Some non-multiplicative properties are l-invariant

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Abstract. A cardinal function φ (or a property $\mathcal{P}$) is called l-invariant if for any Tychonoff spaces X and Y with $C_p(X)$ and $C_p(Y)$ linearly homeomorphic we have $\varphi(X) = \varphi(Y)$ (or the space X has $\mathcal{P}$ ($\equiv X \vdash \mathcal{P}$) iff $Y \vdash \mathcal{P}$). We prove that the hereditary Lindelöf number is l-invariant as well as that there are models of ZFC in which hereditary separability is l-invariant.

Keywords: l-equivalent spaces, l-invariant property, hereditary Lindelöf number

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0. Introduction

There are quite a few equivalences introduced for Tychonoff spaces in the last twenty years. The spaces X and Y are called M-equivalent (A-equivalent) if their free (Abelian) topological groups are topologically isomorphic. A space X is t-equivalent (or u-equivalent, or l-equivalent) to a space Y if there exists a (uniform or linear respectively) homeomorphism between the spaces $C_p(X)$ and $C_p(Y)$. If φ is one of the letters M, A, l, u, t, then the φ-equivalence of X and Y is denoted by $X \overset{\varphi}{\sim} Y$. A property $\mathcal{P}$ (or a cardinal function η) is called φ-invariant if $X \vdash \mathcal{P}$ and $X \overset{\varphi}{\sim} Y$ implies $Y \vdash \mathcal{P}$ (or $\eta(X) = \eta(Y)$ respectively). Here, as before, φ is one of the letters M, A, l, u, t.

It is known ([2]) that

$$X \overset{M}{\sim} Y \Rightarrow X \overset{A}{\sim} Y \Rightarrow X \overset{l}{\sim} Y \Rightarrow X \overset{u}{\sim} Y \Rightarrow X \overset{t}{\sim} Y.$$

There were many attempts to prove φ-invariance of various properties and cardinal functions. Let us mention only that now it is known (see, e.g. [3, Chapter 2]) that

(1) the network weight, the density, the cardinality, the hereditary density of finite powers, the spread of finite powers, the (hereditary) Lindelöf number of finite powers, the discreteness and the σ-compactness are t-invariant;

(2) pseudocompactness, compactness, the Lebesgue covering dimension $\leq n$ are u-invariant;

(3) the Lindelöf property is l-invariant;

(4) the connectedness is M-invariant.

Of course, all properties in (1) are l-invariant as well as u- and M-invariant. The l-invariance of Lindelöf property was recently announced by N.V. Velichko (see [4]).

It is worth mentioning that it is known (see [3] and [9]), that

(5) the weight, the character, the pseudocharacter, the Souslin number and the extent are not l-invariant.

However many questions and hypotheses still remain as to whether some very natural properties and cardinal functions are l-invariant. It is not known, for example, whether countable compactness is l-invariant (while it is not t-invariant [7]). It is not clear whether the spread and the hereditary density are l-invariant. The main obstacle for exploring the l-invariance of such properties is their non-multiplicativity, which makes it impossible to use the fact that the free topological group over X or the space $L_p(X)$ can be represented as countable union of continuous images of something very similar to finite powers of X. Therefore some new methods are required every time one needs to prove that a non-multiplicative property is l-invariant.

In this paper we prove that the hereditary Lindelöf number is l-invariant. We failed to prove the same for the spread and hereditary density, but we establish that there are models of ZFC in which hereditary separability is l-invariant. We also prove that the spread as well as the extent are l-invariant in the class of perfect Tychonoff spaces. It was proved by V.G.Pestov [8], that the spread, the hereditary density and the hereditary Lindelöf number are preserved by M-equivalence.

1. Notations and terminology

Throughout this paper "a space" means "a Tychonoff space". If X is a space then $\mathcal{T}(X)$ is its topology and $\mathcal{T}(x, X) = \{U \in \mathcal{T}(X) : x \in U\}$ for any $x \in X$. An end of a proof of a statement is denoted by $\square$.

