Implicit integral equations with discontinuous right-hand side

Filippo Cammaroto*, Paolo Cubiotti†

Abstract. We consider the integral equation $h(u(t)) = f\big(\int_I g(t,x)\,u(x)\,dx\big)$, with $t \in [0,1]$, and prove an existence theorem for bounded solutions where f is not assumed to be continuous.

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1. Introduction

Let $I = [0,1]$. In this paper we deal with the integral equation

$$(1) \qquad h(u(t)) = f\Big(\int_I g(t,x)\,u(x)\,dx\Big) \quad \text{for a.a.} \quad t \in I,$$

where $g : I \times I \to [0,+\infty[\,$, $h : [\alpha,\beta] \to \mathbf{R}$ and $f : [0,\sigma] \to \mathbf{R}$ $(0 < \alpha < \beta,\ \sigma > 0)$ are given functions. Such problem has been investigated very recently in the paper [6], while for the special case where h is the identity mapping it has been studied in [3], [4], [5], where some sufficient conditions for the existence of integrable solutions have been established. We note that in all the mentioned papers, to which we also refer for some motivations of the equation (1), the continuity of the function f is assumed.

Our aim in this paper, conversely, is to prove an existence result for the equation (1) where we do not assume the continuity of f (Theorem 1 below). In particular, a function f which satisfies the assumptions of our result can be discontinuous at each point of its domain. The key tools in the proof are an existence theorem for inclusions of the type $\Psi(u)(t) \in F(t,\Phi(u)(t))$, due to O. Naselli Ricceri and B. Ricceri [11], and a very recent existence result for Riemann-measurable selections of almost-everywhere lower semicontinuous multifunctions, due to J. Saint Raymond [13].

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† Corresponding author.

2. Preliminaries

We recall that if X and Y are topological spaces, a multifunction $F : X \to 2^Y$ is said to be *lower semicontinuous* at $x_0 \in X$ if for any open $\Omega \subseteq Y$ such that $F(x_0) \cap \Omega \neq \emptyset$, the set $\{x \in X : F(x) \cap \Omega \neq \emptyset\}$ is a neighborhood of x_0 in X. We say that F is lower semicontinuous in X if it is lower semicontinuous at each point $x \in X$. The graph of F is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. For the basic facts about multifunctions, we refer to [2], [9].

In the sequel, for $p \in [1, +\infty]$, p' will be the conjugate exponent of p, and we shall write simply L^p to denote the space $L^p(I)$ with the usual norm $\| \cdot \|_p$. Moreover, we shall denote by $C^0(I)$ the space of all continuous real functions over I, while m will be the Lebesgue measure on the real line $\mathbf{R}$.

If $x \in \mathbf{R}^n$, $r > 0$, we denote by $B(x, r)$ the open ball centered in x with radius r, with respect to the Euclidean norm of $\mathbf{R}^n$. Also, if $A \subseteq \mathbf{R}^n$, we shall write $\operatorname{int} A$, $\overline{A}$ and $\overline{\operatorname{co}}\, A$ to denote the interior, the closure, and the closed convex hull of the set A, respectively. Finally, we put $I_0 =]0, 1[$.

3. Results

The following is our result.

Theorem 1. *Let α, β, σ be positive real numbers, with $\alpha < \beta$. Let $h : [\alpha, \beta] \to \mathbf{R}$, $f : [0, \sigma] \to \mathbf{R}$ and $g : I \times I \to [0, +\infty[$ be three functions, with h continuous, such that*

$$\min_{x \in [\alpha, \beta]} h(x) < \operatorname{ess\,inf}_{x \in [0,\sigma]} f(x), \qquad \operatorname{ess\,sup}_{x \in [0,\sigma]} f(x) < \max_{x \in [\alpha, \beta]} h(x).$$

Moreover, let $\phi_0 \in L^j$, with $j > 1$ and $0 < \|\phi_0\|_1 \leq \frac{\sigma}{\beta}$, and $\phi_1 \in L^1$. Assume that:

(i) *there exists $f_0 : [0, \sigma] \to \mathbf{R}$ such that $f_0 = f$ a.e. in $[0, \sigma]$ and the set*

$$\{x \in [0, \sigma] : f_0 \text{ is discontinuous at } x\}$$

has null Lebesgue measure;

