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Isometric classification of norms in rearrangement-invariant function spaces

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Abstract. Suppose that a real nonatomic function space on $[0,1]$ is equipped with two rearrangement-invariant norms $\|\cdot\|$ and $|||\cdot|||$. We study the question whether or not the fact that $(X,\|\cdot\|)$ is isometric to $(X,|||\cdot|||)$ implies that $\|f\| = |||f|||$ for all f in X. We show that in strictly monotone Orlicz and Lorentz spaces this is equivalent to asking whether or not the norms are defined by equal Orlicz functions, resp. Lorentz weights.

We show that the above implication holds true in most rearrangement-invariant spaces, but we also identify a class of Orlicz spaces where it fails. We provide a complete description of Orlicz functions $\varphi \neq \psi$ with the property that L_φ and L_ψ are isometric.

Keywords: isometries, rearrangement-invariant function spaces, Orlicz spaces, Lorentz spaces

Classification: 46B, 46E

1. Introduction

In this paper we study the following question:

Question 1. Suppose that a nonatomic function space on $[0,1]$ is equipped with two rearrangement-invariant norms $\|\cdot\|$ and $|||\cdot|||$ such that $(X,\|\cdot\|)$ and $(X,|||\cdot|||)$ are isometric. Does it imply that the norms $\|\cdot\|$ and $|||\cdot|||$ are same?

Here the word "the same" could be understood in two ways:

(a) we could say that $\|\cdot\|$ and $|||\cdot|||$ are the same if $\|f\| = |||f|||$ for all f in X, i.e. if the identity map $Id : (X,\|\cdot\|) \longrightarrow (X,|||\cdot|||)$ is an isometry, or

(b) if both $\|\cdot\|$ and $|||\cdot|||$ are Orlicz or Lorentz norms we could say that they are the same if they are defined by equal Orlicz functions or Lorentz weights, respectively.

It is well-known that if we do not require $|||\cdot|||$ to be rearrangement-invariant then the answer to Question 1 is no in either the sense (a) or (b), even when $(X,\|\cdot\|) = L_p[0,1]$ with the usual norm (see e.g. [9]).

Question 1 (a) has been previously studied with the additional assumption that $(X,\|\cdot\|) = (L_p[0,1],\|\cdot\|_p)$ (see [2], [1], [3], [5]).

Question 1 (b) was asked by S. Dilworth and H. Hudzik.

Somewhat to author's surprise the answer to both Question 1 (a) and 1 (b) is negative — there exist Orlicz functions $\varphi \neq \psi$ (then, clearly, $\|\cdot\|_\varphi$, $\|\cdot\|_\psi$ are different also in the sense (a)) so that L_φ and L_ψ are isometric.

In fact, Question 1 (a) and 1 (b) are equivalent for strictly monotone Orlicz-Lorentz spaces X, Y, i.e. $Id : X \longrightarrow Y$ is an isometry if and only if the Orlicz functions φ_X, φ_Y and Lorentz weights w_X, w_Y coincide (Theorem 7; this is not true in general, see [5]). To our surprise the proof is much less obvious than one might expect.

Question 1 (a) for $L_p[0,1]$ has been studied by Abramovich and Zaidenberg [1], [2]. They proved that if Y is a (real or complex) rearrangement-invariant nonatomic function space on $[0,1]$ isometric to $L_p[0,1]$ for some $1 \leq p < \infty$ then the isometric isomorphism can be established via an identity map, i.e. $\|f\|_Y = \|f\|_p$ for all $f \in Y$ (cf. also [5], [3] for the case when we additionally assume that Y is an Orlicz space on $[0,1]$).

Zaidenberg [16], [17], [14] studied the general form of isometries between two complex rearrangement-invariant spaces (r.i. spaces) and Jamison, Kamińska and P.K. Lin [6] studied isometries between two complex Musielak-Orlicz spaces and between two real Nakano spaces. They proved that surjective isometries in such settings have to be weighted composition operators and Zaidenberg characterized situations when the existence of isometry between complex r.i. spaces X and Y implies that the identity map is also an isometry.

Theorem 1 below generalizes these results to real spaces on $[0,1]$. We then use Zaidenberg's characterization of isometry groups of r.i. spaces to characterize when the identity map between real r.i. spaces is an isometry. We also give a full description of the exceptional case of Orlicz spaces which can be isometric even when their Orlicz functions are different (Corollary 9 provides the relation between the Orlicz functions that has to be satisfied in that case).

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2. Preliminaries

Let us suppose that Ω is a Polish space and that μ is a σ-finite Borel measure on Ω. We use the term Köthe space in the sense of [12, p. 28]. Thus a **Köthe function space** X on (Ω, μ) is a Banach space of (equivalence classes of) locally integrable Borel functions f on Ω such that:

(1) If $|f| \leq |g|$ a.e. and $g \in X$ then $f \in X$ with $\|f\|_X \leq \|g\|_X$.

(2) If A is a Borel set of finite measure then $\chi_A \in X$.

The **Köthe dual** of X is denoted X'; thus X' is the Köthe space of all g such that $\int |f||g|\, d\mu < \infty$ for every $f \in X$ equipped with the norm $\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int |f||g|\, d\mu$. Then X' can be regarded as a closed subspace of the dual X^* of X.

A **rearrangement-invariant function space** (**r.i. space**) is a Köthe function space on (Ω, μ) which satisfies the conditions:

(1) X' is a norming subspace of X^*.

(2) If $\tau : \Omega \longrightarrow \Omega$ is any measure-preserving invertible Borel automorphism then $f \in X$ if and only if $f \circ \tau \in X$ and $\|f\|_X = \|f \circ \tau\|_X$.

(3) $\|\chi_B\|_X = 1$ if $\mu(B) = 1$.

The commonly studied r.i. spaces are classical Lebesgue spaces L_p, Orlicz, Lorentz and Orlicz-Lorentz spaces. We recall the definitions below.

