Nonconcentrating generalized Young functionals

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Abstract. The Young measures, used widely for relaxation of various optimization problems, can be naturally understood as certain functionals on suitable space of integrands, which allows readily various generalizations. The paper is focused on such functionals which can be attained by sequences whose "energy" (=pth power) does not concentrate in the sense that it is relatively weakly compact in $L^1(\Omega)$. Straightforward applications to coercive optimization problems are briefly outlined.

Keywords: Young measures, generalizations, relative L^1-weak compactness, coercive optimization problems, nonconcentration of energy

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1. Introduction

More than half a century ago, L.C. Young [19] introduced a tool, called now Young measures, to hold a certain "limit information" about oscillations that may appear in nonconvex variational problems. This was later widely exploited also in optimal control and game theory, where oscillation effects typically arise, as well. Let us recall that a Young measure ν on a domain $\Omega \subset \mathbb{R}^n$ valued on $\mathbb{R}^m$ is a weakly measurable mapping $x \mapsto \nu_x$ from Ω to regular probability measures on $\mathbb{R}^m$; cf. also [2], [14], [16], [17], [19] for details. Alternatively, every Young measure can be considered as a linear continuous functional on the Banach space $L^1(\Omega; C_0(\mathbb{R}^m))$ of Bochner integrable mappings $\Omega \to C_0(\mathbb{R}^m)$, where $C_0(\mathbb{R}^m)$ denotes the space of continuous functions on $\mathbb{R}^m$ vanishing at infinity, prescribed by

$$h \mapsto \int_\Omega \int_{\mathbb{R}^m} h(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x,$$

cf. also the original concept from [20].

This concept was enough general as long as L^∞-apriori estimates were at disposal. Nevertheless, it is an often case that a problem in question is coercive only on some L^p-space with $p < \infty$, and then test integrands vanishing at infinity in general cannot preserve a full relevant information about possible oscillation/concentration effects. The test integrands one can think about should belong to the space $\mathrm{Car}^p(\Omega; \mathbb{R}^m)$, consisting of all Carathéodory functions $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ (i.e. $h(x, \cdot)$ continuous and $h(\cdot, s)$ measurable) satisfying the growth condition $|h(x,s)| \leq a_h(x) + b_h|s|^p$ with some $a_h \in L^1(\Omega)$ and $b_h \in \mathbb{R}$ depending on h. This linear space can be naturally (semi)normed by

$$\|h\| = \inf_{|h(x,s)|\leq a(x)+b|s|^p} \|a\|_{L^1(\Omega)} + b. \tag{1}$$

Nevertheless, the whole space $\mathrm{Car}^p(\Omega;\mathbb{R}^m)$ is not separable, so that often it is more appropriate to work with its subspaces. Let us take some subspace $H \subset \mathrm{Car}^p(\Omega;\mathbb{R}^m)$ and define the natural imbedding i_H of $L^p(\Omega;\mathbb{R}^m)$ into the Banach space H^* by $\langle i_H(u), h\rangle = \int_\Omega h(x,u(x))\,\mathrm{d}x$ with $h \in H$ and $u \in L^p(\Omega;\mathbb{R}^m)$. Then we put

$$Y^p_H(\Omega;\mathbb{R}^m) = \\ \{\eta\in H^*;\ \exists \text{ a net } \{u_\xi\} \text{ bounded in } L^p(\Omega;\mathbb{R}^m),\ \text{w*-}\lim i_H(u_\xi) = \eta\}.$$

It is natural to address the elements of $Y^p_H(\Omega;\mathbb{R}^m)$ as generalized Young functionals; note that for $H = L^1(\Omega;C_0(\mathbb{R}^m))$ we get basically the original Young functionals that can be attained by bounded sequences from $L^p(\Omega;\mathbb{R}^m)$, which justifies our terminology. Let us also remark that the choice $H = H_{\mathcal{R}} := \Big\{h(x,s) = \sum_{l=1}^L g(x)v(s)(1+|s|^p);\ g \in C(\bar\Omega), v \in \mathcal{R}, L \in \mathbb{N}\Big\}$ with $\mathcal{R}$ being a complete ring of continuous bounded functions on $\mathbb{R}^m$ has been used by DiPerna and Majda [7]. For another choice of H see (5) below.

