expresses the fundamental fact that the consequences of investment are very long lived. A finite planning horizon requires some methods of evaluating end-of-period capital stocks and the only proper evaluation is their value in use in the subsequent future (cf. C. Arrow-M. Kurz [2]). However the infinity of the planning horizon is impractical from a computational viewpoint. To actually solve these problems, we must content ourselves with finite horizon approximates and guarantee that their value converges to that of the infinite horizon problem as the end of the planning period goes to infinity.

The earlier deterministic works by B. Peleg-H. Ryder [13] and M.L. Weitzman [15] and the stochastic works by I. Evstigneev [9], M.I. Taksar [14] and N.S. Papageorgiou [12] did not cover the case where the utility $u_k(x,y)$ goes to $-\infty$ as $\|x\|, \|y\| \to 0$ such as $u_k(x,y) = \ln\|x-y\|$ or $u_k(x,y) = \frac{-1}{\|x-y\|}$. This is of special importance in economics, where in many cases the zero input-output combination has to be penalized much more heavily than by simply setting $u_k(0,0) = 0$ or equal to some large but fixed negative value.

2. The model and preliminaries

The model that we use in this study is basically that used by Kravvaritis-Papageorgiou (cf. [10, p. 159]) and Papageorgiou (cf. [12, p. 228]). Kravvaritis-Papageorgiou [10] studied a similar problem with the present work. Namely they investigated the robustness of the model with respect to variations in the data determining it (sensitivity analysis). Here we investigate how finite horizon values of the problem approximate the desired infinite horizon version of it. Such a result is clearly useful because the infinite horizon model is an idealization of the actual economic situation. In reality we can only compute finite horizon approximations of the model and thus we want to know whether these approximations converge to the value of the problem as the time horizon tends to infinity.

The stochastic growth model under consideration is the following: (Ω, Σ, μ) is a complete probability space. As usual each $\omega \in \Omega$ represents one possible state of the environment. The Σ is the set of all possible events and $\mu(\cdot)$ the probability distribution of the states. The time horizon is $N_0 = \{0, 1, 2, \dots\}$, which means that our model is discrete-time, infinite horizon. The uncertainty about the states is described by an increasing sequence $\{\Sigma_n\}_{n\geq 0}$ of sub-σ-fields of Σ such that $\Sigma = \bigvee_{n=0}^{\infty} \Sigma_n$. Each Σ_n represents the information available up until time $n \geq 0$. Our commodity space is a separable reflexive Banach space X. During the last decade, many mathematical economists, in particular those working on equilibrium theory, have considered models involving an infinite dimensional commodity space (we refer to the book by C. Alliprantis-D. Brown-O. Burkinshaw [1] and the references therein). There are many good reasons justifying the use of an infinite dimensional commodity space. In many situations, the same product at different times should be considered as a different good and this naturally leads to an infinite dimensional setting. The same is true with some particular, important models, like the overlapping generations model. But more importantly, working with an infinite dimensional space, which does not provide all the conveniences of the standard finite dimensional case, we achieve a better and deeper understanding of the conventional case. Finally the infinite dimensional model depicts better the situation where we deal with a finite yet very large number of commodities. This is analogous to the situation in equilibrium theory, where problems involving a large number of traders, were studied (following the lead of Aumann [3]) using a continuum of traders and this approach led to some very powerful results and a better understanding of the original situation. We are also given a discount factor

$\delta \in (0,1)$ with which we discount future utilities.

At each time instant $n \geq 1$, the technological possibilities of our economy are described by a multifunction $F_n : \Omega \to 2^{X \times X} \setminus \emptyset$ which has a $\Sigma_n \times B(X) \times B(X)$-measurable graph (here $B(X)$ stands for the Borel σ-field of X). Given a time $n \geq 1$ and a state of the environment $\omega \in \Omega$, the set $F_n(\omega)$ includes all input-output pairs $(x,y) \in X \times X$, which are feasible given the technology at the time period from $n-1$ to n and with the state of the environment being ω, and input x at time $n-1$, produces an output y at time n. The uncertainty in the production process is manifested on the hypothesis that

$$GrF_n = \{(\omega, x, y) \in \Omega \times X \times X : (x,y) \in F_n(\omega)\} \in \Sigma_n \times B(X) \times B(X).$$

At each stage $n \geq 1$, given the state of the environment $\omega \in \Omega$, the utility (gain) realized by operating a production process (x,y), is given by $u_n(\omega, x, y)$, where $u_n : \Omega \times X \times X \to \overline{R} = R \cup \{-\infty\}$ is a $\Sigma_n \times B(X) \times B(X)$-measurable function. Again the uncertainty is expressed via the Σ_n-measurability of u_n. At each stage the utility is discounted by a factor $\delta \in (0,1)$.