We say that $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is an **Orlicz function** if φ is convex and $\varphi(0) = 0$. We define the **Orlicz space** L_φ to be the space of those measurable functions f for which $\|f\|_\varphi$ is finite, where $\|f\|_\varphi$ denotes the **Luxemburg norm** defined by

$$\|f\|_\varphi = \inf\left\{ c : \int_\Omega \varphi(\frac{|f(\omega)|}{c})\, d\mu(\omega) \le 1 \right\}.$$

If f is a measurable function, we define the **non-increasing rearrangement** of f to be

$$f^*(x) = \sup\{\, t : \mu(|f| \ge t) \ge x \,\}.$$

If $1 \le q < \infty$, and if $w : (0, \infty) \to (0, \infty)$ is a non-increasing function, we define the **Lorentz space** $L_{w,q}$ to be the space of those measurable functions f for which $\|f\|_{w,q}$ is finite, where $\|f\|_{w,q}$ denotes the **Lorentz norm** defined by

$$\|f\|_{w,q} = \left(\int_0^\infty w(x) f^*(x)^q \, dx \right)^{1/q}.$$

If φ is an Orlicz function, and if $w : [0, \infty) \to [0, \infty)$ is a non-increasing function, we define the **Orlicz-Lorentz space** $\Lambda_{w,\varphi}$ to be the space of measurable functions f for which $\|f\|_{w,\varphi}$ is finite, where $\|f\|_{w,\varphi}$ denotes the **Orlicz-Lorentz norm** defined by

$$\|f\|_{w,\varphi} = \inf\left\{ c : \int_0^\infty w(x)\varphi(\frac{f^*(x)}{c})\, dx \le 1 \right\}.$$

The Orlicz-Lorentz spaces are a common generalization of the Orlicz spaces and the Lorentz spaces.

An operator $T : X \longrightarrow X$ will be called **elementary** or a **weighted composition operator** if there is a Borel function h and a Borel map $\sigma : \Omega \longrightarrow \Omega$ such that $Tf(s) = h(s)f(\sigma(s))$ a.e. for every $f \in X$. Observe that a necessary condition on a and σ is that if B is a Borel set with $\mu(B) = 0$ then $\mu(\sigma^{-1}B \cap \{|h| > 0\}) = 0$.

We now need to introduce a technical definition. We will say that an r.i. space X has **property** (GP) if there exists $n \in \mathbb{N}$ such that for every $t > 0$,

$$\|\chi_{[0,2^{-n}]}\|_X < \|\chi_{[0,2^{-n}]} + t\chi_{[2^{-n},1]}\|_X.$$

We say that X has property (GP') if X' has property (GP).

Notice that if X is strictly monotone then X has property (GP). The reason for introducing this property, rather than simply dealing with strictly monotone spaces, is that every rearrangement-invariant space X has to satisfy at least one of the properties (GP) or (GP'). Indeed, if X fails both (GP) and (GP'), then for all $n \in \mathbb{N}$, say $n = 1$, there exists $\eta > 0$ small enough so that $\|\chi_{[0,2^{-1}]}\|_X = \|\chi_{[0,2^{-1}]} + \eta\chi_{[2^{-1},1]}\|_X$ and $\|\chi_{[0,2-1]}\|_{X^*} = \|\chi_{[0,2^{-1}]} + \eta\chi_{[2^{-1},1]}\|_{X^*}$. But then

$$\frac{1}{2}(1+\eta^2) = \int (\chi_{[0,2^{-1}]}(s) + \eta\chi_{[2^{-1},1]}(s))^2\, ds \leq \|\chi_{[0,2^{-1}]}\|_X \|\chi_{[0,2^{-1}]}\|_{X^*} = \frac{1}{2}$$

which contradicts the fact that $\eta > 0$.

Notice that for any p, $1 < p < \infty$, L_p satisfies both (GP) and (GP').

An Orlicz space L_φ satisfies (GP) if and only if $\varphi(t) > 0$ for all $t > 0$ and there exists $t > 1$ with $\varphi(t) < \infty$.

A Lorentz space $L_{w,p}$, $1 \leq p < \infty$, satisfies (GP) whenever there exists $t > 0$ with $w(t) > 0$, i.e. whenever $L_{w,p} \neq L_\infty$.

An Orlicz-Lorentz space $\Lambda_{w,\varphi}$ satisfies (GP) whenever $\varphi(t) > 0$ for all $t > 0$, there exists $t > 1$ with $\varphi(t) < \infty$ and there exists $s > 0$ with $w(s) > 0$.

3. Isometries between two different spaces X and Y

We start with the statement of the theorem about the form of surjective isometries between two different rearrangement-invariant spaces X and Y. The proof is a technical refinement of [7, Theorem 6.4] and we present it in Section 5.

Theorem 1. *Let X, Y be nonatomic rearrangement-invariant spaces on $[0,1]$ which are not isometrically equal to $L_2[0,1]$, and such that both X and Y satisfy simultaneously the same property (GP) or (GP'). Suppose that $T : X \longrightarrow Y$ is a surjective isometry. Then there exists a Borel function h on $[0,1]$ with $|h| > 0$ and an invertible Borel map $\sigma : [0,1] \longrightarrow [0,1]$ such that $\lambda(\sigma^{-1}(B)) > 0$ if and only if $\lambda(B) > 0$ for $B \in \mathcal{B}$ and so that $Tf(s) = h(s)f(\sigma(s))$ a.e. for every $f \in X$.*

Remark. Unfortunately we were not able to eliminate a technical requirement for spaces X, Y to satisfy property (GP) or (GP'). On one hand, as discussed above, this is not a very restrictive assumption (and this is the reason presenting it here in this slightly unfinished form). On the other hand there is a general question about the group $\mathcal{W}$ of all invertible weighted composition operations which is interesting in itself and which would immediately imply Theorem 1. Namely if T is an invertible operator such that $T \circ \mathcal{W} \circ T^{-1} \subset \mathcal{W}$ does it imply that $T \in \mathcal{W}$? Our proof shows that the answer to this question is yes if we assume a special form of T.

Now we will apply the analysis of isometry groups of rearrangement-invariant function spaces due to Zaidenberg [16].

Following his notation we denote by $NS = NS[0,1]$ the group of all invertible mappings of $\sigma : [0,1] \to [0,1]$ which, together with their inverses, preserve measurability. All surjective isometries of L_p ($p \neq 2$) have form described in Theorem 1 ([10]). So there is a 1-1 correspondence between the group of all (positive) surjective isometries of L_p and the group NS. Let NS_p denote the group NS equipped with the topology induced by weak convergence of operators in L_p.

If $\sigma : [0,1] \to [0,1]$ preserves measurability then we denote by μ_σ the measure defined by $\mu_\sigma(A) = \mu(\sigma^{-1}(A))$. Since $\mu_\sigma(A) = 0$ iff $\mu(A) = 0$ there exists a Radon-Nikodym derivative $\frac{d\mu_\sigma}{d\mu}$ of μ_σ, and we will denote $\frac{d\mu_\sigma}{d\mu}$ by σ'. If G is a subgroup of NS we denote by $N(G)$ the normalizer of G in NS. Finally if X is a function space we denote by $\mathrm{Iso}(X)$ the group of invertible isometries of X.

The following is due to Zaidenberg ([16, Theorems 2 and 3]) (for proof see [11]).