The above sketched construction falls into a convex-compactification theory developed in [13], [14], [15]. It is known that, if H^* is endowed by the weak* topology, $Y^p_H(\Omega;\mathbb{R}^m)$ is convex, σ-compact subset of H^* into which $L^p(\Omega;\mathbb{R}^m)$ is imbedded continuously and densely. Moreover, if H contains a coercive integrand $h(x,s) \geq a(x) + b|s|^p$ with some $a \in L^1(\Omega)$ and $b > 0$, then $Y^p_H(\Omega;\mathbb{R}^m)$ is locally compact and, if also $1 < p < +\infty$, the imbedding i_H is even homeomorphical, which makes $Y^p_H(\Omega;\mathbb{R}^m)$ a natural hull of the original Lebesgue space, indeed.

Let us emphasize that, unless we care about a concrete nature of the elements of $Y^p_H(\Omega;\mathbb{R}^m)$ (which may be a quite nontrivial task as one can see in case of the DiPerna and Majda measures from [12]), the particular choice of H is basically not intrinsic provided this space is sufficiently "small" to be separable and simultaneously sufficiently "rich" to create a sufficiently fine convex σ-compact hull $Y^p_H(\Omega;\mathbb{R}^m)$ of $L^p(\Omega;\mathbb{R}^m)$ which is needed to extend continuously concrete (class of) problems. The point is that it is usually not difficult to fulfil these antagonistic requirements, cf. (5) below. Without reference to any concrete problem, one can also observe that for an extremely small H the nonconcentration concept introduced here degenerates (cf. Remark 1 below) while for a sufficiently rich H some additional desirable properties may occur (cf. Remark 2 below).

2. Nonconcentrating generalized Young functionals

Sometimes it may happen that concentration effects are essentially excluded so that, likewise in the L^∞-case, only oscillation effects can appear. Then the following class of generalized Young functional is of a particular importance:

Definition 1. *A generalized Young functional $\eta \in Y^p_H(\Omega;\mathbb{R}^m)$ is called p-nonconcentrating if there is a net $\{u_\xi\}_{\xi\in\Xi}$ bounded in $L^p(\Omega;\mathbb{R}^m)$ such that* w*-$\lim_{\xi\in\Xi} i_H(u_\xi) = \eta$ *and the set $\{|u_\xi|^p;\ \xi\in\Xi\}$ is relatively weakly compact in $L^1(\Omega)$.*

The property of being "p-nonconcentrating" is important mainly because every p-nonconcentrating generalized Young functional admits a (not necessarily uniquely determined) Young-measure representation provided the test-integrand space H is separable. More precisely, if $\eta \in Y^p_H(\Omega;\mathbb{R}^m)$ is p-nonconcentrating and H is separable, then there exists a Young measure ν such that

$$\langle \eta, h\rangle \ = \int_\Omega\int_{\mathbb{R}^m} h(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \tag{2}$$

holds for any $h\in H$. This is a consequence of the results by Ball [2] if one realizes that, since H is separable and thus the weak* topology on bounded subsets of H^* is metrizable, η is weakly* attainable by a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ such that $\{|u_k|^p;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$, and that the set $\{h\circ u_k;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$ as well because h has at most p-growth.

Definition 2. *For $\eta,\mathring{\eta} \in Y^p_H(\Omega;\mathbb{R}^m)$, we say that $\mathring{\eta}$ is a p-nonconcentrating modification of η if $\mathring{\eta}$ is p-nonconcentrating and $\langle\mathring{\eta},h\rangle = \langle\eta,h\rangle$ holds for any $h\in H$ such that $|h(x,s)|\le a(x)+o(|s|^p)$ with some $a\in L^1(\Omega)$ and $o:\mathbb{R}^+\to\mathbb{R}$ satisfying* $\lim_{r\to\infty} o(r)/r = 0$.

