Firmly pseudo-contractive mappings and fixed points

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Abstract. We give some fixed point theorems for firmly pseudo-contractive mappings defined on nonconvex subsets of a Banach space. We also prove some fixed point results for firmly pseudo-contractive mappings with unbounded nonconvex domain in a reflexive Banach space.

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1. Introduction

Let X be a real Banach space and D be a nonempty subset of X. An operator $T : D \to X$ is said to be firmly pseudo-contractive if for each $x, y \in D$ and $\lambda > 0$

$$\|x - y\| \leq \|(1 - \lambda)(x - y) + \lambda(T(x) - T(y))\|. \tag{1}$$

If (1) holds locally, i.e. if each $x \in D$ has a neighborhood U such that the restriction of T to U is firmly pseudo-contractive, then T is said to be a local firmly pseudo-contractive.

Following Kato [6], we are able to find an equivalent definition for firmly pseudo-contractive operators. An operator $T : D \to X$ is firmly pseudo-contractive if and only if for every $x, y \in D$ there exists $j \in J(x - y)$ such that

$$\langle T(x) - T(y), j\rangle \geq \|x - y\|^2, \tag{2}$$

where $j : X \to 2^{X^*}$ is the normalized duality mapping which is defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\| = \|u\|\}$$

(see Browder [1] and Kato [6]). It is an immediate consequence of the Hahn-Banach theorem that $J(u)$ is nonempty for each $u \in X$.

The firmly pseudo-contractive mappings are characterized by the fact a mapping $T : D \to X$ is firmly pseudo-contractive if and only if the mappings $f = T - I$ is accretive on D (see Lemma 2.2). Recent interest in mapping theory for accretive operators (e.g. [1], [3], [6], [8], [9]) particularly as it relates to existence theorems for nonlinear ordinary and partial differential equations, has prompted a corresponding interest in the fixed point theory for firmly pseudo-contractive mappings.

We prove approximating fixed point and fixed point theorems for firmly pseudo-contractive nonself mapping $T : D \to X$, where D is a nonconvex closed subset of Banach space X. In Section 3, we present some theorems for firmly pseudo-contractive mappings with unbounded nonconvex domain in Banach space by applying the results derived in Section 2.

Notation 2. Weak (weak*) convergence of a sequence $\{x_n\}$ will be denoted by $x_n \xrightarrow{w} x$ ($x_n \xrightarrow{w^*} x$) and strong convergence by $x_n \to x$. The set of fixed points of a mapping T will be denoted by $F(T)$.

2. Approximating fixed points of firmly pseudo-contractive mappings

Before giving our results, we give some lemmas.

Lemma 2.1. *Let $(X, (\cdot,\cdot))$ be a real Hilbert space, $\phi \neq D \subset X$ and $T : D \to X$. Then the following are equivalent:*

(a) *T is firmly pseudo-contractive;*
(b) *$\|x-y\|^2 + \|(I-T)(x) - (I-T)(y)\|^2 \leq \|T(x) - T(y)\|^2$ for all $x, y \in D$;*
(c) *$T - I$ is monotone.*

Lemma 2.2. *Let X be a real Banach space, $\phi \neq D \subset X$ and $T : D \to X$. The following are equivalent:*

(a) *T is firmly pseudo-contractive;*
(b) *$2I - T$ is pseudo-contractive;*
(c) *$T - I$ is accretive.*

Above lemmas can be shown by simple calculations.

Lemma 2.3. *Let X be a real Banach space, $\alpha, \beta \in R$, $x, y \in X$ and*

$$\|x-y\| \leq \|(1-\alpha)x - (1-\beta)y\|.$$

Then $\langle \alpha x - \beta y, j\rangle \leq 0$ for all $j \in J(x-y)$.

Proof: It follows from Kato [6]. □

Lemma 2.4. *Let X be a real smooth Banach space, $\phi \neq D \subset X$ and $T : D \to X$ is firmly pseudo-contractive. Suppose for $x \in D$ there is a $\lambda > 1$ such that $x = \lambda T(x)$. Then $\langle x, J(y-x)\rangle \geq 0$ for all $y \in F(T)$.*

Proof: Set $r = -(\lambda^{-1} - 1)$. By firmly pseudo-contractivity of T, we have for all $y \in F(T)$

$$\begin{aligned} \langle \lambda^{-1}x - y, j(y-x)\rangle &= \langle T(x) - T(y), J(y-x)\rangle \\ &\leq -\|x-y\|^2 \\ &= \langle x-y, J(y-x)\rangle \end{aligned}$$

yields

$$\langle -rx, J(y-x)\rangle \leq 0,$$

where $r > 0$. Therefore $\langle x, J(y-x)\rangle \geq 0$, completing the proof. □

