The a priori estimate of the maximum modulus to solutions of doubly nonlinear parabolic equations*

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Abstract. The a priori estimate of the maximum modulus of the generalized solution is established for a doubly nonlinear parabolic equation with special structural conditions.

Keywords: doubly nonlinear parabolic equation, generalized solution, bounded solution, maximum modulus, a priori estimate

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Introduction

Since Lions [1] proposed in 1969 to consider the doubly nonlinear parabolic equation,

$$\frac{\partial}{\partial t}\Big(|u|^{\lambda-2}u\Big) - \frac{\partial}{\partial t}\Big(|u_{x_i}|^{p-2}u_{x_i}\Big) = f,$$

there are papers devoted to the existence of generalized solutions of more general doubly nonlinear parabolic equations (see, for example, [2]–[3]). But the investigations concerning the properties of generalized solutions still seldom appear. In [4], Liang Xiting & Liang Xuexin proved the local and global boundedness to generalized solutions of doubly nonlinear parabolic equations with a more general structural condition than (1), (2) below. However, in that paper, it does not supply any a priori estimate for the maximum modulus of solutions. In this present paper, we give a certain a priori estimate of the maximum modulus for the generalized solutions of the doubly nonlinear parabolic equation (1) with structure conditions (2). These are the extensions of the corresponding results for elliptic equations.

Let G be a bounded domain in the n-dimensional Euclidean space E^n and $T > 0$ a finite real number. Consider on $Q = G \times (0,T)$ the following doubly nonlinear parabolic equation

$$\frac{\partial}{\partial t}\Big(|u|^{\lambda-2}u\Big) - \operatorname{div} \underline{A}(x,t,u,\nabla u) + B(x,t,u,\nabla u) = 0, \tag{1}$$

where $2 \le \lambda < np/(n-p)$ as $1 < p < n$ and $2 \le \lambda < \infty$ as $p \ge n$, $\underline{A}(x,t,u,\xi)$ and $B(x,t,u,\xi)$ are defined on $Q \times E^1 \times E^n$, continuous with respect to u and ξ for

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fixed x and t, measurable with respect to x and t for fixed u and ξ and satisfying the following structural conditions, respectively:

$$
\begin{aligned}
\xi \cdot \underline{A}(x,t,u,\xi) &\geq |\xi|^p, \ p>1 \\
|\underline{A}(x,t,u,\xi) &\leq \kappa|\xi|^{p-1}, \ \kappa \geq 1 \\
B(x,t,u,\xi) &= B_1(x,t,u,\xi) + \mu|u|^{\alpha-2}u \\
|B_1(x,t,u,\xi)| &\leq b(x,t)|\xi|^{\beta} + f(x,t)
\end{aligned} \tag{2}
$$

where $\mu \geq 0$;

$$
\begin{aligned}
& q = p(n+\lambda)/(n+p) \leq \alpha \leq p(1+\lambda/n) = l \quad \text{and} \\
& \beta_0 = p - (n+p)/(n+\lambda) \leq \beta < p
\end{aligned}
$$

$$b(x,t) \in L_r(Q) \tag{3}$$

$$
\begin{aligned}
& 1/r = 1 - \beta/p - 1/l \ \text{ as } \ \beta_0 \leq \beta < \beta_1 = p - n/(n+\lambda) \\
& r = \infty \ \text{ as } \ \beta = \beta_1 \qquad (4) \\
& r > (n+p)/(p-\beta) \ \text{ as } \ \beta_1 < \beta < p
\end{aligned}
$$

$$f(x,t) \in L_s(Q), \quad s > (n+p)/p. \tag{5}$$

We call u a generalized solution of (1), if

$$
\begin{gathered}
u \in C(0,T;L_\lambda(G)) \cap L_p(0,T;W_p^1(G)) \ \text{ as } \ \beta_0 \leq \beta \leq \beta_1 \\
u \in C(0,T;L_\lambda(G)) \cap L_p(0,T;W_p^1(G)) \cap L_{t^*}(Q) \\
(1-(p-\beta)(n+\lambda)/(n+p))/t^* + (p-\beta)(n+\lambda)/((n+p)l) + \beta/p + 1/r = 1, \\
\text{as } \ \beta_1 < \beta < p
\end{gathered}
$$