The p-nonconcentrating modification can be straightforwardly exploited to prove (by contradiction arguments) the p-nonconcentration of solutions to various coercive optimization problems, cf. Section 4. The aim of the next section is to develop a needed theoretical background. The main ingredients we will rely on are Chacon's biting lemma [5] (cf. also [3], [17]) and the Dunford-Pettis theorem [8].

3. Main results

Proposition 1. *Every $\eta\in Y^p_H(\Omega;\mathbb{R}^m)$ admits at most one p-nonconcentrating modification $\mathring{\eta}\in Y^p_H(\Omega;\mathbb{R}^m)$.*

Proof: Let us suppose $\mathring{\eta}_1,\mathring{\eta}_2\in Y^p_H(\Omega;\mathbb{R}^m)$ are two p-nonconcentrating modifications of $\eta\in Y^p_H(\Omega;\mathbb{R}^m)$. Let us take $h\in H$ and put $h_r(x,s)=h(x,s)v_r(s)$ with

$$v_r(s) \ = \begin{cases} 1 & \text{for } |s|\le r,\\ 1+r-|s| & \text{for } r\le|s|\le r+1,\\ 0 & \text{for } |s|\ge r+1.\end{cases}$$

Without loss of generality we can suppose that $L^1(\Omega;C_0(\mathbb{R}^m))\subset H$. More in detail, if it is not the case, we can replace H by $\bar{H}=H+L^1(\Omega;C_0(\mathbb{R}^m))$ and extend

$\eta, \mathring{\eta}_1, \mathring{\eta}_2$ on this enlarged space so that again $\eta, \mathring{\eta}_1, \mathring{\eta}_2 \in Y^p_{\bar{H}}(\Omega;\mathbb{R}^m)$. Besides, the extended functionals $\mathring{\eta}_1$ and $\mathring{\eta}_2$ remain p-nonconcentrating as well. If one shows $\mathring{\eta}_1 = \mathring{\eta}_2$ in the sense of $\bar{H}^*$, then it is obvious that it holds for the original functionals on H as well. Thus, adopting the agreement that H contains $L^1(\Omega; C_0(\mathbb{R}^m))$, we may and will suppose $h_r \in H$ because always $h_r \in L^1(\Omega; C_0(\mathbb{R}^m))$.

As h_r has a growth less than p and both $\mathring{\eta}_1$ and $\mathring{\eta}_2$ are p-nonconcentrating modifications of η, we have

$$\langle \mathring{\eta}_1, h_r\rangle \ = \ \langle \eta, h_r\rangle \ = \ \langle \mathring{\eta}_2, h_r\rangle. \tag{3}$$

Now we want to show that

$$\lim_{r\to\infty} \langle \mathring{\eta}_1, h_r\rangle \ = \ \langle \mathring{\eta}_1, h\rangle. \tag{4}$$

As $\mathring{\eta}_1$ is p-nonconcentrating, there is a net $\{u_\xi\}_{\xi\in\Xi}$ bounded in $L^p(\Omega;\mathbb{R}^m)$ such that w*-$\lim_{\xi\in\Xi} i_H(u_\xi) = \mathring{\eta}_1$ and the set $\{|u_\xi|^p;\ \xi\in\Xi\}$ is relatively weakly compact in $L^1(\Omega)$ and therefore, by the Dunford-Pettis theorem, this set is also uniformly integrable. This means that, for any $\varepsilon > 0$, one can find r_ε sufficiently large so that

$$\sup_{\xi\in\Xi} \int\limits_{\{x\in\Omega;\ |u_\xi(x)|^p \ge r_\varepsilon\}} |u_\xi(x)|^p \,\mathrm{d}x \ \le \ \varepsilon.$$

As $h \in H \subset \mathrm{Car}^p(\Omega;\mathbb{R}^m)$, we have $|h(x,s)| \le a(x) + b|s|^p$ for a suitable $a \in L^1(\Omega)$ and $b \in \mathbb{R}$. In particular, a is absolutely continuous in the sense that, for any $\varepsilon > 0$, there is $m_\varepsilon > 0$ small enough so that

$$\sup_{\substack{\tilde{\Omega}\subset\Omega \text{ measurable} \\ |\tilde{\Omega}|\le m_\varepsilon}} \int_{\tilde{\Omega}} a(x)\,\mathrm{d}x \ \le \ \varepsilon.$$