Lemma 2.5. *Let X be a real smooth Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed and $T : D \to X$ continuous firmly pseudo-contractive. Suppose $\{x_n\}$ is a sequence in D with $x_n \xrightarrow{w} x$ and $\{\lambda_n\}$ is a strictly decreasing real sequence in $(1, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ such that $x_n = \lambda_n T(x_n)$ for all $n \in N$. Then $\lim_{n\to\infty} x_n = x$ and $F(T) \neq \phi$.*

Proof: For $x_m, x_n \in D$, $m \geq n$, by inequality (1), we obtain

$$\begin{aligned}\|x_m - x_n\| &\leq \|(1-\lambda)(x_m - x_n) + \lambda(\lambda_m^{-1} x_m - \lambda_n^{-1} x_n)\| \\ &= \|(1 - \lambda(1 - \lambda_m^{-1}))x_m - (1 - \lambda(1 - \lambda_n^{-1}))x_n\|.\end{aligned}$$

Hence, it follows from Lemma 2.3 that

$$\langle (1 - \lambda_m^{-1})x_m - (1 - \lambda_n^{-1})x_n, J(x_m - x_n)\rangle \leq 0,$$

since $(1 - \lambda_m^{-1}) > (1 - \lambda_n^{-1}) \geq 0$ for $m > n$, hence from Lemma 2 of [10] we get

$$\langle x_m, J(x_n - x_m)\rangle \geq 0.$$

For fixed $m \in N$, $(x_n - x_m) \xrightarrow{w} (x - x_m)$, hence by [4] $J(x_n - x_m) \xrightarrow{w^*} J(x - x_m)$ and hence (3) implies

$$0 \leq \lim_{n\to\infty} \langle x_m, J(x_n - x_m)\rangle = \langle x_m, J(x - x_m)\rangle.$$

Therefore,

$$\begin{aligned}\|x - x_m\|^2 &= \langle x, J(x - x_m)\rangle - \langle x_m, J(x - x_m)\rangle \\ &\leq \langle x, J(x - x_m)\rangle.\end{aligned}$$

It follows that $\lim_{m\to\infty} x_m = x$, because $\lim_{m\to\infty}\langle x, J(x - x_m)\rangle = 0$. Since T is continuous and $x_n = \lambda_n T(x_n)$, it follows $T(x) = x$. □

Lemma 2.6. *Let X be a real smooth Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ and $T : D \to X$ firmly pseudo-contractive. Suppose $\{x_n\}$ is a sequence in D with $x_n \xrightarrow{w} x$ and $T(x) = x$ for $x \in D$ and $\{\lambda_n\}$ is a real sequence in $(1, \infty)$ such that $x_n = \lambda_n T(x_n)$ for all $n \in N$. Then*

(a) $\lim_{n\to\infty} x_n = x$;
(b) $\langle x, J(y - x)\rangle \geq 0$ *for all* $y \in F(T)$.

Proof: (a) Since $x = T(x)$ for $x \in D$ and $x_n = \lambda_n T(x_n)$ for all $n \in N$, it follows from Lemma 2.4 that

$$\langle x_n, J(x - x_n)\rangle \geq 0 \quad \text{for all} \quad n \in N.$$

Therefore for all $n \in N$,

$$\begin{aligned} \|x - x_n\|^2 &= \langle x, J(x - x_n)\rangle - \langle x_n, J(x - x_n)\rangle \\ &\le \langle x, J(x - x_n)\rangle. \end{aligned}$$

Since $(x - x_n) \xrightarrow{w} 0$ and J is weakly sequentially continuous at zero, we obtain $\lim_{n\to\infty} \|x - x_n\| = 0$.

