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Antiproximinal sets in the Banach space $c(X)$

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Abstract. If X is a Banach space then the Banach space $c(X)$ of all X-valued convergent sequences contains a nonvoid bounded closed convex body V such that no point in $C(X) \setminus V$ has a nearest point in V.

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The *distance* from an element x of a normed space X to a nonvoid subset M of X is defined by $d(x, M) = \inf\{\|x - y\| : y \in M\}$. An element $y \in M$ such that $\|x - y\| = d(x, M)$ is called a *nearest point* to x in M and the set of all nearest points to x in M is denoted by $P_M(x)$. The set M is called *proximinal* if $P_M(x) \neq \emptyset$ for all $x \in X$, and *antiproximinal* if $P_M(x) = \emptyset$ for all $x \in X \setminus M$. (Observe that $P_M(y) = \{y\}$ for all $y \in M$.)

Let X^* be the conjugate space to X and let M be a nonvoid convex subset of X. A functional $f \in X^*$ is said to *support* M (at x) if there exists $x \in M$ such that $f(x) = \inf f(M)$ or $f(x) = \sup f(M)$. Obviously $f \in X^*$ supports the closed unit ball B_X of X if and only if there exists $x \in B_X$ such that $f(x) = \|f\|$. If $f \neq 0$ then every $x \in B_X$ verifying this equality must be of norm one, i.e. $\|x\| = 1$. We shall denote by $\mathcal{S}(M)$ the set of all support functionals of the set M.

V. Klee [13] called a Banach space X of type N_1 if it contains a nonvoid closed convex antiproximinal set and of type N_2 if it contains a nonvoid bounded closed convex antiproximinal set. A hyperplane $\{x \in X : f(x) = a\}$ with $f \in X^*$, $f \neq 0$, and $a \in \mathbf{R}$, is proximinal if $f \in \mathcal{S}(B_X)$ and antiproximinal if $f \notin \mathcal{S}(B_X)$. Since, by James theorem, a Banach space X is reflexive if and only if $\mathcal{S}(B_X) = X^*$, it follows that a Banach space is of type N_1 if and only if it is non-reflexive.

The first example of a Banach space of type N_2 was exhibited by M. Edelstein and A.C. Thompson [9] — the Banach space c_0 contains a bounded symmetric closed antiproximinal convex body. By a convex body we mean a convex set with nonvoid interior. A bounded symmetric closed convex body is called a *convex cell.* In [4] it was shown that the space c also contains an antiproximinal convex cell and this property is shared by any Banach space of continuous functions isomorphic to c ([5]). The existence of antiproximinal convex cells in more general spaces of continuous functions was proved by V.P. Fonf [10] (see also [11]).

The aim of the present note is to prove the existence of an antiproximinal convex cell in the Banach space $c(X)$ of all X-valued convergent sequences, where X is a non-trivial Banach space. The proof is simpler than the proof in the scalar case given in [4]. The case of the space $c_0(X)$ was considered in [6]. The notation is standard and all spaces will be considered over $\mathbf{R}$.

Let ω be the first infinite ordinal. Then $\mathbf{N} = [1, \omega[$ and $[1, \omega]$ is a compact Hausdorff space with respect to the interval topology (called also ordinal topology). If $X \neq \{0\}$ is a Banach space then $c(X)$ can be identified with the Banach space $C([1, \omega], X)$ of all continuous functions from $[1, \omega]$ to X, equipped with the usual sup-norm. An element $x \in c(X)$ will be denoted by $x = (x(i) : 1 \leq i \leq \omega)$ and sometimes by $(x(\omega)|x(1), x(2), \dots)$. The conjugate of $c(X)$ is the space $l^1(X^*) = l^1([1, \omega], X^*)$ of all sequences $f = (f_i : 1 \leq i \leq \omega)$ such that $\|f\| := \sum_{1 \leq i \leq \omega} \|f_i\| < \infty$, the duality between $c(X)$ and $l^1(X^*)$ being given by the formula

$$f(x) = \sum_{1 \leq i \leq \omega} f_i(x(i)) \tag{1}$$

for $f \in l^1(X^*)$ and $x \in c(X)$. Again the alternate notation $(f_\omega|f_1, f_2, \dots)$ will be used to designate an element of $l^1(X^*)$.

The main result of this paper is:

Theorem 1. *The Banach space $c(X)$ contains a bounded closed antiproximinal convex body.*

The proof will be based on the following characterization of antiproximinal sets.