Let us notice that it certainly holds $|\{x\in\Omega; |u_\xi(x)| \ge r\}| \ \le \ (C/r)^p$ with $C = \sup_{\xi\in\Xi} \|u_\xi\|_{L^p(\Omega;\mathbb{R}^m)}$. Then, for every $r \ge \max(C m_{\varepsilon/2}^{-1/p}, r_{\varepsilon/2b}^{1/p})$ and every $\xi \in \Xi$, we can estimate

$$\begin{aligned}
|\langle i_H(u_\xi), h_r - h\rangle| &= \Big|\int_\Omega h(x,u_\xi(x))(v_r(u_\xi(x)) - 1)\,\mathrm{d}x\Big| \\
&\le \int\limits_{\{x\in\Omega;\ |u_\xi(x)|\ge r\}} \big(a(x) + b|u_\xi(x)|^p\big)\,\mathrm{d}x \\
&\le \int\limits_{\{x\in\Omega;\ |u_\xi(x)|\ge r\}} a(x)\,\mathrm{d}x + \int\limits_{\{x\in\Omega;\ |u_\xi(x)|\ge r\}} b|u_\xi(x)|^p\,\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon.
\end{aligned}$$

Passing to the limit with $\xi \in \Xi$, we obtain $|\langle \overset{\circ}{\eta}_1, h_r - h\rangle| \le \varepsilon$ for every r large enough. As ε was arbitrary, the convergence (4) has been proved.

As the same procedure can be performed also for $\overset{\circ}{\eta}_2$ in place of $\overset{\circ}{\eta}_1$, we get $\langle \overset{\circ}{\eta}_1, h\rangle = \langle \overset{\circ}{\eta}_2, h\rangle$ as a consequence of (3). As it holds for $h \in H$ arbitrary, the identity $\overset{\circ}{\eta}_1 = \overset{\circ}{\eta}_2$ is demonstrated. □

Proposition 2. *Let H be separable. Any $\eta \in Y^p_H(\Omega;\mathbb{R}^m)$ admits its p-nonconcentrating modification $\overset{\circ}{\eta} \in Y^p_H(\Omega;\mathbb{R}^m)$.*

Proof: By the definition of $Y^p_H(\Omega;\mathbb{R}^m)$ and by the separability of H (which causes metrizability of the weak* topology on bounded subsets of H^*, there is a sequence $\{u_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^m)$ such that w*-$\lim_{k\to\infty} i_H(u_k) = \eta$. By the biting lemma (in the version from [17, Theorem 23]) there are measurable $\Omega_k \subset \Omega$ such that $\Omega_k \subset \Omega_{k+1}$ for any $k \in \mathbb{N}$, $|\bigcup_{k\in\mathbb{N}} \Omega_k| = |\Omega|$, and such that, after taking possibly a subsequence (denoted, for simplicity, by the same indices) the set $\{\chi_{\Omega_k}|u_k|^p;\ k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$, where $\chi_{\Omega_k} : \Omega \to \{0,1\}$ denotes the characteristic function of the set Ω_k. Moreover, as the sequence $\{\chi_{\Omega_k}u_k\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^m)$, the sequence $\{i_H(\chi_{\Omega_k}u_k)\}_{k\in\mathbb{N}}$ has a weak* cluster point $\overset{\circ}{\eta}$ in $Y^p_H(\Omega;\mathbb{R}^m)$. By the separability of H, we can take a subsequence (denoted again by the same indices) such that w*-$\lim_{k\to\infty} i_H(\chi_{\Omega_k}u_k) = \overset{\circ}{\eta}$.