If $f : X \to Y$ is a map, and $A \subset X$, then $f \restriction A$ is the restriction of f to A. The symbol τ always stands for a cardinal, and $\mathbf{R}$ is the set of reals with the standard topology. The expression $l(X) \leq \tau$ means the Lindelöf number of the space X does not exceed τ, while $hl(X) \leq \tau$ says all subsets of X have the Lindelöf number $\leq \tau$. By $hd(X) \leq \tau$ is denoted the fact that any subset of X has the density $\leq \tau$. Finally, $s(X) \leq \tau$ (or $ext(X) \leq \tau$) means all discrete (or respectively discrete and closed) subsets of X have the cardinality $\leq \tau$.

All other notions are standard and can be found in [6].

2. Proving the l-invariance of hereditary Lindelöf number and other properties

We shall need the following well known facts [3, Chapter 0].

2.1 Fact. *For every space X the correspondence $x \longmapsto \psi_x$, where $\psi_x(f) = f(x)$ for all $f \in C_p(X)$ embeds X into $C_pC_p(X)$ as a closed linearly independent subspace which we will further on identify with X.*

2.2 Fact. *Let $L_p(X)$ be the linear hull of X in $C_pC_p(X)$. The spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic if and only if $L_p(X)$ and $L_p(Y)$ are linearly homeomorphic.*

2.3 Fact. *For every $f \in C_p(X)$ there is a continuous linear functional $\hat{f} : L_p(X) \to \mathbf{R}$ such that $\hat{f} \restriction X = f$. The space $L_p(X)$ is linearly homeomorphic to the space $L_p(Y)$ iff Y can be closely embedded into $L_p(X)$ in such a way that every $g \in C_p(Y)$ can be extended to a linear continuous functional $\hat{g} : L_p(X) \to \mathbf{R}$.*

2.4 Fact. *Denote by $L_p^0(X)$ the set, consisting of only the trivial (i.e. equal to zero) linear functional on $C_p(X)$. For a natural $n \geq 1$ let $L_p^n(X) = \{z \in L_p(X) :$ there are $\lambda_1, \dots , \lambda_n \in \mathbf{R}$ and $x_1, \dots , x_n \in X$ such that $z = \lambda_1 x_1 + \ldots + \lambda_n x_n\}$. Then the set $L_p^n(X)$ is closed in $L_p(X)$ and $L_p(X) = \cup\{L_p^n(X) : n \in \omega\}$.*

From here on we assume that X and Y are l-equivalent spaces and Y is embedded in $L_p(X)$ like in 2.3.

If $n \geq 1$, let $Y_n = (L_p^n(X)\backslash L_p^{n-1}(X)) \cap Y$. Then for every $y \in Y_n$ we have $y = \lambda_1 x_1 + \ldots + \lambda_n x_n$ where $\lambda_i \neq 0$ for all $i = 1, \dots n$. Denote the set $\{x_1, \dots , x_n\}$ by $\operatorname{supp}(y)$.

Let $E_n(X)$ be the set of all n-element subsets of X with the Vietoris topology. It is easy to see that for any $a = \{a_1, \dots , a_n\} \in E_n(X)$ the sets

$$O(a, U_1, \dots , U_n) = \{b \in E_n(X) : b \in \cup\{U_i : i \leq n\} \text{ and } b \cap U_i \neq \emptyset \text{ for all } i\}$$

constitute a base of a in $E_n(X)$ where the sets $U_i \in \mathcal{T}(a_i, X)$ are chosen arbitrarily with the only restriction that they form a disjoint family.

2.5 Proposition. *The map $d_n : Y_n \to E_n(X)$ defined by $d_n(y) = \operatorname{supp}(y)$ is continuous.*

Proof: Fix any $y = \lambda_1 x_1 + \ldots + \lambda_n x_n \in Y_n$ and $U_i \in \mathcal{T}(x_i, X)$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Find $f_i \in C_p(X)$ with $f_i(x_i) = 1$ and $f_i \restriction (X\backslash U_i) \equiv 0$. Now let

$$V = \cap\{\hat{f}_i^{-1}(\mathbf{R}\backslash\{0\}) : i = 1, \dots , n\}.$$

The set V is open in $L_p(X)$.

Observe first that $y \in V$. Indeed,

$$\hat{f}_i(y) = \hat{f}_i(\lambda_1 x_1 + \ldots + \lambda_n x_n) = \lambda_1 f_i(x_1) + \ldots + \lambda_n f_i(x_n) = \lambda_i f(x_i) = \lambda_i \neq 0,$$

because $f_i(x_j) = 0$ for $i \neq j$. Hence $y \in V$.