(b) Fix $y \in F(T)$, hence by Lemma 2.4 we have

$$\langle x_n, T(y - x_n)\rangle \ge 0 \quad \text{for all} \quad n \in N.$$

Since X is smooth, J is strong-weak* continuous (see e.g. [4]) and $\lim_{n\to\infty}(y - x_n) = (y - x)$, we conclude that $J(y - x_n) \xrightarrow{w^*} J(y - x)$. Therefore,

$$0 \le \lim_{n\to\infty} \langle x_n, J(y - x_n)\rangle = \langle x, J(y - x)\rangle,$$

completing the proof. □

Lemma 2.7. *Let X be a real Banach space, $\phi \neq D \subset X$ be closed, $T : D \to X$ firmly pseudo-contractive and $\{\lambda_n\}$ be a real sequence in $(1, \infty)$. Suppose $\{S_n\}$ be a surjective mapping from X into itself defined by*

$$S_n = \lambda_n T + (\lambda_n - 1)A \quad \textit{for all} \quad n \in N. \tag{4}$$

Then for each $n \in N$ there is exactly one $x_n \in D$ such that

$$x_n = \lambda_n T(x_n) + (\lambda_n - 1)A(x_n) \quad \textit{for all} \quad n \in N.$$

(A stands for specific function defined by $A = u + kI$ for some u in X and for some k in $(-1, \infty)$).

Proof: Since T is firmly pseudo-contractive, then for $x, y \in D$, $n \in N$, there exists $j \in J(x - y)$ so that from (4)

$$\begin{aligned} \langle S_n(x) - S_n(y), j\rangle &= \lambda_n\langle T(x) - T(y), j\rangle + (\lambda_n - 1)k\|x - y\|^2 \\ &\ge [\lambda_n + (\lambda_n - 1)k]\|x - y\|^2 \end{aligned}$$

yields

$$\|S_n(x) - S_n(y)\| \ge a_n\|x - y\|,$$

where $a_n = [\lambda_n - (\lambda_n - 1)k] > 1$. Since $a_n > 1$ for all $n \in N$, it follows from Theorem 1 of [12] that S_n possesses exactly one fixed point x_n in D. It means that

$$x_n = \lambda_n T(x_n) + (\lambda_n - 1)A(x_n) \quad \text{for all} \quad n \in N,$$

completing the proof. □

Now we prove our results as below.

Theorem 2.1. *Let X be a real reflexive Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed and bounded, $T : D \to X$ continuous firmly pseudo-contractive and $\{\lambda_n\}$ is a strictly decreasing real sequence in $(1, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$. Suppose $\{S_n\}$ is a sequence of surjective mappings from X into itself defined by*

$$S_n = \lambda_n T + (\lambda_n - 1)A \quad \text{for all} \quad n \in N,$$

where A is a linear operator on D into X defined by $Ax = k'x$ for all $x \in D$ and for some $k' \in (-1, \infty)$. Then

(a) *for each $n \in N$ there is exactly one $x_n \in D$ such that*

$$x_n = (\lambda_n/(1 - (\lambda_n - 1)k'))\; T(x_n);$$

(b) *$\{x_n\}$ converges strongly to some fixed point of T.*

Proof: Part (a) follows from Lemma 2.7, so (b) remains to be proved. Since X is reflexive and D is bounded, there exists $z \in X$ and a subsequence $\{x_{\mu_n}\}$ of $\{x_n\}$ such that $x_{\mu_n} \xrightarrow{w} z$ (Pettis' theorem). Applying Lemma 2.5, we conclude that $\lim_{n\to\infty} x_{\mu_n} = z$ and $z = Tz$. Again applying Lemma 2.6, we get

$$\langle z, J(y - z)\rangle \geq 0 \quad \text{for all} \quad y \in F(T),$$

and the result follows by Theorem 1.7 of [11]. □

Theorem 2.2. *Let X be a real smooth Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed and $T : D \to X$ continuous firmly pseudo-contractive. Suppose $\{x_n\}$ is a sequence in D with $x_n \xrightarrow{w} x$ and $\{\lambda_n\}$ a strictly increasing real sequence in $(0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$ such that*

$$(2\lambda_n - 1)x_n = \lambda_n T(x_n) \quad \text{for all} \quad n \in N.$$

Then $\lim_{n\to\infty} x_n = x$ and $x \in F(T)$.

Proof: Defining $\delta_n = \lambda[1 - (2\lambda_n - 1)\lambda_n^{-1}]$ for all $n \in N$, hence for $m > n$, $\delta_n > \delta_m \geq 0$ from (1), we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|(1 - \lambda)(x_n - x_m) + \lambda[(2\lambda_n - 1)\lambda_n^{-1}x_n - (2\lambda_m - 1)\lambda_m^{-1}x_m]\| \\ &= \|(1 - \delta_n)x_n - (1 - \delta_m)x_m\|.\end{aligned}$$

Using Lemma 2.3, we obtain

$$\langle \delta_n x_n - \delta_m x_m, J(x_n - x_m)\rangle \leq 0,$$

it follows from Lemma 2 and 3 of [10] that $\lim_{n\to\infty} x_n = x$. Since T is continuous and $(2\lambda_n - 1)x_n = \lambda_n T(x_n)$, the result follows. □