Built into $F_n(\omega)$ and $u_n(\omega, \cdot, \cdot)$ may be constraints as $x, y \in X_+$, with X_+ being a closed, convex proper cone in X, furnishing X with a partial order and also constraints like $u_n(\omega, x, y) = -\infty$ if $(\omega, x, y) \notin GrF_n$ (i.e. the utility function u_n is only defined on GrF_n and extended to all of $\Omega \times X \times X$, by setting it equal to $-\infty$ outside GrF_n). So our very general formulation accommodates a variety of important situations.

A **program** (policy, path) is a sequence $\widehat{x} = \{x_n\}_{n \geq 0}$ such that $x_n \in L^1(\Sigma_n, X)$, with $L^1(\Sigma_n, X)$ being the Lebesgue-Bochner space of integrable X-valued functions defined on (Ω, Σ_n, μ). Let $v \in L^1(\Sigma_0, X)$ be the initial capital stock of our economy.

A program $\widehat{x}$ is said to be **feasible** if $x_0 = v$ and $(x_n(\omega), x_{n+1}(\omega)) \in F_{n+1}(\omega)$ μ-a.e. We will denote the set of all feasible programs by $H(v)$.

Given any $N > 0$, a finite sequence $\widehat{x}_N = \{x_k\}_{k=0}^{N+1}$ is said to be N-**feasible program** if $x_0 = v$ and $(x_n(\omega), x_{n+1}(\omega)) \in F_{n+1}(\omega)$ μ-a.e. for all $n \in \{0, 1, 2, \ldots, N\}$. Of course such a finite sequence is associated with the N-finite horizon approximation of the original infinite horizon problem. We will denote the set of all N-feasible programs by $H_N(v)$.

Our goal is to find $\widehat{x} \in H(v)$ which maximizes the expected intertemporal discounted utility $U(\widehat{x}) = \sum_{k=0}^{\infty} \delta^{k+1} J_{k+1}(x_k, x_{k+1})$, where $\widehat{x} = \{x_k\}_{k \geq 0} \in H(v)$ and $J_{k+1}(u,v) = \int_\Omega u_{k+1}(\omega, u(\omega), v(\omega))\, d\mu(\omega)$, for all $(u,v) \in L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$, the expected instantaneous utility at time $k+1$. Therefore our optimization problem is:

$$(1) \qquad \begin{cases} U(\widehat{x}) = \sum_{k=0}^{\infty} \delta^{k+1} J_{k+1}(x_k, x_{k+1}) \to \sup = m \\ \text{s.t. } (x_k(\omega), x_{k+1}(\omega)) \in F_{k+1}(\omega) \;\; \mu\text{-a.e., } \;\; k \in N_0 \\ x_0(\omega) = v(\omega) \;\; \mu\text{-a.e. (i.e. } \;\widehat{x} \in H(v)). \end{cases}$$

The corresponding finite horizon approximation to it is the following problem:

$$(2)\qquad \begin{cases} U_N(\widehat{x}_N) = \sum_{k=0}^{N} \delta^{k+1} J_{k+1}(x_k, x_{k+1}) \to \sup = m_N \\ \text{s.t. } (x_k(\omega), x_{k+1}(\omega)) \in F_{k+1}(\omega) \ \ \mu\text{-a.e.}, \ \ k \in \{0,1,2,\dots N\} \\ x_0(\omega) = v(\omega) \ \ \mu\text{-a.e. (i.e. } \ \widehat{x}_N \in H_N(v)). \end{cases}$$

The precise mathematical hypotheses on the data of our model are the following:

H(1): $F_k : \Omega \to 2^{X \times X}$, $k \geq 1$ is a multifunction with nonempty, closed and convex values and $GrF_k = \{(\omega, x, y) \in \Omega \times X \times X : (x,y) \in F_k(\omega)\} \in \Sigma_k \times B(X) \times B(X)$ (graph measurability of $F_k(\cdot)$);

H(2): $u_k : \Omega \times X \times X \to \overline{R} = R \cup \{-\infty\}$, $k \geq 1$, is an integrand such that
- (i) $(\omega, x, y) \mapsto u_k(\omega, x, y)$ is $\Sigma_k \times B(X) \times B(X)$-measurable,
- (ii) $(x,y) \mapsto u_k(\omega, x, y)$ is concave and u.s.c.,
- (iii) $u_k(\omega, x, y) \leq \varphi(\omega, \|x\|, \|y\|)$ μ-a.e. with $\varphi : \Omega \times R^+ \times R^+ \to R^+$ measurable, $\varphi(\omega, \cdot, \cdot)$ is nondecreasing and for all $r_1, r_2 \in L^1(\Omega, R^+)$ $\int_\Omega \varphi(\omega, r_1(\omega), r_2(\omega))\, d\mu(\omega) < +\infty$;

H(3): there exists a sequence $\beta_k \geq 0$ such that $\sum_{k=0}^{\infty} \delta^{k+1} \beta_{k+1} < \infty$ and for any $\widehat{x}_N \in H_N(v)$ we have $J_{k+1}(x_k, x_{k+1}) \leq \beta_{k+1}$ for all $k \in \{0,1,2,\dots,N\}$;

H(4): for every $k \in N_0$ and every $M \in R$, there exists $c_{k+1}(M)(\cdot) \in L^1(\Sigma_{k+1}, R)$ such that for all $N \geq k$ and all $\widehat{x}_N \in H_N(v)$ for which we have $M < \sum_{n=0}^{N} \delta^{n+1} J_{n+1}(x_n, x_{n+1})$, we have $\|x_{k+1}(\omega)\| \leq c_{k+1}(M)(\omega)$ μ-a.e.

Hypothesis H(1) is a very general one. Note that I. Evstigneev [9] and M.I. Taksar [14] assumed that $G_k(\omega) \subset K$ for all $(k, \omega) \in N_0 \times \Omega$ with K being a fixed compact and convex subset of R^m (in their model the commodity space was R^m). Also they assumed that $(0,0) \in F_k(\omega)$ for all $(k,\omega) \in N_0 \times \Omega$ (possibility of inaction). On the other hand, N.S. Papageorgiou [12] (who had an infinite dimensional space) assumed the free disposability hypothesis (i.e. if $(x,y) \in F_k(\omega)$ and $x \leq x'$, $y' \leq y$ then $(x', y') \in F_k(\omega)$; see hypothesis H(P)(3), [12, p. 229]).

In this work we do not need such hypotheses. Note that the convexity on the sets $F_k(\omega)$ follows from the well-known "law of diminishing returns", while the closedness hypothesis on the sets $F_k(\omega)$ is primarily a mathematical one, which is though consistent with economic principles and very common in models of growth theory.

Hypothesis H(2) is also very general and incorporates many cases existing in the relevant growth theory literature. In contrast to I. Evstigneev [9] we do not assume that $u_k(\omega, \cdot, \cdot)$ is continuous. Also our growth hypothesis H(2)(iii) is considerably more general than that of M.I. Taksar [14], where it is assumed that $|u_k(\omega, x, y)| \leq \alpha_k$ for all $(\omega, x, y) \in \Omega \times X \times X$ with $\alpha_k \geq 0$ such that $\sum_{k \geq 0} \alpha_k <$

∞, and also incorporates the growth hypothesis employed by N.S. Papageorgiou [12] (cf. hypothesis H(u)(3) in [12]). None of above mentioned works covers the case where the utility function goes to $-\infty$ as $\|x\|, \|y\| \to 0$. This is the case if for example $u_k(\omega, x, y) = a_k(\omega) \ln \|x - y\|$ with $a_k \in L^\infty(\Sigma_k, R)$, $a_k \geq 0$ or $u_k(\omega, x, y) = -a_k(\omega)\|x - y\|^\alpha$, $0 < \alpha < 1$, $a_k \in L^\infty(\Sigma_k, R)$, $a_k \geq 0$.