and the following holds

$$
\begin{aligned}
& \int_0^t \int_G \{-\nu_t|u|^{\lambda-2}\mu + \nabla\nu \cdot \underline{A}(x,t,u,\nabla u) + \nu B(x,t,u,\nabla u)\}\, dx\, dt \\
& \quad + \int_G \nu(x,t)|u(x,t)|^{\lambda-2}u(x,t)\,|_{t=0}^{t=t}\, dx = 0 \\
& \forall\, t \in (0,T), \ \phi \in W_\lambda^1(0,T;L_\lambda(G)) \cap L_p(0,T;\mathring{W}_p^1(G));
\end{aligned} \tag{1$'$}
$$

our results are the following:

Theorem 1. *Suppose* (2)–(5) *are fulfilled in which* $\mu > 0$ *and* $s = \infty$. *Let* u *be a generalized solution of* (1). *In addition, there is a constant* $M > 0$ *such that*

$$(u-M)^+ = \max(u-M,0) \in L_p(0,T;\mathring{W}_p^1(G)) \ \text{ and } \ (u-M)^+\,|_{t=0} = 0. \tag{6}$$

Then holds:

$$\operatorname*{ess\,sup}_Q u^+ \le \max\{M, (\|f\|_{L_\infty(Q)}/\mu)^{1/(\alpha-1)}\}. \tag{7}$$

In particular, if $1 < p < 2n/(n+\lambda-2)$ *holds,*

$$\operatorname*{vrai\,max}_Q u^+ \le M \quad \text{or} \quad \operatorname*{vrai\,max}_Q u^+ < (\|f\|_{L_\infty(Q)}/\mu)^{1/(\alpha-1)} \tag{8}$$

and there exists a constant $\theta > 0$ *depending only on* $n, p, \mu, \lambda, \kappa$, $b(x,t)$ *and* $|Q|$, *the* $n+1$ *dimensional Lebesgue measure of* Q, *such that*

$$\operatorname*{ess\,sup}_Q u^+ \le \max\{M, (\|f\|_{L_\infty(Q)}/\mu(1+\theta))^{1/(q-1)}\}, \tag{9}$$

provided $\alpha = q$ *and* $\beta = \beta_0$.

Theorem 2. *Suppose* (2)–(5) *are fulfilled in which* $\mu \ge 0$ *and* $\beta = \beta_0$. *Let* u *be a generalized solution of* (1) *satisfying* (6). *Then there exists a constant* C *depending only on* $n, p, \mu, \lambda, \kappa$, $b(x,t)$ *and* $|Q|$ *such that*

$$\operatorname*{ess\,sup}_Q u^+ \le M + C\|f\|_{L_s(Q)}^{1/(q-1)}. \tag{10}$$

Proof of Theorems

In order to prove the theorems we need the following lemmata. The proof of Lemma 1 is essentially the same as in [5, Chapter II, § 3] and Lemma 2 is a special case of [6, Chapter II, Lemma 5.1].

Lemma 1. *Let* $u \in C(0,T;L_\lambda(G)) \cap L_p(0,T;\mathring{W}_p^1(G))$. *Then*

$$\|u\|_{L_l(Q)} \le C(n,p,\lambda)|||u|||_Q^{1/q},\ l = p(1+\lambda/n),\ q = p(n+\lambda)/(n+p),$$
$$|||u|||_Q = \operatorname*{ess\,sup}_{t\in(0,T)} \int_G |u|^\lambda\,dx + \int\!\!\int_Q |\nabla u|^p\,dx\,dt.$$