We claim that $d_n(V \cap Y_n) \subset W = O(\{x_1, \dots , x_n\}, U_1, \dots , U_n)$. To show this let $z = \mu_1 t_1 + \ldots + \mu_n t_n \in V \cap Y_n$. Then $\hat{f}_i(z) \neq 0$ so that there must be a $\sigma(i) \in \{1, \dots , n\}$ such that $f_i(t_{\sigma(i)}) \neq 0$. Consequently, $t_{\sigma(i)} \in U_i$. The sets U_i being disjoint we have $\sigma(i) \neq \sigma(j)$ if $i \neq j$ so σ is a bijection of $\{1, \dots , n\}$ onto itself. Thus $d_n(z) = \{t_1, \dots , t_n\} \cap U_i \neq \emptyset$ for all i and $\{t_1, \dots , t_n\} \subset U_1 \cup \ldots \cup U_n$ so $d_n(V \cap Y_n) \subset W$. $\square$

2.6 Proposition. *Let $\Delta_n = \{x \in X^n :$ there are different i, j with $x_i = x_j\}$ and $X(n) = X^n \backslash \Delta_n$. Let $e_n : X(n) \to E_n(X)$ be "the order forgetting map", that is $e_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$. Then*

(1) *e_n is continuous;*
(2) *e_n is open;*
(3) *$|e_n^{-1}(a)| = n!$ for all $a \in E_n(X)$.*

PROOF: Let $x = (x_1, \ldots, x_n) \in X(n)$. Then $e_n(x) = a = \{x_1, \ldots, x_n\}$. Pick any $U_i \in \mathcal{T}(x_i, X)$ such that $U_i \cap U_j = \emptyset$ for different i, j. Let $U = U_1 \times \ldots \times U_n$. Then $U \in \mathcal{T}(x, X(n))$ and (1) and (2) follow from the equality $e_n(U) = W$, where $W = O(a, U_1, \ldots, U_n)$.

So let us prove this equality. If $y = (y_1, \ldots, y_n) \in U$ then $y_i \in U_i$ so that $e_n(y) \in W$. Now let $b = \{y_1, \ldots, y_n\} \in W$. Then $b \subset U_1 \cup \ldots \cup U_n$ and $b \cap U_i \neq \emptyset$ for al i. Therefore for every $i \in \{1, \ldots, n\}$ there is a $\sigma(i) \in \{1, \ldots, n\}$ with $y_{\sigma(i)} \in U_i$. The sets U_i are disjoint so σ is a bijection of $\{1, \ldots, n\}$ onto itself. Hence $b = e_n(y)$, where $y = (y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \in U$. This proves the equality $e_n(U) = W$. □

2.7 Corollary. *Let T be a subset of $E_n(X)$. Denote the set $e_n^{-1}(T)$ by S and let $e_T : S \to T$ be the restriction of e_n to S. Then*

(1) *e_T is open;*
(2) *e_T is closed;*
(3) *e_T is a local homeomorphism, which means that for any $s \in S$ there is a $U_s \in \mathcal{T}(s, S)$ such that $e_T \restriction U_s$ is a homeomorphism of U_s onto an open subset of T.*

PROOF: A restriction of an open map to a saturated set is again an open map, so e_T is open. Any open map with fibers, which are finite and have the same number of elements is closed and locally homeomorphic by [5, Chapter 6, Problems 124 and 125]. □

2.8 Theorem. *Let X and Y be l-equivalent spaces with Y contained in $L_p(X)$ as in 2.3. Then for every $n \in \omega \backslash \{0\}$ we can choose the sets X_n, X'_n and Y'_n with the following properties:*

(1) *$X_n \subset X$, $X'_n \subset X(n)$ and $Y'_n \subset E_n(X)$;*
(2) *X_n is a continuous image of X'_n;*
(3) *$X'_n = e_n^{-1}(Y'_n)$ for all $n \in \omega \backslash \{0\}$;*
(4) *$Y'_n = d_n(Y_n)$, where $Y_n = (L_p^n(X) \backslash L_p^{n-1}(X)) \cap Y$;*
(5) *$\cup\{X_n : n \in \omega \backslash \{0\}\} = X$.*

PROOF: The properties (3) and (4) define Y'_n and X'_n for all n. Let $p_n : X(n) \to X$ be the natural projection of $X(n)$ onto the first coordinate. Denote the set $p_n(X'_n)$ by X_n. Then for the sets we constructed the properties (1)-(4) hold. Let us show that (5) is also true.