Hypothesis H(3) will enable us to establish some bounds for feasible programs of (2) and eventually show that $m_N \to m$ as $N \to \infty$. This hypothesis is automatically satisfied in the context of the models of I. Evstigneev [9], M.I. Taksar [14], Kravvaritis-Papageorgiou [10] and N.S. Papageorgiou [12] because of the stronger requirements that they imposed on $F_k(\omega)$ and $u_k(\omega, x, y)$.

Finally hypothesis H(4) is very mild indeed, since the bounding function depends on $(k, \omega) \in N_0 \times \Omega$.

By w_{seq} we will denote the topology on $L^1(\Sigma_k, X)$, $k \in N_0$, whose closed sets are the sequentially weakly closed in $L^1(\Sigma_k, X)$ (this is indeed a topology; see G. Buttazzo [5, pp. 8–9]). In general w_{seq} is stronger than the weak topology on $L^1(\Sigma_k, X)$. From the Eberlein-Smulian theorem we know that these two topologies coincide on weakly compact subset of $L^1(\Sigma_k, X)$. Also note that w_{seq} is first countable.

Finally if Z is a Banach space and $v : Z \to \overline{R} = R \cup \{-\infty\}$ is a concave function, the directional derivative of $v(\cdot)$ at $z \in Z$ is defined by

$$g'(z; h) = \lim_{\lambda \to 0^+} \frac{v(z + \lambda h) - v(z)}{\lambda}.$$

Note that because of the concavity of $v(\cdot)$, $\lambda \to \frac{v(z+\lambda h)-v(z)}{\lambda}$ is an increasing function and so the limit in the above definition makes sense.

3. Existence result

In this section we prove an existence result for problem (1) and we also show that its values can be obtained as the limit of the values of the finite horizon problems (2).

Theorem 1. *If hypotheses* H(1), H(2), H(3) *and* H(4) *hold and* $m > -\infty$ *then problem* (1) *admits a solution and furthermore* $m_N \to m$ *as* $N \to \infty$.

Proof: We start by observing that under our assumptions the function U is well posed. Moreover note that because of hypothesis H(3) we have that

$$m \leq \sum_{k=0}^{\infty} \delta^{k+1} \beta_{k+1} < \infty.$$

Since by hypothesis $-\infty < m$, given $\varepsilon > 0$ we can find $\widehat{z} = \{z_k\}_{k \geq 0} \in H(v)$ such that

$$-\infty < m - \varepsilon < \sum_{k=0}^{\infty} \delta^{k+1} J_{k+1}(z_k, z_{k+1}) \leq m < \infty.$$

Hence $\{\delta^{k+1}J_{k+1}(z_k, z_{k+1})\}$ is a summable sequence in R and so

$$\sum_{k=0}^{\infty} \delta^{k+1} J_{k+1}(z_k, z_{k+1}) = \lim_{N\to\infty} \sum_{k=0}^{N} \delta^{k+1} J_{k+1}(z_k, z_{k+1}).$$

Let $\widehat{z}_N$ be the finite sequence obtained from $\widehat{z}$ by stopping at $N+1$. Clearly then $\widehat{z}_N = \{z_k\}_{k=0}^{N+1} \in H_N(v)$. So we have

$$\begin{aligned}
&\sum_{k=0}^{N} \delta^{k+1} J_{k+1}(z_k, z_{k+1}) \le m_N \\
&\Rightarrow \lim_{N\to\infty} \sum_{k=0}^{N} \delta^{k+1} J_{k+1}(z_k, z_{k+1}) = \sum_{k=0}^{\infty} \delta^{k+1} J_{k+1}(z_k, z_{k+1}) \le \liminf_{N\to\infty} m_N \\
&\Rightarrow m - \varepsilon \le \liminf_{N\to\infty} m_N.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get

$$m \le \liminf_{N\to\infty} m_N. \tag{3}$$

Next let $w = \limsup_{N\to\infty} m_N$ and let $N_n \to \infty$ be such that

$$w - \frac{1}{n} < m_{N_n}$$

(its existence follows from the definition of $\limsup_{N\to\infty} m_N = w$). Hence for each $n \ge 1$ we can find $\widehat{z}^n = \{z_k^n\}_{k=0}^{N_n+1} \in H_{N_n}(v)$ such that