Lemma 2. *Let* $u \in L_1(Q)$ *satisfying*

$$\int\!\!\int_Q (u-k)^+\,dx\,dt \le F|Q\cap\{u>k\}|^{1+\tau}\ \forall\, k \ge k_0 \ge 0,$$

where F, τ *are positive constants. Then*

$$\operatorname*{ess\,sup}_Q u^+ \le k_0 + (1+1/\tau)F|Q|^\tau.$$

Proof of Theorem 1: We take for $\theta \geq 0$

$$k(\theta) = (\|f\|_{L_\infty(Q)}/(\mu(1+2\theta)))^{1/(\alpha-1)}.$$

If $M < k(\theta)$, then there is a $\theta' > 0$ such that $M < k(\theta)$ for $\theta \in (0, \theta')$. Let $k_0 = \max\{M, k(\theta)\}$. For any $k \geq k_0$, we have on the set $Q \cap \{u > k\}$ that

$$f(x,t) - \mu|u|^{\alpha-2} \leq F = \begin{cases} 0 & \text{as } M \geq k(\theta), \\ (2\theta/(1+2\theta))\|f\|_{L_\infty(Q)} & \text{as } M < k(\theta). \end{cases} \tag{11}$$

For the sake of simplicity we assume $u \in W^1_\lambda(0,T; L_\lambda(G))$. Then for $k \geq k_0$, we may take $\nu = (u-k)^+$ as a test function (in general we should take the time average of ν as a test function and invoke a limit process). Inserting such a ν into $(1)'$ we obtain by integrating by part with respect to t that

$$\begin{aligned} \int_0^t \int_G (u-k)^+(|u|^{\lambda-2}u)_t \, dx\, dt + \int_0^t \int_{G\cap\{u>k\}} |\nabla u|^p \, dx\, dt \\ \leq \int_0^t \int_G (u-k)^+(b(x,t)|\nabla u|^\beta + F)\, dx\, dt,\ t \in (0,T). \end{aligned} \tag{12}$$

Represent

$$I = \int_0^t \int_G (u-k)^+(|u|^{\lambda-2}u)_t \, dx\, dt. \tag{13}$$

In order to estimate I we put

$$\tilde{u} = \begin{cases} u & \text{as } u > k, \\ k & \text{as } u \leq k; \end{cases}$$

then

$$\begin{aligned} I &= \int_0^t \int_G (\tilde{u}-k)(|u|^{\lambda-2}\tilde{u})_t \, dx\, dt \\ &= (\lambda-1)\int_0^t \int_G \Big(\frac{\overline{u}^\lambda}{\lambda} - \frac{k\overline{u}^{\lambda-1}}{\lambda-1} + \frac{k^\lambda}{\lambda(\lambda-1)}\Big)_t \, dx\, dt \\ &= (\lambda-1)\int_G \Big(\frac{\overline{u}^\lambda}{\lambda} - \frac{k\overline{u}^{\lambda-1}}{\lambda-1} + \frac{k^\lambda}{\lambda(\lambda-1)}\Big) dx \\ &\geq (\lambda-1)\int_{G\cap\{k<u<h\}} \Big(\frac{u^\lambda}{\lambda} - \frac{ku^{\lambda-1}}{\lambda-1} + \frac{k^\lambda}{\lambda(\lambda-1)}\Big) dx. \end{aligned}$$

Let

$$g(\eta) = \frac{1}{\lambda} - \frac{\eta}{\lambda-1} + \frac{\eta^\lambda}{\lambda(\lambda-1)} - \frac{(1-\eta)^\lambda}{\lambda^2}, \quad \eta \in (0,1).$$

On account of $g(0) = 1/\lambda - 1/\lambda^2 > 0$ and $g(1) = 0$ and

$$\eta^{\lambda-1} + \Big(\frac{\lambda-1}{\lambda}\Big)(1-\eta)^{\lambda-1} < 1 \quad \text{for} \quad \lambda \geq 2 \quad \text{and} \quad \eta \in (0,1)$$

we have

$$g'(\eta) = \frac{1}{\lambda-1} + \frac{\eta^{\lambda-1}}{\lambda-1} + \frac{(1-\eta)^{\lambda-1}}{\lambda} < 0 \quad \text{for} \quad \eta \in (0,1).$$