Pick any $x \in X$. The set Y is a basis in $L_p(X)$ so there are $y_1, \dots, y_k \in Y$ and $\lambda_1, \dots, \lambda_k \in \mathbf{R}\backslash\{0\}$ with $x = \lambda_1 y_1 + \ldots + \lambda_k x_k$. This implies $x \in \operatorname{supp}(y_i)$ for some $i \in \{1, \dots, k\}$. Pick the $n \in \omega$ such that $y_i \in Y_n$. Evidently, $x \in d_n(y_i) = \{x_1, \dots, x_n\}$. The set $e_n^{-1}(d_n(y_i))$ contains all possible permutations of the set $\{x_1, \dots, x_n\}$ so there is a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $x_{\sigma(1)} = x$. Therefore $z = (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in X'_n$ and $p_n(z) = x$ so we are done. □

In what follows, given a pair of l-equivalent spaces X and Y, we are going to use the notation of Theorem 2.8 (that is the symbols X_n, X'_n, Y_n and Y'_n) without explicit reference.

2.9 Proposition. *Let $f : S \to T$ be a locally homeomorphic and perfect map between the spaces S and T. Then $hl(S) = hl(T)$.*

Proof: We only need to prove that $hl(S) \le hl(T) = \tau$. The perfect preimages do not raise the Lindelöf number, so $l(S) \le \tau$. Each point $s \in S$ has a neighbourhood $V_s \in \mathcal{T}(s, S)$ with $hl(V_s) \le \tau$ because f is a local homeomorphism. Now pick a subcover of cardinality $\le \tau$ from the open cover $\{V_s : s \in S\}$ of the space S. We have $hl(S) \le \tau \cdot \sup\{hl(V_s) : s \in S\} \le \tau$. □

2.10 Corollary. *If $X \overset{l}{\sim} Y$, then $hl(X) = hl(Y)$.*

Proof: It suffices to prove that $hl(X) \le hl(Y)$. We assume that Y is embedded in $L_p(X)$ like in 2.3. Let $hl(Y) = \tau$. Then $hl(Y_n) \le \tau$ and $hl(Y'_n) \le \tau$. The space X'_n is a perfect locally homeomorphic preimage of Y'_n by 2.7 so $hl(X'_n) \le \tau$ by 2.10. Therefore $hl(X_n) \le \tau$ and $hl(X) \le \tau$ because the hereditary Lindelöf number is not raised by continuous images and countable unions. □

2.11 Theorem. *If $X \overset{l}{\sim} Y$ then $ext(Y) \le s(X)$ and $ext(X) \le s(Y)$.*

Proof: By symmetry of the situation it suffices to prove that $ext(X) \le s(Y)$. Let $\tau = s(Y)$. Then $s(Y_n) \le \tau$ and $s(Y'_n) \le \tau$ for the spread is hereditary and continuous maps do not raise it. It follows that $ext(X'_n) \le \tau$ because, evidently, closed finite-to-one maps can not lower the extent, which in its turn does not exceed the spread. Thus $ext(X_n) \le \tau$ and $ext(X) \le \tau$ the extent being countably additive and not raised by continuous maps. □

2.12 Corollary. *If X and Y are l-equivalent perfect spaces then $s(X) = s(Y)$ and $ext(X) = ext(Y)$.*

Proof: We have $ext(X) \le s(Y)$ and $ext(Y) \le s(X)$. But in perfect spaces the extent coincides with the spread, so $ext(X) = s(X) = s(Y) = ext(Y)$. □

2.13 Theorem. *Assume that $X \overset{l}{\sim} Y$ and $s(Y) \le \tau$. Then the space X contains a dense subset Z with $hl(Z) \le \tau$.*

Proof: Any space with spread $\le \tau$ contains a dense subset with hereditary Lindelöf number $\le \tau$ ([1]). Thus there is a dense $\tilde{Y}_n \subset Y_n$ with $hl(\tilde{Y}_n) \le \tau$.