$$w - \frac{1}{n} < \sum_{k=0}^{N_n} \delta^{k+1} J_{k+1}(z_k^n, z_{k+1}^n).$$

Let $M = w - \frac{1}{n}$ and for each $k \in N_0$ define $V_{K+1} = \{y \in L^1(\Sigma_{k+1}, X) : \|y(\omega)\| \le c_{k+1}(M)(\omega)\ \mu\text{-a.e.}\}$. Since X is reflexive, from Dunford's theorem (cf. [7, Theorem 1, p. 101]) we have that V_{K+1} is w_{seq}-compact in $L^1(\Sigma_{k+1}, X)$. Let $V = \prod_{k\ge 0} V_{K+1} \subset \prod_{k\ge 0} L^1(\Sigma_{k+1}, X)$. Then from Tichonov's theorem V equipped with the product w_{seq}-topology is compact and in fact sequentially compact (cf. [8, Theorem 3.6, p. 230]). Incidentally note that this topology on V coincides with the relative weak topology on V as a subset of $\prod_{k\ge 0} L^1(\Sigma_{k+1}, X)$ (cf. [6, p. 43]). Then let $\widehat{x}^n = \{x_k^n\}_{k\ge 0} \in \{v\} \times V$ be defined by

$$x_k^n = \begin{cases} z_k^n & \text{for } \ k \in \{0, 1, 2, \dots, N_n + 1\} \\ 0, & \text{otherwise.} \end{cases}$$

Then we can find a subsequence $\{\widehat{x}^{n_i}\}_{i\geq 0}$ of $\{\widehat{x}^{n}\}_{n\geq 1}$ such that $x^{n_i} \to x^*$ in $\{v\} \times V$ equipped with the product w_{seq}-topology. From the properties of the product topology (cf. [8]) we have $x_k^n \to x_k^*$ in V_k furnished with the w_{seq}-topology, for every $k \in \{0, 1, \dots\}$. Using Theorem 3.1 of [11] we deduce that $\widehat{x}^* \in H(v)$.

Next we claim that for every $k \in N_0$, $J_{k+1}(\cdot,\cdot)$ is weakly-u.s.c. on $L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$. To this end we need to show that for every $\lambda \in R$, $L_\lambda = \{(x, y) \in L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X) : \lambda \leq J_{k+1}(x, y)\}$ is weakly closed. Note that because of hypothesis H(2)(ii) L_λ is convex. So L_λ is weakly closed if and only if it is strongly closed. Thus let $\{(x_n, y_n)\}_{n\geq 1}$ be a sequence in L_λ such that $(x_n, y_n) \to (x, y)$ in $L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$.

As a by-product of completeness of the Lebesgue-Bochner spaces (cf. [4, Theorem 17.11, p. 378]) we know that we can find a subsequence $\{(x_{n_i}, y_{n_i})\}_{i\geq 1}$ such that $(x_{n_i}(\omega), y_{n_i}(\omega)) \to (x(\omega), y(\omega))$ μ-a.e. in $X \times X$ and $\|x_{n_i}(\omega)\| \leq a_1(\omega)$, $\|y_{n_i}(\omega)\| \leq a_2(\omega)$ μ-a.e., for all $i \geq 1$, with $a_1, a_2 \in L^1(\Omega, R^+)$. Then because of hypothesis H(2)(iii) we have

$$u_{k+1}(\omega, x_{n_i}(\omega), y_{n_i}(\omega)) \leq \varphi(\omega, a_1(\omega), a_2(\omega)) \quad \mu\text{-a.e.}$$

and

$$\int_\Omega \varphi(\omega, a_1(\omega), a_2(\omega))\, d\mu(\omega) < \infty.$$