By the very definition, $\overset{\circ}{\eta}$ is p-nonconcentrating. Let us take $h \in H$ such that $|h(x,s)| \le a(x) + o(|s|^p)$ with some $a \in L^1(\Omega)$. Then we can estimate

$$\begin{aligned}\big|\langle i_H(u_k) - i_H(\chi_{\Omega_k}u_k), h\rangle\big| &= \Big|\int_{\Omega\setminus\Omega_k} (h(x,u_k(x)) - h(x,0))\,\mathrm{d}x\Big| \\ &\le \|2|a| + o(0)\|_{L^1(\Omega\setminus\Omega_k)} + \|o(|u_k|^p)\|_{L^1(\Omega\setminus\Omega_k)} \\ &\equiv\ T_{k,1} + T_{k,2}.\end{aligned}$$

The first term converges to zero for $k \to \infty$ because of $|\Omega \setminus \Omega_k| \to 0$. As o has a sub-linear growth and the sequence $\{|u_k|^p\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, there is some $b : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{r\to\infty} b(r)/r = +\infty$ and $\{b(o(|u_k|^p))\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. By La Vallée-Pousin's criterion, the set $\{o(|u_k|^p);\ k \in \mathbb{N}\}$ is weakly* relatively compact in $L^1(\Omega)$. Then, by the Dunford-Pettis theorem, this set is equi-continuous with respect to the Lebesgue measure, which eventually causes also the term $T_{k,2}$ tending to zero for $k \to \infty$. Altogether, we have got $\lim_{k\to\infty}\langle i_H(u_k) - i_H(\chi_{\Omega_k}u_k), h\rangle = 0$.

On the other hand, we have also

$$\lim_{k\to\infty}\langle i_H(u_k) - i_H(\chi_{\Omega_k}u_k), h\rangle \ =\ \langle \eta - \overset{\circ}{\eta}, h\rangle,$$

so that we showed that $\langle \eta, h\rangle = \langle \overset{\circ}{\eta}, h\rangle$. As this holds for any $h \in H$ with the growth less that p, we have shown that $\overset{\circ}{\eta}$ is the p-nonconcentrating modification of η. □

Proposition 3. *Let H be separable, $\eta \in Y_H^p(\Omega;\mathbb{R}^m)$, and $\mathring{\eta} \in Y_H^p(\Omega;\mathbb{R}^m)$ its p-nonconcentrating modification. Then:*

(i) $\langle \eta - \mathring{\eta}, h\rangle \geq 0$ *provided $h \in H$ such that $h(x,s) \geq a_0(x)$ for some $a_0 \in L^1(\Omega)$,*

(ii) $\langle \eta - \mathring{\eta}, h\rangle > 0$ *provided $\eta \neq \mathring{\eta}$ and $h \in H$ is coercive in the sense $h(x,s) \geq a_0(x) + b|s|^p$ with some $a_0 \in L^1(\Omega)$ and $b > 0$.*

Proof: Let us take the sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{\chi_{\Omega_k} u_k\}_{k\in\mathbb{N}}$ as in the proof of Proposition 2. For $h \in H$ satisfying $a_0(x) \leq h \leq a(x) + c|s|^p$ we can estimate

$$\begin{aligned}\langle i_H(u_k) - i_H(\chi_{\Omega_k} u_k), h\rangle &= \int_{\Omega\setminus\Omega_k} (h(x,u_k(x)) - h(x,0))\,\mathrm{d}x \\ &\geq \int_{\Omega\setminus\Omega_k} (a_0(x) - a(x))\,\mathrm{d}x,\end{aligned}$$

from which we obtain in the limit $\langle \eta - \mathring{\eta}, h\rangle \geq 0$. The point (i) is proved.