Theorem 2.

(a) *If X is a rearrangement-invariant function space on $[0,1]$ and $X \neq L_2[0,1]$ then the group of invertible isometries of X coincides with one of the following closed subgroups of $NS = NS_p$:*
   (i) *NS;*
   (ii) *$NS(\mathbb{R}_+; a) = \{\sigma \in NS \mid \exists\, b > 0; \sigma'(t) \in \{ba^k\}_{k=-\infty}^{\infty}$ for almost all $t \in [0,1]\}$, where $a > 1$;*
   (iii) *$NS(a; d) = \{\sigma \in NS \mid \exists\, s \in \mathbb{Z} : \sigma'(t) \in \{a^{s+kd}\}_{k=-\infty}^{\infty}$ for almost all $t \in [0,1]\}$, where $a > 1$, $d \in \mathbb{Z}$, $d \geq 1$;*
   (iv) *$U = \{\sigma \in NS \mid \sigma$ is measure preserving$\}$.*

(b) *Two subgroups of NS from the list* (i)–(iv) *are conjugate if and only if they coincide.*

(c) *If G is a subgroup of type* (i), (ii) *or* (iv) *then $N(G) = G$, and $N(NS(a; d)) = NS(\mathbb{R}_+; a^d)$.*

Theorem 2 allows us to precisely identify which weighted composition operators can act as surjective isometries between r.i. spaces.

Proposition 3. *Suppose that X, Y are rearrangement invariant function spaces and $T : X \xrightarrow{\text{onto}} Y$ is an isometry such that $Tf = hf \circ \sigma$. Then either σ is measure preserving and $|h| = 1$ a.e. or there exists p, $1 \leq p \leq \infty$, such that X, Y are equal to L_p with some equivalent norms and T is also an isometry as an operator from L_p to L_p, i.e. $|h(t)|^p = \sigma'(t)$ a.e.*

Proof: We consider two cases. First we assume that $\mathrm{Iso}(X) = U$ (or, symmetrically, $\mathrm{Iso}(Y) = U$). If $\tau \in NS$ we denote by V_τ the composition map defined on Y by

$$V_\tau f = f \circ \tau\ .$$

If τ is measure preserving then $T^{-1}V_\tau T$ is an isometry of X. Hence we have

$$T^{-1}V_\tau Tf(t) = \frac{h \circ \tau(t)}{h \circ \sigma^{-1}(t)}\ f \circ \sigma \circ \tau \circ \sigma^{-1}(t)$$

and $\sigma \circ \tau \circ \sigma^{-1} \in U$ and $|(h \circ \tau(t))/(h \circ \sigma^{-1}(t))| = 1$ a.e. Since $\tau \in U$ is arbitrary we conclude that $|h| \equiv$ const a.e. so $|h| \equiv 1$ a.e. and by Theorem 2 (c) $\sigma \in U$, i.e. σ is measure preserving.

If $\mathrm{Iso}(X) \neq U$ then by [7, Theorem 7.2] (cf. also [16]) X equals L_p for some p, $1 \leq p \leq \infty$, with an equivalent renorming and Boyd indices p_X, q_X of X are both equal to p. Similarly, $\mathrm{Iso}(Y) \neq U$ implies that $Y \cong L_p$ and $p_Y = q_Y = p$ with clearly the same p, since T is an isometry.

Now, by [7, Proposition 7.1] (cf. also [7, Theorem 5.1]) T is an isometry of L_p. We should remark that [7, Proposition 7.1] is formulated for isometries acting on only one space $X \to X$ but in fact it uses only the equality of Boyd indices of the range and domain spaces. Since there is no change in the proof we will not repeat it here. □

As an immediate consequence of Theorem 1 and Proposition 3 we obtain the following corollary.

Corollary 4. *Suppose that X, Y are nonatomic rearrangement-invariant spaces on $[0,1]$ not isometrically equal to $L_2[0,1]$, and both satisfying the same property (GP) or (GP') and that there exists an isometry $T : X \xrightarrow{\text{onto}} Y$. Then*

$$\mathrm{Iso}(X) = \mathrm{Iso}(Y).$$

Further, if $\mathrm{Iso}(X) \neq NS(a,d)$ then $Id : X \xrightarrow{\text{onto}} Y$ is also an isometry.

In particular identity map is an isometry if one of the spaces X or Y is equal to L_p for some p, $1 < p < \infty$, $p \neq 2$, since then L_p satisfies both (GP) and (GP'). Thus we generalize the result of Abramovich and Zaidenberg [2] who proved Corollary 4 in the case when one of the spaces X or Y equals L_p, for some $1 \leq p \leq \infty$.

Also notice that, by [7, Theorem 7.2], if X, Y are not isomorphic to any L_p then $\mathrm{Iso}(X) = \mathrm{Iso}(Y) = U$ and hence Corollary 4 holds.

Furthermore, since $N(NS(a;d)) \supsetneq NS(a;d)$ (Theorem 2 (c)) there are isometric spaces X, Y such that $\mathrm{Iso}(X) = \mathrm{Iso}(Y) = NS(a;d)$ and the identity map is not an isometry — notice that identity map is well defined since X, Y are both equal to L_p with an equivalent norm. Thus, as a consequence of Theorem 2 (c), we get a positive answer to Question 1 (a). In fact, we even can get two Orlicz spaces with this property.

Example 1. (See [10], [16] and Corollary 9 below.)

Consider the Orlicz function $\varphi(t) = t^5 \exp(\sin \ln t)$. Take any $\sigma \in NS(\mathbb{R}_+, e^{2\pi\cdot 5}) \setminus NS(e^{2\pi\cdot 5}; 1)$ and let h be such that $Tf = h\, f \circ \sigma$ defines the isometry on L_5, i.e. $|h(t)|^5 = \sigma'(t)$ for almost all t. Now define a new norm $\|\cdot\|_X$ on L_5 by $\|f\|_X = \|Tf\|_\varphi$. Then, clearly X and L_φ are isometric. Moreover

$$\begin{aligned}\|f\|_X = \|h \cdot f \circ \sigma\|_\varphi &= \inf\left\{c \ :\ \int \varphi\left(\frac{|h(t)f(\sigma(t))|}{c}\right)\, dt \leq 1\right\}\\ &= \inf\left\{c \ :\ \int \frac{|h(t)|^5|f(\sigma(t))|^5}{c^5} \cdot \exp\left[\sin\left(\ln|h(t)| + \ln\frac{|f(\sigma(t))|}{c}\right)\right]\, dt \leq 1\right\}\end{aligned}$$

and letting $u = \sigma(t)$, since $|h(t)|^5 = \sigma'(t)$

$$= \inf\left\{c \ : \ \int \frac{|f(u)|^5}{c^5} \cdot \exp\left[\sin\left(\frac{1}{5}\ln(\sigma'(t)) + \ln\frac{|f(u)|}{c}\right)\right] dt \leq 1\right\}$$

since for all t there exists $k(t) \in \mathbb{Z}$ with $\sigma'(t) = be^{2\pi\cdot 5k(t)}$

$$= \inf\left\{c \ : \ \int \frac{|f(u)|^5}{c^5} \cdot \exp\left[\sin\left(\ln(b^{\frac{1}{5}} \cdot \frac{|f(u)|}{c})\right)\right] dt \leq 1\right\}$$