(ii) $\operatorname{int} h^{-1}(t) = \emptyset$ *for all $t \in \operatorname{int} h([\alpha, \beta])$;*

(iii) *for each $t \in I$, the function $g(t, \cdot\,)$ is measurable;*

(iv) *for a.a. $x \in I$, the function $g(\,\cdot\,, x)$ is continuous in I, differentiable in I_0 and*

$$g(t, x) \leq \phi_0(x), \qquad 0 < \frac{\partial g}{\partial t}(t, x) \leq \phi_1(x) \quad \text{for all} \quad t \in I_0.$$

Then there exists $\hat{u} \in L^\infty$ which is a solution of (1).

In the proof of Theorem 1 we shall need the following lemma.

Lemma 2. *Let $f : [0, \sigma] \to \mathbf{R}$ $(\sigma > 0)$ and let $\gamma, \delta \in \mathbf{R}$ be such that*

$$\delta < \operatorname{ess\,inf}_{x\in[0,\sigma]} f(x) \leq \operatorname{ess\,sup}_{x\in[0,\sigma]} f(x) < \gamma.$$

Assume that there exists $f_0 : [0, \sigma] \to \mathbf{R}$ such that $f_0 = f$ a.e. in $[0, \sigma]$ and the set

$$D = \{x \in [0, \sigma] : f_0 \ \textit{ is discontinuous at } \ x\}$$

has null Lebesgue measure.
Then there exists $\hat{f} : [0, \sigma] \to \mathbf{R}$ such that $\hat{f} = f$ a.e. in $[0, \sigma]$, the set $\{x \in [0, \sigma] : \hat{f}$ is discontinuous at $x\}$ has null Lebesgue measure and also

$$\delta \leq \hat{f}(x) \leq \gamma \quad \textit{for all} \quad x \in [0, \sigma].$$

Proof: Let $A = \{x \in [0, \sigma] : f_0(x) \leq \delta\}$, $B = \{x \in [0, \sigma] : f_0(x) \geq \gamma\}$. Of course, if $A \cup B = \emptyset$, our claim follows. Assume $A \cup B \neq \emptyset$, and let $x^* \in (A \cup B) \setminus \{0, \sigma\}$. We claim that f_0 is not continuous at x^*. To this aim, assume $x^* \in A$ (if $x^* \in B$, the argument is analogous). Arguing by contradiction, assume that f_0 is continuous at x^*. Then there exists $\epsilon > 0$ such that

$$f_0(z) < \operatorname{ess\,inf}_{x\in[0,\sigma]} f(x) \quad \text{for all} \quad z \in B(x^*, \epsilon) \subseteq \,]0, \sigma[\,.$$

Therefore, there exists $H \subseteq [0, \sigma]$, with $m(H) = 0$, such that $f_0(z) \neq f(z)$ for all $z \in B(x^*, \epsilon) \setminus H$, and this contradicts the assumption since $m(B(x^*, \epsilon) \setminus H) > 0$. Consequently, we get $A \cup B \subseteq D \cup \{0, \sigma\}$ and $m(A \cup B) = 0$. Now, define $\hat{f} : [0, \sigma] \to \mathbf{R}$ by setting

$$\hat{f}(x) = \begin{cases} \delta & \text{if } x \in A \cup B, \\ f_0(x) & \text{otherwise.} \end{cases}$$

Of course, we have $\delta \leq \hat{f}(x) \leq \gamma$ for all $x \in [0, \sigma]$, and also $\hat{f} = f$ a.e. in $[0, \sigma]$. Now, choose $x_0 \in]0, \sigma[\setminus D$, and let us prove that $\hat{f}$ is continuous at x_0. Since $x_0 \in]0, \sigma[\setminus D \subseteq]0, \sigma[\setminus (A \cup B)$, we have that f_0 is continuous at x_0 and also $\delta < f_0(x_0) < \gamma$. By continuity, there exists $\eta > 0$ such that $\delta < f_0(x) < \gamma$ for all $x \in B(x_0, \eta) \subseteq]0, \sigma[$. Therefore, $B(x_0, \eta) \cap (A \cup B) = \emptyset$ and $\hat{f}(x) = f_0(x)$ for all $x \in B(x_0, \eta)$, hence $\hat{f}$ is continuous at x_0. Therefore, we have $\{x \in]0, \sigma[: \hat{f}$ is discontinuous at $x\} \subseteq D$ and our claim follows. □