This implies

$$\begin{aligned} &\frac{u^\lambda}{\lambda} - \frac{ku^{\lambda-1}}{\lambda-1} + \frac{k^\lambda}{\lambda(\lambda-1)} \geq \frac{(u-k)^\lambda}{\lambda^2} \quad \text{on} \quad Q \cap \{u > k\}, \\ &\qquad\qquad I \geq C(\lambda) \int_{G\cap\{u>k\}} (u-k)^\lambda\,dx. \end{aligned} \tag{14}$$

Combining (13), (14) with (12) we get that

$$\begin{aligned} &\int_{G\cap\{u>k\}} (u-k)^\lambda\,dx + \int_0^t \int_{G\cap\{u>k\}} |\nabla u|^p\,dx\,dt \\ &\qquad \leq C \int_0^t \int_G (u-k)^+ (b(x,t)|\nabla u|^\beta + F)\,dx\,dt. \end{aligned} \tag{15}$$

Take the supremum for $t \in (0,T)$ yields

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,T)} \int_G |(u-k)^+|^\lambda\,dx + \int\int_{Q\cap\{u>k\}} |\nabla u|^p\,dx\,dt \\ &\qquad \leq C \int\int_Q (u-k)^+ (b(x,t)|\nabla u|^\beta + F)\,dx\,dt. \end{aligned} \tag{16}$$

In what follows we write

$$F(k) = \int\int_Q (u-k)^+ F\,dx\,dt \tag{17}$$

which is a nondecreasing function of k. In virtue of $(u-k)^+ \in L_p(0,T; \mathring{W}_p^1(G))$ for $k \geq M$ we have by the use of Lemma 1 that

$$\begin{aligned} &\int\int_Q (u-k)^+ b(x,t)|\nabla u|^\beta\,dx\,dt \\ &= \int\int_Q |(u-k)^+|^{1-(p-\beta)(n+p)/(n+\lambda)+(p-\beta)(n+p)/(n+\lambda)} b(x,t)|\nabla u|^\beta\,dx\,dt \\ &\leq \varepsilon(k)\Big(\int\int_Q |(u-k)^+|^l\,dx\,dt\Big)^{(p-\beta)(n+p)/(n+\lambda)l} \times \\ &\quad \times \Big(\int\int_Q |\nabla(u-k)^+|^p\,dx\,dt\Big)^{\beta/p} \\ &\leq C\varepsilon(k)|||(u-k)^+|||_Q, \end{aligned} \tag{18}$$

where $\varepsilon(k) = \|b(x,t)\|_{L_r(Q\cap\{u>k\})}U$,

$$U = \|u\|_{L_l(Q)}^{1-(p-\beta)(n+p)/(n+\lambda)} \quad \text{as} \quad \beta_0 \le \beta \le \beta_1 \quad \text{and}$$
$$U = \|u\|_{L_{l^*}(Q)}^{1-(p-\beta)(n+p)/(n+\lambda)} \quad \text{as} \quad \beta_1 < \beta < p. \tag{19}$$

In virtue of

$$|Q \cap \|u > k\|| \le k^{-l} \iint_Q |u|^l \, dx\, dt \to 0 \quad \text{as} \quad k \to \infty$$

and the absolute continuity of a Lebesgue integral we have $\varepsilon(k) \to 0$ as $k \to \infty$. So, we can take an $h_0 > k_0$ such that

$$C\varepsilon(k) \le 1/2 \quad \text{as} \quad k \ge h_0. \tag{20}$$

From (16)–(19) it follows

$$\|\|(u-h_0)^+\|\|_Q \le CF(h_0). \tag{21}$$

Let $h > k \ge k_0$ be arbitrary; it follows from (16) that

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in(0,T)} \int_G ((u-k)^+ - (u-h)^+)\, dx + \iint_{Q\cap\{k<u<h\}} |\nabla u|^p \, dx\, dt \\ \le C\Big\{ \iint_Q (u-k)^+ b(x,t)|\nabla u|^\beta \, dx\, dt + F(k)\Big\}. \end{aligned} \tag{22}$$