Thus $\|\cdot\|_X$ is an Orlicz norm defined by the Orlicz function $\varphi_\sigma(t) = t^5\exp(\sin\ln b^{1/5}t)$, where $b \in \mathbb{R}_+ \setminus \{e^{10\pi k}; k \in \mathbb{Z}\}$ is such that $\sigma'(t) \in \{be^{2\pi\cdot 5k} : k \in \mathbb{Z}\}$.

4. Isometric uniqueness of norm

In this section we study Question 1 (b) for Orlicz-Lorentz spaces. To our surprise the proof, while definitely not difficult, is not completely obvious.

We will work not with the usual definition of Orlicz-Lorentz spaces (which was presented in the preliminaries) but with the equivalent one introduced by Montgomery-Smith [13]. We feel that this definition is very close to the spirit of Krasnosel'skii-Rutickii's definition of Orlicz spaces ([8]) and it is convenient for us for technical reasons. We briefly remind the notations from [13] below.

First we define φ-functions. These replace the notion of Orlicz functions in our discussions.

A **φ-function** is a continuous, strictly increasing function $F : [0,\infty) \longrightarrow [0,\infty)$ such that $F(0) = 0$; $F(1) = 1$ and $\lim_{n\to\infty} F(t) = \infty$.

The definition of a φ-function is slightly more restrictive than that of an Orlicz function in that we insist that F be strictly increasing.

If F is a φ-function, we will define the function $\widetilde{F}(t)$ to be $1/F(1/t)$ if $t > 0$, and 0 if $t = 0$.

If $(\Omega, \mathcal{F}, \mu)$ is a measure space, and F and G are φ-functions, then we define the **Orlicz-Lorentz functional** of a measurable function f by

$$\|f\|_{F,G} = \|f^* \circ \tilde{F} \circ \widetilde{G}^{-1}\|_G.$$

The **Orlicz-Lorentz space** $L_{F,G}(\Omega, \mathcal{F}, \mu)$ (or $L_{F,G}$ for short) is the vector space of measurable functions f for which $\|f\|_{F,G} < \infty$, modulo functions that are zero almost everywhere.

Notice that if $\Omega = [0,1]$ then functions F and G do not need to be defined on $[0,\infty)$. We require only that $(\widetilde{F} \circ \widetilde{G}^{-1})^{-1}$ is defined on $[0,1]$, and thus F and G are really defined on $[0,1]$.

The Orlicz-Lorentz spaces defined here are equivalent to the definition given in the preliminaries. Namely, if w is a weight function, and G is a φ-function, then $\Lambda_{w,G} = L_{\tilde{W}^{-1}\circ G,G}$, where

$$W(t) = \int_0^t w(s)\,ds.$$

We will also need the following two simple lemmas from [13]. For the sake of completeness we present their short proofs with adjustments for our situation.

Lemma 5. *Suppose that F, G_1 and G_2 are φ-functions. Then $\|f\|_{F,G_1} = \|f\|_{F,G_2}$ for all measurable f if and only if $\|f\|_{1,G_1} = \|f\|_{1,G_2}$ for all measurable f.*

Proof: This follows because $\|f\|_{F,G} = \|f^* \circ \tilde{F}\|_{1,G}$. $\square$

Lemma 6. *Suppose that G_1, G_2 and H are φ-functions. Then if $\|f\|_{1,G_1} = \|f\|_{1,G_2}$ for all measurable f, then $\|f\|_{1,G_1\circ H} = \|f\|_{1,G_2\circ H}$ for all measurable f.*

Proof: Suppose that $\|f\|_{1,G_1} = \|f\|_{1,G_2}$ for all measurable f. Fix $\varepsilon > 0$ and let g be any simple function such that $1/(1+\varepsilon) \le \|g\|_{1,G_2\circ H} < 1$ and $g^* = \sum_{i=1}^n a_i\chi_{A_i}$.

Then there exists $\lambda < 1$ such that

$$\begin{aligned}1 &\ge \int G_2 \circ H \circ \frac{1}{\lambda} g^* \circ \widetilde{H}^{-1} \circ \widetilde{G}_2^{-1}(x)\,dx = \sum_{i=1}^n \int G_2 \circ H \left(\frac{1}{\lambda} a_i \chi_{\widetilde{G}_2\circ\widetilde{H}(A_i)}(x)\right) dx \\ &= \sum_{i=1}^n G_2 \circ H\left(\frac{1}{\lambda}a_i\right)\mu(\widetilde{G}_2 \circ \widetilde{H}(A_i)) > \sum_{i=1}^n G_2 \circ H(a_i)\mu(\widetilde{G}_2 \circ \widetilde{H}(A_i)) \\ &= \int G_2 \circ H \circ g^* \circ \widetilde{H}^{-1} \circ \widetilde{G}_2^{-1}(x)\,dx,\end{aligned}$$

since H is strictly increasing.

Therefore, $\|H \circ g^* \circ \widetilde{H}\|^{-1}{}_{1,G_2} < 1$, and hence $\|H \circ g^* \circ \widetilde{H}^{-1}\|_{1,G_1} < 1$. Therefore,

$$\int G_1\bigl(H \circ g^* \circ \widetilde{H}^{-1} \circ \widetilde{G}_1^{-1}(x)\bigr)\,dx \le 1,$$

that is, we have

$$\int G_1 \circ H\bigl(g^* \circ \tilde{H}^{-1} \circ \widetilde{G}_1^{-1}(x)\bigr)\,dx \le 1,$$

that is, $\|g\|_{1,G_1\circ H} \le 1$. Hence $\|g\|_{1,G_1\circ H} \le (1+\varepsilon)\|g\|_{1,G_2\circ H}$. Since ε is arbitrary and by symmetry we get that $\|g\|_{1,G_1\circ H} = \|g\|_{1,G_2\circ H}$ for all simple functions. $\square$

We are now ready to prove the main result of this section.