Let us suppose that (ii) does not hold, so that $\langle \eta - \mathring{\eta}, h\rangle = 0$ for some $h \in H$ such that $h(x,s) \geq a_0(x) + b|s|^p$ with $b > 0$. Our task is to deduce that inevitably $\eta = \mathring{\eta}$. Let us take some $\tilde{h} \in H$. As $H \subset \mathrm{Car}^p(\Omega;\mathbb{R}^m)$ and h is coercive, $\tilde{h}$ can be majorized by h in the sense $\tilde{h} \leq \tilde{a} + \tilde{c}h$ with some $\tilde{a} \in L^1(\Omega)$ and some $\tilde{c} \geq 0$. By the point (i), we can see that $\tilde{c}h - \tilde{h} \geq -\tilde{a}$ implies $\langle \eta - \mathring{\eta}, \tilde{c}h - \tilde{h}\rangle \geq 0$. Taking into account also our assumption $\langle \eta - \mathring{\eta}, h\rangle = 0$, we obtain

$$\langle \eta - \mathring{\eta}, \tilde{h}\rangle \;\leq\; \tilde{c}\langle \eta - \mathring{\eta}, h\rangle \;=\; 0.$$

Making the same procedure for $-\tilde{h}$, we can see that $\langle \eta - \mathring{\eta}, \tilde{h}\rangle = 0$. As $\tilde{h} \in H$ is arbitrary, we have got $\eta = \mathring{\eta}$. The point (ii) has thus been demonstrated. □

Remark 1. Note that the concept of the p-nonconcentrating modification is sensible only if H contains integrands which have (in absolute value) the growth precisely p because otherwise every generalized Young measure is, by the very definition, automatically the p-nonconcentrating modification of itself.

Remark 2. If H contains $H_{\mathcal{R}}$ defined in Section 1 for some ring $\mathcal{R}$, then even every sequence $\{u_k\}_{k\in\mathbb{N}}$ such that every cluster point of $\{i_H(u_k)\}_{k\in\mathbb{N}}$ is p-nonconcentrating must have a relatively L^1-weakly compact energy $\{|u_k|^p;\ k \in \mathbb{N}\}$; cf. [14, Proposition 3.4.15] for details. Let us emphasize that this obviously does not hold if H is too small.

Remark 3. For an example of a procedure realizing the mapping $\eta \mapsto \mathring{\eta}$ in a concrete case we refer to [11, Theorem 3] where an explicit formula is isolated for the case of the generalization of Young measures developed by DiPerna and Majda [7].

4. Applications in brief

The above results have straightforward applications to various optimization problems which are coercive in an L^p-space with $p < +\infty$ but not with $p = +\infty$. Such problems arise typically in variational calculus, but also in optimal control theory and game theory if the control (or strategies) are not apriori bounded. For illustration, let us demonstrate it briefly (we refer to [14, Chapter 4] for details) on a simple optimal control problem for a system of ordinary differential equations

$$
(\mathsf{P}) \qquad \begin{cases} \text{minimize} & \int_0^T \varphi(t, y(t), u(t))\,\mathrm{d}t \\ \text{subject to} & \mathrm{d}y/\mathrm{d}t = \Phi(t, y(t), u(t)), \quad y(0) = y_0 \\ & y \in W^{1,q}(0,T;\mathbb{R}^n), \quad u \in L^p(0,T;\mathbb{R}^m), \end{cases}
$$

with some $\varphi : (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\Phi : (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, and $p, q > 1$. Note that the admissible controls are not bounded in $L^\infty(0,T;\mathbb{R}^m)$ as usual; for such kind of problems we refer also to [4], [18]. Under suitable qualification imposed on Φ (in particular, a Lipschitz continuity of $\Phi(t,\cdot,s)$ and at most p/q-growth of $|\Phi(t,r,\cdot)|$), the controlled system possesses for any control $u \in L^p(0,T;\mathbb{R}^m)$ a unique solution y in the Sobolev space $W^{1,q}(0,T;\mathbb{R}^n)$ of functions $(0,T) \to \mathbb{R}^n$ with the time derivative in $L^q(0,T;\mathbb{R}^n)$. Then we take a separable subspace $H \subset \mathrm{Car}^p(0,T;\mathbb{R}^m)$ which is rich enough so that $(\varphi \circ y) \in H$ and $g \cdot (\Phi \circ y) \in H$ for any $y \in W^{1,q}(0,T;\mathbb{R}^n)$ and $g \in L^{q/(q-1)}(0,T;\mathbb{R}^n)$, where $[\varphi \circ y](t,s) = \varphi(t, y(t), s)$ and $[g \cdot (\Phi \circ y)](t,s) = g(t) \cdot \Phi(t, y(t), s)$. The smallest linear subspace of this property is obviously

$$
H = \{c(\varphi \circ y_1) + g \cdot (\Phi \circ y_2);\ c \in \mathbb{R},\ g \in L^{q/(q-1)}(0,T;\mathbb{R}^n),\ y_1, y_2 \in W^{1,q}(0,T;\mathbb{R}^n)\}. \tag{5}
$$