Proof of Theorem 1. By Lemma 1, there exists $f_1 : [0, \sigma] \to \mathbf{R}$ such that $f_1 = f$ a.e. in $[0, \sigma]$, the set $E_0 = \{x \in [0, \sigma] : f_1$ is discontinuous at $x\}$ has null Lebesgue measure and also

$$\min_{x\in[\alpha,\beta]} h(x) \leq f_1(x) \leq \max_{x\in[\alpha,\beta]} h(x) \quad \text{for all} \quad x \in [0, \sigma].$$

Now, observe that by (ii) and Theorem 2.4 of [12] the function h is inductively open. That is, there exists a Borel measurable $Y \subseteq [\alpha, \beta]$ such that $h|_Y$ is open and $h(Y) = h([\alpha, \beta])$. Consequently, it is not difficult to check that the multifunction $T : h([\alpha, \beta]) \to 2^Y$ defined by $T(z) = h^{-1}(z) \cap Y$ is lower semicontinuous with nonempty values. Let $Q : [0, \sigma] \to 2^Y$ be defined by $Q(x) = T(f_1(x))$ (such definition makes sense since $f_1([0, \sigma]) \subseteq h([\alpha, \beta])$). By the continuity of f_1, it can be easily checked that Q is lower semicontinuous at each point $x \in [0, \sigma] \setminus E_0$ and consequently the multifunction $x \in [0, \sigma] \to \overline{Q(x)} \subseteq [\alpha, \beta]$ is lower semicontinuous at each point $x \in [0, \sigma] \setminus E_0$. Moreover, it has nonempty closed values. By Theorem 3 of [13], there exists $k : [0, \sigma] \to [\alpha, \beta]$ such that $k(x) \in \overline{Q(x)}$ for all $x \in [0, \sigma]$ and the set $\{x \in [0, \sigma] : k \text{ is discontinuous at } x\}$ has null Lebesgue measure. By the continuity of h we easily get

$$k(x) \in h^{-1}(f_1(x)) \quad \text{for all} \quad x \in [0, \sigma]. \tag{2}$$

Now, let $\psi : \mathbf{R} \to \mathbf{R}$ be defined by

$$\psi(x) = \begin{cases} k(|x|) & \text{if } |x| \leq \sigma, \\ \beta & \text{otherwise.} \end{cases}$$

We want to apply Theorem 1 of [11], taking $T = I$, $X = Y = \mathbf{R}$, $p = s = +\infty$, $q = j'$, $V = L^\infty$, $\Psi(u) = u$, $\Phi(u)(t) = \int_I g(t,x)\, u(x)\, dx$, $\varphi(\lambda) = \lambda \|\phi_0\|_1$, $r = \frac{\sigma}{\|\phi_0\|_1}$, and $F : \mathbf{R} \to 2^{\mathbf{R}}$ defined by

$$F(x) = \bigcap_{\epsilon > 0} \; \bigcap_{m(N)=0} \overline{\text{co}}\, \psi(B(x, \epsilon) \setminus N).$$

To this aim, observe the following facts:

(a) $\Phi(L^\infty) \subseteq C^0(I)$ (this fact comes from (iv) and the Lebesgue dominated convergence theorem);

(b) if $\{v_n\}$ is a sequence in L^∞, $v \in L^\infty$, with $\{v_n\}$ weakly convergent to v in $L^{j'}$, then $\{\Phi(v_n)\} \to \Phi(v)$ strongly in L^1 (see Theorem 2 at p. 359 of [8]);

(c) for each $u \in L^\infty$, one has

$$\text{ess sup}_{t \in I} |\Phi(u)(t)| \leq \|\phi_0\|_1 \|u\|_\infty = \varphi(\|u\|_\infty);$$

(d) By Proposition 1 at p. 102 of [1], F has nonempty convex values and closed graph. Moreover, one has $F(x) \subseteq [\alpha, \beta]$ for all $x \in \mathbf{R}$, hence, in particular,

$$\sup_{|x| \leq \sigma} d(0, F(x)) \leq \beta \leq \frac{\sigma}{\|\phi_0\|_1}.$$