Theorem 7. *Suppose that L_{F_1,G_1}, L_{F_2,G_2} are Orlicz-Lorentz spaces and that the identity map is an isometry, i.e. for all $f \in L_{F_1,G_1}$, $\|f\|_{F_1,G_1} = \|f\|_{F_2,G_2}$. Then $F_1(t) = F_2(t)$ and $G_1(t) = G_2(t)$ for all t.*

Remark. Notice that, by Corollary 4, whenever L_{F_1,G_1}, L_{F_2,G_2} are isometric and $\mathrm{Iso}(L_{F_1,G_1}) \neq N(a,d)$, for any $a > 1$, $d \in \mathbb{N}$, then the identity map is an isometry. In particular this holds whenever L_{F_1,G_1}, L_{F_2,G_2} are not isomorphic to any L_p, and also when $L_{F_1,G_1} = L_p$.

Proof: Let $A \subset \Omega$. Then for $i = 1, 2$

$$\|\chi_A\|_{F_i,G_i} = \widetilde{F}_i(\mu(A) = \frac{1}{F_i((\mu(A))^{-1})}.$$

Hence if $\Omega = [0,\infty)$ then $F_1(t) = F_2(t)$ for all $t \geq 0$ and if $\Omega = [0,1]$ then $F_1(t) = F_2(t)$ for $t \geq 1$, but as we mentioned in the definition of $L_{F,G}([0,1])$, $[1,\infty)$ is the required domain of F in this case. Now by Lemma 5 we get that for all f $\|f\|_{1,G_1} = \|f\|_{1,G_2}$ and by Lemma 6 $\|f\|_{1,G_1\circ G_2^{-1}} = \|f\|_{1,1}$. So it is enough to prove that if $\|f\|_{1,G} = \|f\|_1$ for all $f \in L_1$ then $G(t) = t$ for all t. For any s, $0 < s < 1$ and a, $0 < a \leq 1 < \frac{1}{s}$ consider $f = \frac{1-as}{1-s}\chi_{[0,1-s]} + a\chi_{[1-s,1]}$. Then $\|f\|_1 = 1$ and $f^* = f$. Hence $\|f\|_{1,G} = 1$. Since

$$f \circ \widetilde{G}^{-1}(t) = \frac{1-a+as}{s}\chi_{[0,\frac{1}{G(1/s)}]} + a\chi_{[\frac{1}{G(1/s)},1]}$$

and since G is continuous, $\|f\|_{1,G} = 1$ implies that for all s, $0 < s < 1$ and a, $0 < a \leq 1 < \frac{1}{s}$:

$$\frac{1}{G(1/s)}G\left(\frac{1-a+as}{s}\right) + \left(1 - \frac{1}{G(1/s)}\right)G(a) = 1. \tag{1}$$

Now, for a fixed s, take a left derivative of equation (1) with respect to a. We get

$$\frac{1}{G(1/s)}G'\left(\frac{1-a+as}{s}\right)\left(\frac{-1+s}{s}\right) + \left(1 - \frac{1}{G(1/s)}\right)G'(a) = 0$$

where $G'(t)$ denotes the left derivative of G. In particular when $a = 1$ we have

$$\frac{1}{G(1/s)}G'(1)\left(\frac{-1+s}{s}\right) + \left(1 - \frac{1}{G(1/s)}\right)G'(1) = 0.$$

Since G is convex and strictly increasing $G'(1) \neq 0$, so

$$\frac{1}{G(1/s)}\left(\frac{-1+s}{s}\right) + \left(1 - \frac{1}{G(1/s)}\right) = 0$$

and, after simplification $G(1/s) = 1/s$ for all s, $0 < s < 1$, i.e. $G(t) = t$ for all $t > 1$ which ends the proof in the case when $\Omega = [0,1]$ since in that case G is considered as a function from $[1,\infty)$.

If $\Omega = [0,\infty)$ we consider any $a < 1$. Then $\frac{1-a+as}{s} > 1$ and (1) can be rewritten as:

$$s \cdot \frac{1-a+as}{s} + (1-s)G(a) = 1.$$

and therefore

$$\begin{aligned} 1 - a + as + (1-s)G(a) &= 1 \\ (1-s)G(a) &= a(1-s) \\ G(a) &= a \text{ for all } a < 1. \end{aligned}$$

By continuity $G(t) = t$ for all $t > 0$. □

As a corollary of Theorem 7 we obtain the characterizations of isometric Lorentz spaces and Orlicz spaces.

Corollary 8. *Two Lorentz spaces $L_{w_1,p_1}[0,1]$, $L_{w_2,p_2}[0,1]$ are isometric if and only if $w_1(t) = w_2(t)$ a.e. and $p_1 = p_2$, or L_{w_1,p_1}, L_{w_2,p_2} are isometric to L_∞, i.e. for $i = 1,2$, $p_i = \infty$ or $w_i(t) = 0$ for all $t > 0$.*

Proof: If there exist t_1, t_2 such that $w_i(t_i) > 0$ for $i = 1,2$, and $p_1, p_2 \neq \infty$, then L_{w_1,p_1}, L_{w_2,p_2} satisfy property (GP). Also, by [4], $\mathrm{Iso}(L_{w_1,p_1}) = \mathrm{Iso}(L_{w_2,p_2}) = U$. Hence, by Corollary 4, $Id : L_{w_1,p_1} \longrightarrow L_{w_2,p_2}$ is an isometry.

If $w_1(t), w_2(t) > 0$ for all $t > 0$, we can apply Theorem 7 to get the conclusion. Otherwise let

$$F_{s,i}(a) = \|\chi_{[0,s]} + a\chi_{[s,1]}\|_{w_i,p_i}$$

for $i = 1,2$. Since the identity map is an isometry $F_{s,1}(a) = F_{s,2}(a)$ for all s, $a \leq 1$. In particular

$$\left(\int_{[0,s]} w_1(t)\,dt\right)^{\frac{1}{p_1}} = F_{s,1}(0) = F_{s,2}(0) = \left(\int_{[0,s]} w_2(t)\,dt\right)^{\frac{1}{p_2}}. \tag{2}$$

Moreover

$$\int_{[s,1]} w_1(t)\,dt = F'_{s,1}(1) = F'_{s,2}(1) = \int_{[s,1]} w_2(t)\,dt. \tag{3}$$

Since (2) and (3) hold for all $s \leq 1$ we conclude that $p_1 = p_2$ and $w_1(t) = w_2(t)$ a.e.