The effective domain of the integral appearing on the right hand side of (22) excludes any set of the form $Q \cap \{u = \text{const}\}$ having a positive measure. For simplicity we assume $|Q \cap \{u = \text{const}\}| = 0$. For any $k > k_0$, we take

$$h_{-1} = \infty > h_0 > h_1 > \cdots > h_m > h_{m+1} = k$$

such that $Q_i = Q \cap \{h_{i-1} > u > h_i\}$ satisfies

$$|Q_i|^{1/r} = \varepsilon/N \quad (i = 1, 2, \ldots, m) \quad \text{and} \quad |Q_{m+1}|^{1/r} \le \varepsilon/N,$$

where ε will be specified and N is defined by

$$\varepsilon(N) = \Big(\iint_{Q\cap\{b(x,t)>N\}} b(x,t)^r \, dx\, dt\Big)^{1/r} \le \varepsilon. \tag{23}$$

It follows from (23) that

$$\begin{aligned} \|b(x,t)\|_{L_r(Q_i)} &\le \Big(\iint_{Q\cap\{b(x,t)>N\}} b(x,t)^r \, dx\, dt\Big)^{1/r} + N|Q_i|^{1/r} \\ &\le \varepsilon(N) + N|Q_i|^{1/r} \le 2\varepsilon \quad i = 1, 2, \ldots, m+1. \end{aligned} \tag{24}$$

Denote for $i = 1, 2, \ldots, m+1$

$$u_i = (u - h_i)^+ - (u - h_{i-1})^+. \tag{25}$$

Then for almost everywhere $(x,t) \in Q$ we have

$$(u - h_i)^+ = \sum_{j=0}^{i} u_j \quad \text{on} \quad Q \cap \{u > h_i\},$$

$\nabla u_i = \nabla u$ as $(x,t) \in Q_i$ and $\nabla u_i = 0$ as (x,t) otherwise. It follows from (22) by taking $k = h_{i+1}$ and $h = h_i$ that

$$|||u_{i+1}|||_Q \leq C(II + F(k)), \tag{26}$$

where

$$\begin{aligned} II &= \iint_{Q\cap\{u>h_{i+1}\}} (u - h_{i+1}) b(x,t) |\nabla u|^\beta \, dx \, dt \\ &= \iint_{Q\cap\{u>h_{i+1}\}} \Big(u_{i+1} + \sum_{j=0}^{i} u_j \Big) b(x,t) |\nabla u|^\beta \, dx \, dt = III + IV \end{aligned} \tag{27}$$

$$\begin{aligned} III &= \iint_{Q\cap\{u>h_{i+1}\}} u_{i+1} b(x,t) |\nabla u|^\beta \, dx \, dt \\ &\leq \iint_{Q\cap\{u>h_{i+1}\}} u_{i+1} b(x,t) \Big(\sum_{j=0}^{i+1} |\nabla u_j| \Big)^\beta \, dx \, dt \\ &\leq 2^\beta \iint_{Q\cap\{u>h_{i+1}\}} u_{i+1} b(x,t) \Big[|\nabla u_{i+1}|^\beta + \Big(\sum_{j=0}^{i} |\nabla u_j| \Big)^\beta \Big] \, dx \, dt \\ &\leq 2^\beta \iint_{Q_{i+1}} u_{i+1} b(x,t) |\nabla u_{i+1}|^\beta \, dx \, dt \\ &\quad + (2m)^\beta \sum_{j=0}^{i} \iint_{Q_i \cap Q_j} u_{i+1} b(x,t) |\nabla u_j|^\beta \, dx \, dt \\ &\leq 2^\beta U \|b(x,t)\|_{L_r(Q_i)} \|u_{i+1}\|_{L_l(Q)}^{(p-\beta)(n+p)/(n+\lambda)} \|\nabla u_{i+1}\|_{L_p(Q)}^\beta \\ &\quad + C(m,U) \|b(x,t)\|_{L_r(Q_i)} \|u_{i+1}\|_{L_l(Q)}^{(p-\beta)(n+p)/(n+\lambda)} \sum_{j=0}^{i} \|\nabla u_i\|_{L_p(Q)}^\beta \\ &\leq 2^\beta U 2\varepsilon C |||u_{i+1}|||_Q + C(m,U) |||u_{i+1}|||_Q^{(p-\beta)/p} \sum_{j=0}^{i} |||u_j|||_Q^{\beta/p}; \end{aligned} \tag{28}$$