Such H is also separable if endowed by the natural norm (1). Of course, we can also take a large space, e.g. $H + H_{\mathcal{R}}$ which is still separable provided $\mathcal{R}$ is so; then Remark 2 is effective.

As neither linearity of $\Phi(t,r,\cdot)$ nor convexity of $\varphi(t,r,\cdot)$ is required, the problem (P) need not have any solution and we must introduce naturally the relaxed (=extended) problem:

$$
(\mathsf{RP}) \qquad \begin{cases} \text{minimize} & \langle \eta, \varphi \circ y \rangle \\ \text{subject to} & \mathrm{d}y/\mathrm{d}t = (\Phi \circ y) \bullet \eta, \quad y(0) = y_0 \\ & y \in W^{1,q}(0,T;\mathbb{R}^n), \quad \eta \in Y_H^p(\Omega;\mathbb{R}^m), \end{cases}
$$

where $(\Phi \circ y) \bullet \eta \in L^q(0,T;\mathbb{R}^n)$ is defined by the identity $\langle (\Phi \circ y) \bullet \eta, g \rangle = \langle \eta, g \cdot (\Phi \circ y) \rangle$ which is to be valid for any $g \in L^{q/(q-1)}(0,T;\mathbb{R}^n)$. Under suitable qualification imposed on the data φ and Φ, the relaxed problem (RP) has a solution which can be understood in a natural way as a generalized solution to the original problem (P). Especially, we have to suppose that φ is coercive: $\varphi(t,r,s) \geq a(t) + b|s|^p$ with some $a \in L^1(\Omega)$ and $b > 0$. Since H is separable, we can immediately claim that, for every solution (y, η) to (RP), the relaxed control η is p-nonconcentrating. Indeed, if it would not be true, then the p-nonconcentrating modification $\overset{\circ}{\eta}$ of η (which does exist thanks to Proposition 2) would drive the controlled system to the same state y because Φ has the growth p/q lesser than p but, by Proposition 3, $\langle \overset{\circ}{\eta}, \Phi \circ y \rangle < \langle \eta, \Phi \circ y \rangle$ so that the pair (y, η) would not be optimal, a contradiction.

Remark 4. The p-nonconcentration of optimal solutions to relaxed problems enables us, using the Young-measure representation (2) of such solutions, to analyze in details the respective optimality conditions which take typically the form of (extended) maximum principles of the Pontryagin type (for optimal control or game-theoretical problems) or of the Weierstrass type (for problems of variational calculus), cf. [14].

Remark 5. The fact that every solution to a relaxed problem is p-nonconcentrating has an impact on the minimizing sequences for the original problems: by Remark 2 we can here deduce that every minimizing sequence $\{(y_k, u_k)\}_{k \in \mathbb{N}}$ for (P) does not concentrate energy in the sense that the set $\{|u_k|^p\}_{k \in \mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$. In optimal-control context, such property had to be essentially only supposed in Berliocchi and Lasry [4, Theorem 5]. In the variational-calculus context, the results of such type were recently obtained by Kinderlehrer and Pedregal [9], cf. also Kristensen [10]. Yet, such results usually rely on fairly advanced techniques, e.g. a calculation of the lower semicontinuous envelope of the minimized functional, which may be a nontrivial task; cf. [1], [6]. Contrary to this, the results presented here make possible to overcome readily these nontrivial technicalities and can be applied by a routine way in many cases in optimal control, variational calculus, and non-cooperative game theory, cf. [14].

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