If $w_1(t) = 0$ for all $t > 0$ or if $p_1 = \infty$, then $L_{w_1,p_1} = L_\infty$ and, by [2], $Id : L_{w_1,p_1} = L_\infty \longrightarrow L_{w_2,p_2}$ is an isometry, and thus also $p_2 = \infty$ or $w_2(t) = 0$ for all $t > 0$. □

Corollary 9. *Let φ, ψ be two Orlicz functions such that $0 < \varphi(t) < \infty$ and $0 < \psi(t) < \infty$ for all $t > 0$. Then Orlicz spaces L_φ and L_ψ are isometric if and only if one of the two possibilities holds:*

(1) *$\varphi(t) = \psi(t)$ for all t;*

(2) *there exist $b > 0$ and $p \geq 1$, such that $\varphi(t) = (1/b)\psi(b^{(1/p)}t)$ for all t. In this case L_φ, L_ψ are isomorphic to L_p.*

Remark. If $\varphi(t) = \psi(t) = \infty$ for all $t > 1$, then L_φ and L_ψ are isometric with L_∞ via an identity map ([5, Corollary 1]).

For the proof of the above corollary we need a description of isometry groups of Orlicz spaces, which is due to Zaidenberg [15], [16]. However, since the English translation of [16] has an unfortunate misprint (the author has not seen the original Russian version) and since the proof is not presented in full there, we will enclose both the exact statement and its proof (we use the same notation as in Theorem 2).

Lemma 10. *If L_φ is a (real or complex) Orlicz space ($L_\varphi \neq L_2$) then the group G_φ of all surjective isometries of L_φ equals:*

(a) *NS, when $\varphi(t) = t^p$ for some $p \geq 1$, $p \neq 2$,*

(b) *$NS(\bar{a}^p, 1)$, when the multiplier group of φ is generated by $\bar{a}$, $\bar{a} > 1$, and L_φ is isomorphic to L_p,*

(c) *U, otherwise.*

Proof of Lemma 10: By [15, Theorem 4] (cf. also [7]) every surjective isometry S of L_φ has form

$$Sf = h\, f \circ \sigma,$$

where σ an invertible Borel map $\sigma : [0,1] \longrightarrow [0,1]$ and $h : [0,1] \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are such that

$$\varphi(|h(t)| \cdot x) = \sigma'(t)\varphi(x) \quad \text{for a.e. } t \text{ and all } x \geq 0. \tag{4}$$

By Theorem 2 (a) the isometry group G_φ coincides with one of the following groups: NS, $NS(\mathbb{R}_+, a)$, $NS(a, 1)$, U.

As mentioned above, it is well known that $G_\varphi = NS$ if and only if $L_\varphi = L_p$.

If $G_\varphi \neq U$ then L_φ is isomorphic to L_p ([16, Theorem 4], [7, Theorem 7.2]). Moreover, in this case we have:

$$|h(t)|^p = \sigma'(t) \quad \text{a.e.} \tag{5}$$

Suppose that $G_\varphi \supsetneq U$. Then $G_\varphi \subseteq NS(\mathbb{R}_+, a)$ for some $a > 1$, i.e. there exists $b > 0$ with $\sigma'(t) \in \{ba^k\}^{k \in \mathbb{Z}}$ for all t.

Consider $\sigma \in G_\varphi \setminus U$. Without loss of generality we can assume that $\mu(\{t : \sigma'(t) = b\}) > 0$. Then by (5) and (4) we get

$$\varphi(b^{1/p}x) = b\varphi(x) \quad \text{for all } x.$$

In particular, when $x = 1$, we get $\varphi(b^{1/p}) = b$. Therefore

$$\varphi(b^{1/p}x) = \varphi(b^{1/p})\varphi(x) \quad \text{for all } x. \tag{6}$$

Hence $b^{1/p}$ belongs to the multiplier group of φ. If the multiplier group of φ is trivial (i.e. $= \{1\}$) then $b = 1$ for all $\sigma \in G_\varphi \setminus U$, which is impossible. Thus the multiplier group of φ is cyclic, generated, say, by $\bar{a}$.

Let $k \in \mathbb{Z}$ be such that $\mu(\{t : \ \sigma'(t) = ba^k\}) > 0$. Similarly as above we get by (5), (4) and (6)

$$\begin{aligned}\varphi(b^{1/p}a^{k/p}x) &= ba^k\varphi(x) = a^k\varphi(b^{1/p}x)\\ &= \varphi(a^{k/p})\varphi(b^{1/p}x)\end{aligned}$$

for all $x \geq 0$.

Thus $a^{k/p}$ belongs to the multiplier group of φ, i.e. $a^{k/p} = \bar{a}^n$ for some $n \in \mathbb{N}$. Hence $G_\varphi = NS(\bar{a}^p, 1)$. □

Proof of Corollary 9: Assume first that L_φ and L_ψ are isometric.

Since $0 < \varphi(t) < \infty$ and $0 < \psi(t) < \infty$ for all $t > 0$, L_φ and L_ψ both satisfy (GP). Thus, by Corollary 4, $\mathrm{Iso}(L_\varphi) = \mathrm{Iso}(L_\psi)$.

Further, if $\mathrm{Iso}(L_\varphi) = \mathrm{Iso}(L_\psi) \neq NS(a,1)$ for any $a > 1$, then the identity map between L_φ and L_ψ is also an isometry and thus, by Theorem 7, we have (1).

Thus suppose that $\mathrm{Iso}(L_\varphi) = \mathrm{Iso}(L_\psi) = NS(a,1)$ for some $a > 1$. Then, by Lemma 10, there exists p ($1 \leq p < \infty$) so that L_φ, L_ψ are isomorphic to L_p and $\bar{a} = a^{1/p}$ is a generator of the multiplier group for ψ (resp. φ) i.e.

$$\psi(\bar{a}t) = \psi(\bar{a})\psi(t), \quad \text{for all } t$$

and if $\psi(ct) = \psi(c)\psi(t)$ for all t, then $c = \bar{a}^n$ for some $n \in \mathbb{Z}$.

Now, since L_ψ is isomorphic to L_p, we can represent ψ as:

$$\psi(t) = t^p \cdot \tilde{\psi}(t) \quad \text{for all } t,$$

where, by [8, Theorem 8.1]

$$C_1 \leq \tilde{\psi}(t) \leq C_2 \quad \text{for all } t \geq t_0, \tag{7}$$

for some C_1, C_2, $t_0 > 0$ (and $\tilde{\psi}(t) \not\equiv const$ since L_ψ is not isometric to L_p).

It is easy to see that

$$\tilde{\psi}(ct) = \tilde{\psi}(c)\tilde{\psi}(t) \iff \psi(ct) = \psi(c)\psi(t).$$

Thus $\bar{a}$ is a generator of the multiplier group for $\tilde{\psi}$. Further

$$\tilde{\psi}(\bar{a}^n) = (\tilde{\psi}(\bar{a}))^n \quad \text{for all } \ n \in \mathbb{N}.$$

Therefore, by (7), we get $\tilde{\psi}(\bar{a}) = 1$.