Theorem 2.3. *Let X be a real reflexive Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed, bounded and starshaped with respect to zero and $T : D \to X$ continuous firmly pseudo-contractive. Then $F(T) \neq \phi$.*

Proof: For $n \in N$, define $T_n = \lambda_n(2I - T) : D \to D$, and $\lambda_n = 1 - \frac{1}{n}$. Then by Lemma 2.2 T_n is strictly pseudo-contractive and hence it follows from Corollary 1 of [3] that T_n possesses exactly one fixed point $x_n \in D$. Since X is reflexive and $\{x_n\}$ is bounded, there exists an $x \in D$ and some subsequence $\{x_{\psi_n}\}$ of $\{x_n\}$ such that $x_{\psi_n} \xrightarrow{w} x$. The result follows from Theorem 2.2. □

3. Fixed points of firmly pseudo-contractive mappings with unbounded nonconvex domain

In [5], Goebel and Kuczumow proved a result for nonexpansive mappings on a closed convex subset of a Hilbert space which is expanded in [2], [7], [9], [13].

Thus it is interesting to investigate the existence of fixed points for firmly pseudo-contractive mappings defined on closed unbounded nonconvex subset in Banach space. We begin with the following lemma.

Lemma 3.1. *Let X be a real Banach space, $\phi \neq D \subset X$ and $T : D \to X$ firmly pseudo-contractive. Suppose the set*

$$G(z) = \{u \in D : (r-1)\|u-z\|^2 + r\langle T(z), j\rangle \leq 0 \text{ for some } j \in J(u-z) \text{ and } r > 1\} \tag{5}$$

is bounded for some z in D. Then the set $H = \{x \in D : x = \lambda T(x)$ for some $\lambda > 1\}$ is bounded.

Proof: Without loss of generality we may assume that $z = 0$ and $T(0) \neq 0$. Let $x \in H$, then $x = \lambda T(x)$ for some $\lambda > 1$. Since T is firmly pseudo-contractive, there exists $j \in J(x)$ such that

$$\langle T(x) - T(0), j\rangle \geq \|x - 0\|^2,$$

i.e.

$$\lambda^{-1}\|x\|^2 - \langle T(0), j\rangle \geq \|x\|^2$$

yielding

$$\langle T(0), j\rangle + t\|x\|^2 \leq 0 \text{ for some } j \in J(x),$$

where $t = (1 - \lambda^{-1}) < 1$, hence $x \in G(0)$. Since $G(0)$ is bounded, therefore H is bounded. □

Theorem 3.1. *Let X be a real smooth Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed and $T : D \to X$ continuous firmly pseudo-contractive. Suppose $\{\lambda_n\}$ is a strictly decreasing real sequence in $(1, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ and $\{S_n\}$ is a sequence of surjective mappings from X into itself defined by*

$$S_n = \lambda_n T + (\lambda_n - 1)A \quad \text{for all} \quad n \in N,$$

where $A : D \to X$ is a linear operator on D into X defined by $Ax = hx$ for all $x \in D$ and for some $h \in (-1, \infty)$. Also suppose that the set $G(z)$ is bounded for some $z \in D$. Then $F(T) \neq \phi$.

Proof: For $n \in N$, by Lemma 2.7, we obtain

$$x_n = (1 - (\lambda_n - 1)h)^{-1} \lambda_n T(x_n).$$

Set $c_n = (1 - (\lambda_n - 1)h)^{-1} \lambda_n$ for all $n \in N$. Since $c_n > 1$, $n \in N$, then we conclude from Lemma 3.1 that $\{x_n\}$ is bounded. Applying Lemma 2.5, we get the result. □

Theorem 3.2. *Let X be a real reflexive Banach space possessing a weakly sequentially continuous duality mapping $J : X \to X^*$, $\phi \neq D \subset X$ be closed and starshaped with respect to zero and $T : D \to X$ continuous firmly pseudo-contractive. Suppose that the set $G(z)$ is bounded for some $z \in D$. Then $F(T) \neq \phi$.*

Proof: As in proof of Theorem 2.4, for each $n \in N$ there exists a unique $x_n \in D$ such that $x_n = (2\lambda_n - 1)^{-1} \lambda_n T(x_n)$, where $\lambda_n = 1 - \frac{1}{n}$. Hence, it follows from Lemma 3.1 that $\{x_n\}$ is bounded. Thus the result follows by Theorem 2.2. □

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