Consequently, $hl(\tilde{Y}_n') \leq \tau$, where $\tilde{Y}_n' = d_n(\tilde{Y}_n)$. Clearly $\tilde{Y}_n'$ is dense in Y_n'. The map e_n is open so the set $\tilde{X}_n' = e_n^{-1}(\tilde{Y}_n')$ is dense in X_n'. Use 2.7 and 2.9 to conclude that $hl(\tilde{X}_n') \leq \tau$. Hence $hl(\tilde{X}_n) \leq \tau$, where $\tilde{X}_n = p_n(\tilde{X}_n')$. Thus, the set $\tilde{X} = \cup\{\tilde{X}_n : n \in \omega\backslash\{0\}\}$ is dense in X and $hl(\tilde{X}) \leq \tau$. □

It was proved in [10], that there exist models of ZFC in which the statement $SA =$ "there are no regular S-spaces" holds.

2.14 Theorem (SA). *If Y is a hereditarily separable space and $X \overset{l}{\sim} Y$ then X is hereditarily separable.*

Proof: Any Y_n is hereditarily separable and hence so is Y_n'. By SA the space Y_n' is Lindelöf as well as the space X_n' being a perfect preimage of Y_n'. The space X_n' is locally homeomorphic to Y_n' and hence locally hereditary separable. It is clear that a Lindelöf locally hereditarily separable space is hereditarily separable, so that X_n' is hereditarily separable. Now it is easy to see that X_n is hereditarily separable for all n, so X is hereditarily separable. □

3. Unsolved problems

Here is the list of the problems the author did not succeed in solving while working on this paper. The fact that they occurred to him does not mean, of course, that he was the first to discover them. The topic is so popular that it is quite possible that some of them have been published or orally announced before.

In the following text the letters X and Y stand for Tychonoff topological spaces.

3.1 Problem. *Let $X \overset{t}{\sim} Y$. Is it true that $hl(X) = hl(Y)$?*

3.2 Problem. *Let $X \overset{u}{\sim} Y$. Is it true that $hl(X) = hl(Y)$?*

3.3 Problem. *Let $X \overset{l}{\sim} Y$. Is it true that $s(X) = s(Y)$?*

3.4 Problem. *Let $X \overset{u}{\sim} Y$. Is it true that $s(X) = s(Y)$?*

3.5 Problem. *Let $X \overset{t}{\sim} Y$. Is it true that $s(X) = s(Y)$?*

3.6 Problem. *Let X and Y be compact t-equivalent spaces. Is it true that $t(X) = t(Y)$? Here $t(Z)$ is the tightness of a space Z.*

3.7 Problem. *Let X and Y be compact u-equivalent spaces. Is it true that $t(X) = t(Y)$?*

3.8 Problem. *Let $X \overset{l}{\sim} Y$. Is it true that $hd(X) = hd(Y)$?*

3.9 Problem. *Let $X \overset{u}{\sim} Y$. Is it true that $hd(X) = hd(Y)$?*

3.10 Problem. *Let $X \overset{t}{\sim} Y$. Is it true that $hd(X) = hd(Y)$?*

3.11 Problem. *Let $n \geq 1$ be a natural number and let $f : X \to Y$ be an open onto map with $|f^{-1}(y)| = n$ for all $y \in Y$. Is then $s(X) = s(Y)$?*

3.12 Problem. *Let $n \geq 1$ be a natural number and let $f : X \to Y$ be an open onto map with $|f^{-1}(y)| = n$ for all $y \in Y$. Is then $hd(X) = hd(Y)$?*

3.13 Problem. *Let X be l-equivalent to Y. Suppose that X has a dense hereditarily Lindelöf subset. Is it true that Y also contains a dense hereditarily Lindelöf subset?*

3.14 Problem. *Let X be u-equivalent to Y. Suppose that X has a dense hereditarily Lindelöf subset. Is it true that Y also contains a dense hereditarily Lindelöf subset?*

3.15 Problem. *Let X be l-equivalent to Y. Suppose that X has a dense hereditarily separable subset. Is it true that Y also contains a dense hereditarily separable subset?*

3.16 Problem. *Let X be u-equivalent to Y. Suppose that X has a dense hereditarily separable subset. Is it true that Y also contains a dense hereditarily separable subset?*

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