So we can apply Fatou's lemma and since by hypothesis H(2)(ii) $u_{k+1}(\omega,\cdot,\cdot)$ is u.s.c., we obtain that

$$\begin{aligned}\limsup_{i\to\infty} \int_\Omega u_{k+1}(\omega, x_{n_i}(\omega), y_{n_i}(\omega))\, d\mu(\omega) &\leq \int_\Omega u_{k+1}(\omega, x(\omega), y(\omega))\, d\mu(\omega)\\ \Rightarrow \limsup_{i\to\infty} J_{k+1}(x_{n_i}, y_{n_i}) &\leq J_{k+1}(x, y)\\ \Rightarrow \lambda \leq J_{k+1}(x, y);& \quad \text{i.e.} \quad (x, y) \in L_\lambda.\end{aligned}$$

Therefore $J_{k+1}(\cdot,\cdot)$ is weakly-u.s.c. on $L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$. Hence if for economy in the notation we denote $\widehat{x}^{n_i} = \{x_k^{n_i}\}_{k\geq 0}$ by $\widehat{x}^i = \{x_k^i\}_{k\geq 0}$, we have

$$\limsup_{i\to\infty} J_{k+1}(x_k^i, x_{k+1}^i) \leq J_{k+1}(x_k^*, x_{k+1}^*).$$

Now define a following sequence in R:

$$R_{k+1}^i = \begin{cases} J_{k+1}(x_k^i, x_{k+1}^i) & \text{if } \ 0 \leq k \leq N_{n_i} = N_i \\ 0, & \text{otherwise,} \end{cases}$$

for all $i \in N_0$. Then we have

$$\limsup_{i\to\infty} R_{k+1}^i \leq J_{k+1}(x_k^*, x_{k+1}^*)$$

and by hypothesis H(3) $R^i_{k+1} \le \beta_{k+1}$ for all $k \in N_0$. Hence via Fatou's lemma we get, recalling that $\widehat{x}^* \in H(v)$:

$$\begin{aligned} &\limsup_{i\to\infty} \sum_{k\ge 0} \delta^{k+1} R^i_{k+1} \le \sum_{k\ge 0} \delta^{k+1} J_{k+1}(x^*_k, x^*_{k+1}) \le m \\ &\Rightarrow \limsup_{i\to\infty} \sum_{k=0}^{N_i} \delta^{k+1} J_{k+1}(x^*_k, x^*_{k+1}) \le m \\ &\Rightarrow w - \lim_{i\to\infty} \frac{1}{n_i} \le \sum_{k\ge 0} \delta^{k+1} J_{k+1}(x^*_k, x^*_{k+1}) \le m \\ &\Rightarrow \limsup_{N\to\infty} m_N \le \sum_{k\ge 0} \delta^{k+1} J_{k+1}(x^*_k, x^*_{k+1}) \le m. \end{aligned} \tag{4}$$

Combining inequalities (3) and (4) and recalling that $\widehat{x}^* \in H(v)$ we conclude that $\widehat{x}^*$ is an optimal program and furthermore $m_N \to m$ as $N \to \infty$. □

4. A transversality condition

In this section we present a primal (i.e. no dual variables (prices) are involved) transversality condition for optimality.

In what follows by $J'_{k+1}(x^*_k, x^*_{k+1}; h_1, h_2)$ we will denote the directional derivative of the concave functional $J_{k+1}(\cdot,\cdot)$ at the point $(x^*_k, x^*_{k+1}) \in L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$ in the direction $(h_1, h_2) \in L^1(\Sigma_k, X) \times L^1(\Sigma_{k+1}, X)$. Furthermore if $\int_\Omega u_{k+1}(\omega, x^*_k(\omega) + h_1(\omega), x^*_{k+1}(\omega) + h_2(\omega))\, d\mu(\omega) > -\infty$ from the monotone convergence theorem we get that

$$J'_{k+1}(x^*_k, x^*_{k+1}; h_1, h_2) = \int_\Omega u'_{k+1}(\omega, x^*_k(\omega), x^*_{k+1}(\omega); h_1(\omega), h_2(\omega))\, d\mu(\omega)$$

with $u'_{k+1}(\omega, x, y; \overline{h}_1, \overline{h}_2)$ being the directional derivative of the concave function $u_{k+1}(\omega,\cdot,\cdot)$ at the point $(x,y) \in X \times X$ in the direction $(\overline{h}_1, \overline{h}_2) \in X \times X$.