Therefore, by Theorem 1 of [11], there exists $\hat{u} \in L^{\infty}$, with $\|\hat{u}\|_{\infty} \leq \frac{\sigma}{\|\phi_0\|_1}$, and $E \subseteq I$, with $m(E) = 0$, such that

$$\hat{u}(t) \in F(\Phi(\hat{u})(t)) \quad \text{for all} \quad t \in I \setminus E. \tag{3}$$

Since $F(\mathbf{R}) \subseteq [\alpha, \beta]$, we get $\alpha \leq \hat{u} \leq \beta$ a.e. in I. Taking into account (iv), the last fact easily implies that $\Phi(\hat{u}) : I \to [0, \sigma]$ is strictly increasing. Moreover, by (iii), (iv), and Lemma 2.2 at p. 226 of [10], we have

$$\frac{d}{dt}\, \Phi(\hat{u})(t) = \int_I \frac{\partial g}{\partial t}(t, x)\, \hat{u}(x)\, dx > 0$$

for all $t \in I_0$. By Theorem 2 of [14] (taking into account (a)), the function $\Phi(\hat{u})^{-1}$ is absolutely continuous. Now, put

$$N^* = \{x \in [0, \sigma] : k \ \text{ is discontinuous at } \ x\} \cup \\ \cup \{0, \sigma\} \cup \{x \in [0, \sigma] : f_1(x) \neq f(x)\}. \tag{4}$$

Of course, $m(N^*) = 0$. By Theorem 18.25 of [7], we get $m(\Phi(\hat{u})^{-1}(N^*)) = 0$. Now, choose $t \in I \setminus (E \cup \Phi(\hat{u})^{-1}(N^*))$. By (4), we have that $\Phi(\hat{u})(t) \in]0, \sigma[$ and k is continuous at $\Phi(\hat{u})(t)$. Since $\psi = k$ in $]0, \sigma[$, the function ψ is continuous at $\Phi(\hat{u})(t)$, hence, by Proposition 1 at p. 102 of [1] we get

$$F(\Phi(\hat{u})(t)) = \big\{\psi(\Phi(\hat{u})(t))\big\} = \big\{k(\Phi(\hat{u})(t))\big\}.$$

By (2) and (3), we have that $h(\hat{u}(t)) = f_1(\Phi(\hat{u})(t))$. Again by (4), we have $f_1(\Phi(\hat{u})(t)) = f(\Phi(\hat{u})(t))$, hence

$$h(\hat{u}(t)) = f\Big(\int_I g(t, x)\, \hat{u}(x)\, dx\Big).$$

Since $m(E \cup \Phi(\hat{u})^{-1}(N^*)) = 0$, our conclusion follows. □

The following simple example shows that Theorem 1 is no longer true if in assumption (iv) we assume that $0 \leq \frac{\partial g}{\partial t}(t, x) \leq \phi_1(x)$.

Example. Let $\sigma = 1$, $\alpha = \frac{1}{4}$, $\beta = \frac{4}{3}$, h the identity mapping,

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{2} & \text{if } x \in]\frac{1}{2}, 1] \end{cases} \qquad g(t, x) = \frac{3}{4}.$$

Choose any $j > 1$, $\phi_0(x) \equiv \frac{3}{4}$, $\phi_1(x) \equiv 1$. Assume that there exists $u \in L^1$ which solves problem (1). Hence we have $u > 0$ a.e. in I, and thus $u(t) = f(\frac{3}{4}\|u\|_1)$ a.e. in I. Consequently, we have that either $u(t) = \frac{1}{2}$ a.e. in I, or $u(t) = 1$ a.e. in I. In the former case, we get $u(t) = f(\frac{3}{8}) = 1$ a.e. in I, a contradiction. In the latter case we get $u(t) = f(\frac{3}{4}) = \frac{1}{2}$ a.e. in I, another contradiction. Consequently, there exists no solution $u \in L^1$ to problem (1).

Remark. We note that in the paper [6], besides the continuity of f and h, it is assumed that the function h is nondecreasing. Anyway, the results proved in [6] are independent from ours.

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Department of Mathematics, University of Messina, 98166 Sant'Agata-Messina, Italy

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