in the deduction of (28) we have used (24) and for the sake of simplicity we have absorbed the norm $\|b(x,t)\|_{L_r(Q)}$ into the constant $C(m,U)$, the U is appearing in (19). Similarly

$$\begin{aligned}
IV &= \int\int_{Q\cap\{u>h_{i+1}\}} \sum_{j=0}^{i} u_j b(x,t)|\nabla u|^{\beta}\,dx\,dt \\
&\le 2^{\beta}\int\int_{Q\cap\{u>h_{i+1}\}} \sum_{j=0}^{i} u_j b(x,t)\Big[|\nabla u_{i+1}|^{\beta} + \Big(\sum_{j=0}^{i}|\nabla u_j|\Big)^{\beta}\Big]\,dx\,dt \\
&\le 2^{\beta}U\|b(x,t)\|_{L_r(Q)} \sum_{j=0}^{i} \|u_j\|_{L_l(Q)}^{(p-\beta)(n+p)/(n+\lambda)} \|\nabla u_{i+1}\|_{L_p(Q)}^{\beta} \\
&\quad + (2m)^{\beta}U\|b(x,t)\|_{L_r(Q)} \sum_{j=0}^{i} \|u_j\|_{L_l(Q)}^{(p-\beta)(n+p)/(n+\lambda)} \|\nabla u_j\|_{L_p(Q)}^{\beta} \\
&\le 1/4|||u_{i+1}|||_Q + C(m,U)\sum_{j=0}^{i}|||u_j|||_Q.
\end{aligned} \tag{29}$$

Now we see from (28) that if at the beginning we take ε so small that

$$2^{\beta+1}CU\varepsilon \le 1/4, \tag{30}$$

then from (26)–(29) we infer

$$|||u_{i+1}|||_Q \le C(m,U)\Big\{\sum_{j=0}^{i}|||u_j|||_Q + F(k)\Big\}, \quad i = 1,2,\ldots,m. \tag{31}$$

Because of (30) the ε is controlled by C, U and β. By the definition of Q_i we have

$$m(\varepsilon/N)^r \le \sum_{i=1}^{m}|Q_i| \le |Q|.$$

So, m is finite and is also controlled by C, U and β. (31) is now written as

$$|||u_{i+1}|||_Q \le C(U)\Big\{\sum_{j=0}^{i}|||u_j|||_Q + F(k)\Big\}, \quad i = 1,2,\ldots,m. \tag{32}$$

Combining this with (21), the latter can be written as

$$|||u_0|||_Q \le CF(h_0) \le CF(k), \tag{33}$$

we deduce by induction that

$$\||u_i\||_Q \le C(U)F(k), \quad i = 1, 2, \ldots, m+1. \tag{34}$$

Thus

$$\begin{aligned}\|(u-k)^+\|^{q/l}_{L_l(Q)} &\le C\||(u-k)^+\||_Q \le C\||\sum_{i=0}^{m+1} u_i\||_Q \\ &\le C\sum_{i=0}^{m+1}\||u_i\||_Q \le C(U)F(k) \\ &\le C(U)F\|(u-k)^+\|_{L_l(Q)}|Q\cap\{u>k\}|^{1-1/l},\end{aligned} \tag{35}$$

i.e.

$$\begin{gathered}\|(u-k)^+\|_{L_l(Q)} \le C(U)F^{1/(q-1)}|Q\cap\{u>k\}|^{(1-1/l)/(q-1)}, \\ \int\int_Q (u-k)^+\,dx\,dt \le C(U)F^{1/(q-1)}|Q\cap\{u>k\}|^{1+\tau}, \quad \forall\, k \ge k_0\end{gathered}$$

where $\tau = p/(n(p-1)+p(\lambda-1)) > 0$. By Lemma 2 we have

$$\begin{aligned}\operatorname*{ess\,sup}_Q u^+ &\le k_0 + (1+1/\tau)C(U)F^{1/(q-1)}|Q|^\tau \\ &\le \max\{M, k(\theta)\} + C(U)F^{1/(q-1)}\end{aligned} \tag{36}$$

(for simplicity we have absorbed $(1+1/\tau)|Q|^\tau$ into the constant $C(U)$). From (36), it follows (7) by letting $\theta \to 0$.