Now suppose that $T : L_\varphi \longrightarrow L_\psi$ is a surjective isometry. Then by Theorems 1 and 2 (c) there exist $\sigma \in NS(\mathbb{R}_+, a)$ and a map h on $[0,1]$ with $|h(t)|^p = \sigma'(t)$ a.e., such that

$$Tf = h \cdot f \circ \sigma \quad \text{for all} \quad f \in L_\varphi.$$

Thus for all $f \in L_\varphi$ we have:

$$\begin{aligned}
\|f\|_\varphi &= \inf\left\{c \ : \int_0^1 \psi\left(\frac{|h(t)|\ |f(\sigma(t))|}{c}\right) dt \le 1\right\} \\
&= \inf\left\{c \ : \int_0^1 \frac{|h(t)|^p\ |f(\sigma(t))|^p}{c^p} \tilde{\psi}\left(\frac{|h(t)|\ |f(\sigma(t))|}{c}\right) dt \le 1\right\}
\end{aligned}$$

letting $s = \sigma(t)$, since $|h(t)|^p = \sigma'(t)$, we get

$$\begin{aligned}
&= \inf\left\{c \ : \int_0^1 \frac{|f(s)|^p}{c^p} \tilde{\psi}\left(\frac{|h(\sigma^{-1}(s))|\ |f(s)|}{c}\right) ds \le 1\right\} \\
&= \inf\left\{c \ : \int_0^1 \frac{|f(s)|^p}{c^p} \tilde{\psi}\left(\frac{b^{1/p} a^{k(s)/p}\ |f(s)|}{c}\right) ds \le 1\right\}
\end{aligned}$$

where $k(s) \in \mathbb{Z}$ is such that $\sigma'(\sigma^{-1}(s)) = ba^{k(s)}$ (since $\sigma \in NS(\mathbb{R}_+, a)$)

$$= \inf\left\{c \ : \int_0^1 \frac{|f(s)|^p}{c^p} \tilde{\psi}\left(\frac{b^{1/p}\ |f(s)|}{c}\right) ds \le 1\right\}$$

since $a^{1/p} = \bar{a}$ is a multiplier for $\tilde{\psi}$ and $\tilde{\psi}(a^{1/p}) = 1$,

$$\begin{aligned}
&= \inf\left\{c \ : \int_0^1 \psi_1\left(\frac{|f(s)|}{c}\right) ds \le 1\right\} \\
&= \|f\|_{\psi_1},
\end{aligned}$$

where $\psi_1(x) = (1/b)\psi(b^{1/p}x)$, for all x.

Thus $Id : L_\varphi \longrightarrow L_{\psi_1}$ is an isometry and by Theorem 7, $\varphi(x) = \psi_1(x)$ a.e., i.e. $\varphi(x) = (1/b)\psi(b^{1/p}x)$. $\square$

5. Proof of Theorem 1

The proof of Theorem 1 follows the same essential steps as that of [7, Theorem 6.4].

Notice that, similarly as in [7], it is enough to assume that spaces X, Y satisfy property (GP). Otherwise X', Y' satisfy (GP) and we prove that $T' : Y' \longrightarrow X'$ is elementary. Then by [7, Lemma 2.4] we get that also T is elementary (see also final remarks in the proof of Theorem 6.4 in [7, p. 322]).

We will need an analogue of [7, Proposition 6.3]. The proof is the same so we omit it.

Proposition 11. *Let X be an r.i. space on $[0,1]$ with property (GP), and such that $X \neq L_2$ (isometrically) and let Y be any Köthe function space on $[0,1]$ for which Y' is norming. Suppose $T : X \longrightarrow Y$ is a surjective isometry. Then there exists a sequence of Borel maps $\sigma_n : [0,1] \longrightarrow [0,1]$ and Borel functions a_n on $[0,1]$ so that $|a_n(s)| \geq |a_{n+1}(s)|$ a.e. for every n and $\sigma_m(s) \neq \sigma_n(s)$ whenever $m \neq n$ and $s \in [0,1]$ and for which*

$$Tf(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n(s)) \quad \text{a.e. for all} \quad f \in L_\infty,$$

and the above random measure representation (also called abstract kernel) is unique.

Further, there exists a constant C, independent of the isometry T, such that

$$\int_0^1 \left(\sum_{n=1}^{\infty} |a_n(s)| \right) ds \leq C.$$

Remark. The reader might have noticed a slight difference in the assumptions of Proposition 11 here and Proposition 6.3 in [7]. In fact property (GP) described here is weaker than property (P) from [7]. Because of the excessive technical terminology we do not wish to fully analyze the slight differences that this produces — we just want to point out that property (P) was introduced to ensure the validity of [7, Lemma 5.3], which provides the crucial step for the argument in [7]. The generalized property (GP) even though weaker than property (P) still is sufficient for [7, Lemma 5.3] to hold (with identical proof) and hence there is no difference in any subsequent arguments including the proof of the proposition quoted above.

Next we need to show that $|a_2(s)| = 0$ a.e. We follow the construction very similar to that presented in [7]. We will recall all the necessary notations.

First, by Proposition 11, T^{-1} has a random measure representation with all the properties described above:

$$T^{-1} f(s) = \sum_{n=1}^{\infty} b_n(s) f(\varrho_n(s)) \quad \text{a.e. for all} \quad f \in L_\infty,$$

and there exists $K > 0$, depending only on Y, such that

$$\int_0^1 \left(\sum_{n=1}^{\infty} |b_n(s)| \right) ds \leq K.$$

Let $M_N(s)$ be the greatest index such that $\sigma_1(s), \dots, \sigma_M(s)$ belong to distinct dyadic intervals $D(N,k)$. Then $M_N(s) \longrightarrow \infty$ for all s and it follows easily that given $\varepsilon > 0$ we can find M, N and a Borel subset E of $[0,1]$ with $\lambda(E) > 1-\varepsilon$ and such that $M_N(s) \geq M$ for $s \in E$, and

$$\int_{[0,1]\setminus E} \sum_{n=1}^{\infty} |a_n(t)|\, dt < \varepsilon$$

$$\int_E \sum_{n=M+1}^{\infty} |a_n(s)|\, ds < \varepsilon.$$

Next we use the group $\mathcal{T}$ defined in [7]. We recall the definition.