We have the following necessary condition for optimality:

Theorem 2. *If hypotheses* H(1), H(2), H(3) *and* H(4) *hold,* $m > -\infty$ *and* $\widehat{x}^* \in H(v)$ *is an optimal program then for every* $\widehat{x} \in H(v)$ *such that* $-\infty < \sum_{k\ge 0} \delta^{k+1} J_{k+1}(x_k, x_{k+1})$, *we have that the real sequence* $\{\delta^{k+1} J'_{k+1}(x^*_k, x^*_{k+1}; x_k - x^*_k, x_{k+1} - x^*_{k+1})\}_{k\ge 0}$ *is summable and*

$$\begin{aligned} -\infty < \sum_{k\ge 0} \delta^{k+1} J'_{k+1}(x^*_k, x^*_{k+1}; x_k - x^*_k, x_{k+1} - x^*_{k+1}) \\ = \sum_{k=0}^{\infty} \delta^{k+1} \int_\Omega u'_{k+1}(\omega, x^*_k(\omega), x^*_{k+1}(\omega); \\ x_k(\omega) - x^*_k(\omega), x_{k+1}(\omega) - x^*_{k+1}(\omega))\, d\mu(\omega) \le 0. \end{aligned}$$

Proof: Let $\widehat{x} \in H(v)$ be such that $-\infty < \sum_{k\geq 0} \delta^{k+1} J_{k+1}(x_k, x_{k+1})$. For $\lambda \in (0,1)$ define $\widehat{x}^{\lambda} = \{x_k^{\lambda}\}_{k\geq 0}$ by $x_k^{\lambda} = (1-\lambda)x_k^* + \lambda x_k$ for all $k \in N_0$. Clearly $\widehat{x}^{\lambda} \in H(v)$. From the concavity of $J_{k+1}(\cdot,\cdot)$ (cf. hypothesis H(2)(ii)) we have

$$\begin{aligned}
&\delta^{k+1}(J_{k+1}(x_k, x_{k+1}) - J_{k+1}(x_k^*, x_{k+1}^*)) \\
&\leq \frac{\delta^{k+1}}{\lambda}(J_{k+1}(x_k^{\lambda}, x_{k+1}^{\lambda}) - J_{k+1}(x_k^*, x_{k+1}^*)),\ k \in N_0, \\
&\Rightarrow \sum_{k\geq 0} \delta^{k+1} J_{k+1}(x_k, x_{k+1}) - J_{k+1}(x_k^*, x_{k+1}^*)) \\
&\leq \sum_{k\geq 0} \delta^{k+1} \frac{1}{\lambda}\Big[J_{k+1}(x_k^{\lambda}, x_{k+1}^{\lambda}) - J_{k+1}(x_k^*, x_{k+1}^*)\Big] \leq 0,
\end{aligned}$$

the last inequality being a consequence of the optimality of $\widehat{x}^*$. In the above inequality the left hand side is bigger than $-\infty$ (hence finite) because of the choice of $\widehat{x} \in H(v)$. On the right hand side apply Fatou's lemma to get

$$\begin{aligned}
-\infty &< \limsup_{\lambda\to 0} \sum_{k\geq 0} \delta^{k+1} \frac{1}{\lambda}\Big[J_{k+1}(x_k^{\lambda}, x_{k+1}^{\lambda}) - J_{k+1}(x_k^*, x_{k+1}^*)\Big] \\
&\leq \sum_{k\geq 0} \delta^{k+1} \limsup_{\lambda\to 0} \frac{1}{\lambda}\Big[J_{k+1}(x_k^{\lambda}, x_{k+1}^{\lambda}) - J_{k+1}(x_k^*, x_{k+1}^*)\Big] \\
&= \sum_{k\geq 0} \delta^{k+1} J'_{k+1}(x_k^*, x_{k+1}^*; x_k - x_k^*, x_{k+1} - x_{k+1}^*) \\
&= \sum_{k\geq 0} \delta^{k+1} \int_{\Omega} u'_{k+1}(\omega, x_k^*(\omega), x_{k+1}^*(\omega); \\
&\qquad\qquad x_k(\omega) - x_k^*(\omega), x_{k+1}(\omega) - x_{k+1}^*(\omega))\, d\mu(\omega) \leq 0.
\end{aligned}$$

$\square$

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