In particular, if $1 < p < 2n/(n+\lambda-2)$, from (36) and (11) we have

$$\operatorname*{ess\,sup}_Q u^+ \le \max\{M.k(\theta)\} = M \quad \text{as} \quad M \ge k(0) = (\|f\|_{L_\infty(Q)}/\mu)^{1/(\alpha-1)}$$

and

$$\begin{aligned}\operatorname*{ess\,sup}_Q u^+ &\le k(\theta) + C(U)F^{1/(q-1)} \\ &= (\|f\|_{L_\infty(Q)}/(\mu(1+2\theta)))^{1/(\alpha-1)} \\ &\quad + C(U)(\|f\|_{L_\infty(Q)}2\theta/(1+2\theta))^{1/(q-1)} \quad \text{as} \quad M < k(0).\end{aligned} \tag{37}$$

Observing $q = p(n+\lambda)/(n+p) \in (1,2)$ as $1 < p < 2n/(n+\lambda-2)$, we can show that for $\theta > 0$ small enough, which depends on $C(U)$, $\|f\|_{L_\infty(Q)}$, μ, λ, α and p,

$$\begin{gathered}(\|f\|_{L_\infty(Q)}/\mu(1+2\theta))^{1/(\alpha-1)} + C(U)(\|f\|_{L_\infty(Q)}2\theta/(1+2\theta))^{1/(q-1)} \\ \le (\|f\|_{L_\infty(Q)}/\mu(1+\theta))^{1/(\alpha-1)} < (\|f\|_{L_\infty(Q)}/\mu)^{1/(\alpha-1)}.\end{gathered}$$

The conclusion (8) then follows.

Finally, (37) can be written as

$$\operatorname*{vrai\,max}_{Q} u^{+} \leq (\|f\|_{L_\infty(Q)}/\mu(1+2\theta))^{1/(q-1)} + C(\|f\|_{L_\infty(Q)}2\theta/(1+2\theta))^{1/(q-1)}$$

provided $\alpha = q$ and $\beta = \beta_0$ owing to the fact that

$$U = \|u\|_{L_l(Q)}^{1-(p-\beta)(n+p)/(n+\lambda)} = 1 \quad \text{as} \quad \beta = \beta_0.$$

For $q \in (1,2)$ and θ small enough (θ is now independent of u) we have

$$C(2\theta/(1+2\theta))^{1/(q-1)} \leq (\mu(1+\theta))^{-1/(q-1)} - (\mu(1+2\theta))^{-1/(q-1)}$$

and then the conclusion (9) follows. □

Proof of Theorem 2: Under the assumptions of Theorem 2 instead of (11) we have

$$(11)' \qquad f(x,t) - \mu|u|^{\alpha-2}u \leq f(x,t) \quad \text{on} \quad Q \cap \{u > k\}$$

and then (35) is replaced by

$$\begin{aligned}\|(u-k)^{+}\|_{L_l(Q)}^{q/l} &\leq C \int\int_{Q} (u-k)^{+} f(x,t)\,dx\,dt \\ &\leq C\|f\|_{L_s(Q)}\|(u-k)^{+}\|_{L_l(Q)}|Q \cap \{u>k\}|^{1-1/l-1/s},\end{aligned}$$

where the constant C is independent of u owing to $\beta = \beta_0$. Then it follows

$$\int\int_{Q} (u-k)^{+}\,dx\,dt \leq C\|f\|_{L_s(Q)}^{1/(q-1)}|Q \cap \{u>k\}|^{1+t_1},$$

where $\tau_1 = (p-(n+p)/s)/(n(p-1)+p(\lambda-1)) > 0$. By the use of Lemma 2 we infer (10). □

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