Set $P = 2^N$ and let us identify the circle group $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z} = [0,1)$ in the natural way. For $\theta \in [0,1)^P$ we define a measure preserving Borel automorphism $\gamma = (\gamma_1, \dots, \theta_P)$ given by $\gamma(0) = 0$ and then

$$\gamma(s) = s + (\theta_k - \rho)2^{-N}$$

for $(k-1)2^{-N} < s \leq k2^{-N}$ where $\rho = 1$ if $2^N s + \theta_k > k$ and $\rho = 0$ otherwise. Thus γ leaves each $D(N,k)$ invariant. The set of all such γ is a group of automorphisms Γ which we endow with the structure of the topological group $\mathbb{T}^P = [0,1)^P$. We denote Haar measure on Γ by $d\gamma$. For each k let Γ_k be the subgroup of all $\gamma(\theta)$ for which $\theta_i = 0$ when $i \neq k$. Thus $\Gamma = \Gamma_1 \dots \Gamma_P$.

We also let the finite permutation group Π_P act on $[0,1]$ by considering a permutation π as inducing an automorphism also denoted π by $\pi(0) = 0$ and then $\pi(s) = \pi(k) - k + s$ for $(k-1)2^{-N} < s \leq k2^{-N}$. We again denote normalized Haar measure on Π_P by $d\pi$. Finally note that the set $\Gamma\Pi_P = \mathcal{T}$ also forms a compact group when we endow this with the product topology and Haar measure $d\tau = d\gamma\, d\pi$ when $\tau = \gamma\pi$.

We now wish to consider the isometries $V_\tau : X \longrightarrow X$ for $\tau \in \mathcal{T}$ defined by $V_\tau x = x \circ \tau$. For every $\tau \in \mathcal{T}$ the operator $S(\tau) = TV_\tau T^{-1}$ is a surjective isometry of Y and so, by [7, Theorem 6.4], the unique random measure representation of $S(\tau)$ consists of exactly one term i.e. there exist Borel function h on $[0,1]$ and an invertible Borel map $\sigma : [0,1] \longrightarrow [0,1]$ such that $S(\tau)f = hf \circ \sigma$ for all f in Y.

On the other hand the abstract kernels of T^{-1}, V_τ and T 'multiply' i.e., more precisely, the following version of [7, Lemma 6.5] holds (with the identical proof).

Lemma 12. *For almost every $\tau \in \mathcal{T}$ we have that*

$$\int_0^1 \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |a_j(s)||b_n(\tau\sigma_j s)|\,ds < \infty \tag{8}$$

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_j(s)b_n(\tau\sigma_j s)\delta(\varrho_n\tau\sigma_j s) = \nu_s^{S(\tau)} \text{ a.e.} \tag{9}$$

Finally, we state an equivalent of [7, Lemma 6.6] in the formulation more convenient for our present purpose.

Lemma 13. *For almost every $(s,\tau) \in E\times\mathcal{T}$ and for any two distinct pairs (n,j), (m,i) where $m,n \in \mathbb{N}$ and $i,j \le M$ if $\varrho_n\tau\sigma_j s = \varrho_m\tau\sigma_i s$ then $a_j(s)b_n(\tau\sigma_j s) = 0$ or $a_i(s)b_m(\tau\sigma_i s) = 0$.*

We now complete the proof that $|a_2(s)| = 0$ a.e.

By Lemma 12 we have that for almost every $(s,\tau) \in E\times\mathcal{T}$ the series

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_j(s)b_n(\tau\sigma_j s)\delta(\varrho_n\tau\sigma_j s)$$

has length one. Hence for all, except one pair $(j,n) \in \mathbb{N}\times\mathbb{N}$ we have

$$a_j(s)b_n(\tau\sigma_j s) = -\sum_{k,l\in I(j,n,\tau,s)} a_k(s)b_l(\tau\sigma_k s), \tag{10}$$

where $I(j,n,\tau,s) = \{(k,l) : \varrho_n\tau\sigma_j s = \varrho_l\tau\sigma_k s\}$.

We restrict our attention to $j = 1,2$ and $n = 1$. Then (10) is valid for at least one of the pairs $(1,1)$, $(2,1)$. By Lemma 13 for almost every $(s,\tau) \in E\times\mathcal{T}$ if $(k,l) \in I(j,1,\tau,s)$ then $k > M$. Hence, if (10) holds for $(j,1)$ we have

$$\begin{aligned}
\int_{E\times\mathcal{T}} |a_j(s)b_1(\tau\sigma_j s)|\,ds\,d\tau &= \int_{E\times\mathcal{T}} \Big|\sum_{k,l\in I(j,n,\tau,s)} a_k(s)b_l(\tau\sigma_k s)\Big|\,ds\,d\tau \\
&\le \int_{E\times\mathcal{T}} \sum_{k>M}\sum_{l=1}^{\infty} |a_k(s)||b_l(\tau\sigma_k s)|\,ds\,d\tau \\
&= \int_E \left(\sum_{k>M} |a_k(s)| \int_{\mathcal{T}} \sum_{l=1}^{\infty} |b_l(\tau\sigma_k s)|\,d\tau\right) ds
\end{aligned}$$

But for almost every s $\int_{\mathcal{T}} |b_l(\tau\sigma_k s)|\,d\tau = \int_0^1 |b_l(t)|\,dt$. Thus we obtain:

$$\int_{E\times\mathcal{T}} |a_j(s)b_1(\tau\sigma_j s)|\,ds\,d\tau \le \int_E \sum_{k>M} |a_k(s)|\,ds \int_0^1 \sum_{l=1}^{\infty} |b_l(t)|\,dt \le \varepsilon K.$$

On the other hand,

$$\int_{E\times\mathcal{T}} |a_j(s)b_1(\tau\sigma_j s)|\,ds\,d\tau = \int_E |a_j(s)| \int_{\mathcal{T}} |b_1(\tau\sigma_j s)|\,d\tau\,ds$$
$$= \int_E |a_j(s)|\,ds \int_0^1 |b_1(t)|\,dt = \beta \int_E |a_j(s)|\,ds,$$

where $\beta = \int_0^1 |b_1(t)|\,dt$ $(0 < \beta \le K)$.

Therefore

$$\int_E |a_j(s)|\,ds \le \frac{\varepsilon K}{\beta}$$

and

$$\int_0^1 |a_j(s)|\,ds = \int_E |a_j(s)|\,ds + \int_{[0,1]\setminus E} |a_j(s)|\,ds \le \frac{\varepsilon K}{\beta} + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily we see that (10) implies that $\int_0^1 |a_j(s)|\,ds = 0$. Therefore (10) cannot hold for $(1,1)$ and, thus, it holds for $(2,1)$ and $|a_2(s)| = 0$ a.e. $\square$

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