Lemma 2 ([9])**.** *A nonvoid closed convex subset M of a Banach space X is antiproximinal if and only if*

$$\mathcal{S}(M) \cap \mathcal{S}(B_X) = \{0\}, \tag{2}$$

where B_X denotes the closed unit ball of X.

The following lemma gives some information about the support functionals of the unit ball of $c(X)$. The characterization of support functionals of the unit ball of $C(T)$, for a compact Hausdorff space T, was given by S.I. Zukhovickij [19] in the scalar case and by V.L. Chakalov [1] for vector-valued functions. For characterization of support functionals of the unit balls in other concrete Banach spaces, see [7], [14] and [15].

Lemma 3. *Let B_c be the closed unit ball of $c(X)$ and let $f = (f_i : 1 \leq i \leq \omega)$, $f \neq 0$, be an element in $l^1(X^*)$.*

(a) *If $f = (f_i : 1 \leq i \leq \omega) \in \mathcal{S}(B_c) \setminus \{0\}$ and $x = (x(i); 1 \leq i \leq \omega) \in B_c$ is such that $f(x) = \|f\|$, then $f_i(x(i)) = \|f_i\|$ for all $i \in [1, \omega]$ and $\|x(i)\| = 1$ for all $i \in [1, \omega]$ such that $f_i \neq 0$.*

(b) *Let $\mathbf{N} = [1, \omega[$ and let $\sigma_i : \mathbf{N} \to \mathbf{N}$, $i = 1, 2$, be two strictly increasing functions such that $\sigma_1(\mathbf{N}) \cap \sigma_2(\mathbf{N}) = \emptyset$. Let $h \in X^*$, $h \neq 0$, and $\alpha_j, \beta_j > 0$, $j \in \mathbf{N}$.*

If $f = (f_i : 1 \le i \le \omega) \in l^1(X^*)$ *is such that* $f_{\sigma_1(j)} = \alpha_j h$ *and* $f_{\sigma_2(j)} = -\beta_j h$ *for all* $j \in \mathbf{N}$*, then* $f \notin \mathcal{S}(B_c)$*.*

Proof: (a) Let $f \in \mathcal{S}(B_c) \setminus \{0\}$ and let $x \in B_c$ be such that $f(x) = \|f\|$. Since $f_i(x(i)) \le \|f_i\| \cdot \|x(i)\|$, for all $i \in [1,\omega]$, it follows that

$$\sum_{1\le i\le\omega} \|f_i\| = \|f\| = f(x) =$$

$$= \sum_{1\le i\le\omega} f_i(x(i)) \le \sum_{1\le i\le\omega} \|f_i\| \cdot \|x(i)\| \le \sum_{1\le i\le\omega} \|f_i\|,$$

implying $f_i(x(i)) = \|f_i\|$, for all $i \in [1,\omega]$, and $\|x(i)\| = 1$ for all $i \in [1,\omega]$ such that $f_i \ne 0$.

(b) Let $h \in X^*$, $h \ne 0$, α_j, β_j, σ_1, σ_2 and $f \in l^1(X^*)$ fulfill the hypotheses of the lemma and suppose, on the contrary, that there exists an element $x = (x(i) : 1 \le i \le \omega) \in B_c$ such that $f(x) = \|f\|$. Taking into account the first point of the lemma it follows that

$$\alpha_j\|h\| = \|f_{\sigma_1(j)}\| = \alpha_j h(x(\sigma_1(j)))$$

and

$$\beta_j\|h\| = \|f_{\sigma_2(j)}\| = -\beta_j h(x(\sigma_2(j)))$$

implying $h(x(\sigma_1(j))) = \|h\|$ and $h(x(\sigma_2(j))) = -\|h\|$, for all $j \in \mathbf{N}$. Since $\sigma_k(j) \to \omega$ for $j \to \omega$, $k = 1,2$, and the functions x and h are continuous, the above equalities yield, for $j \to \omega$, the contradiction $h(x(\omega)) = \|h\| > 0$ and $h(x(\omega)) = -\|h\| < 0$. □

Other result we need for the proof of the Theorem 1 is the following one, emphasizing the behaviour of support functionals under linear isomorphisms. If X, Y are Banach spaces and $A : X \to Y$ is an isomorphism then its conjugate $A^* : Y^* \to X^*$ is an isomorphism too and $(A^*)^{-1} = (A^{-1})^*$ ([8, Lemma VI 3.7]). The support functionals of a set $M \subseteq X$ and of the set $A(M) \subset Y$ are related as follows:

Lemma 4 ([9, Lemma 1])**.** *Let* X*,* Y *be Banach spaces,* M *a nonvoid closed convex subset of* X *and* $A : X \to Y$ *an isomorphism. Then*

$$\mathcal{S}(M) = A^*(\mathcal{S}(A(M))). \tag{3}$$

More exactly

$$g \in \mathcal{S}(A(M)) \;\Leftrightarrow\; A^*g \in \mathcal{S}(M). \tag{4}$$

Now we are in position to pass to:

Proof of Theorem 1: First we construct an isomorphism $A : c(X) \to c(X)$ in the following way. For an element $x = (x(i) : 1 \le i \le \omega) \in c(X)$ define $Ax : [1, \omega] \to X$ by

$$Ax(\omega) = x(\omega) + \sum_{1 \le j < \omega} (-1)^j 2^{-j-2} x(2j-1) \tag{5}$$

and

$$\begin{aligned} Ax(i) = x(i) + \sum_{1 \le j \le 2^i} (-1)^j 2^{-j-2} x(2j-1) + \\ +2^{-i-1} \sum_{1 \le j < \omega} (-1)^j 2^{-j} x(2^i(2j-1)) \end{aligned} \tag{6}$$

for $1 \le i < \omega$. Since the series in the right hand sides of the equalities (5) and (6) are norm convergent and X is a Banach space, it follows that the definition of Ax makes sense. Since

$$\begin{aligned} \|Ax(\omega) - Ax(i)\| \le \|x(\omega) - x(i)\| + 2^{-i-1} \sum_{1 \le j < \omega} 2^{-j} \|x\| = \\ = \|x(\omega) - x(i)\| + 2^{-i-1} \|x\|, \end{aligned}$$

and $\lim_{i \to \omega} x(i) = x(\omega)$, it follows that $\lim_{i \to \omega} Ax(i) = Ax(\omega)$, i.e. Ax is an element of $c(X)$. Obviously the operator $A : c(X) \to c(X)$ is linear. By (5) and (6) we have

$$\|Ax(\omega)\| \le \|x\| + 2^{-2} \|x\| = (5/4) \|x\|$$

and, respectively,

$$\|Ax(i)\| \le \|x\| + 2^{-2} \|x\| + 2^{-i-1} \|x\| \le (3/2) \|x\|$$

for $1 \le i < \omega$, implying

$$\|Ax\| \le (3/2) \|x\|, \tag{7}$$

for all $x \in c(X)$, which is equivalent to the continuity of the operator A.

Now let $x \in c(X)$, $x \ne 0$, and let $i_0 \in [1, \omega]$ be such that $\|x(i_0)\| = \|x\| := \sup\{\|x(i)\| : 1 \le i \le \omega\}$. If $i_0 = \omega$, then, by (5), $\|Ax\| \ge \|Ax(\omega)\| \ge \|x(\omega)\| - 2^{-2} \|x\| = (3/4) \|x\|$.

If $1 \le i_0 < \omega$, then by (6)

$$\|Ax\| \ge \|Ax(i_0)\| \ge \|x(i_0)\| - (2^{-2} + 2^{-i-1}) \|x\| \ge (1/2) \|x\|.$$

It follows that

$$\|Ax\| \ge (1/2) \|x\|, \tag{8}$$

for all $x \in c(X)$. The inequalities (7) and (8) show that A is an isomorphism of $c(X)$ onto $c(X)$. Its conjugate A^* will be an isomorphism of $l^1(X^*)$ onto $l^1(X^*)$ acting by the formula

$$A^* f(x) = f(Ax) = \sum_{1 \le i \le \omega} f_i(Ax(i)), \tag{9}$$

for $f \in l^1(X^*)$ and $x \in c(X)$. Taking into account the formulae (5) and (6), defining the operator A, one obtains

$$f_\omega(Ax(\omega)) = f_\omega(x(\omega)) + \sum_{1 \le j < \omega} (-1)^j 2^{-j-2} f_\omega(x(2j-1)) \tag{10}$$

and

$$\begin{aligned} f_i(Ax(i)) &= f_i(x(i)) + \sum_{1 \le j \le 2^i} (-1)^j 2^{-j-2} f_i(x(2j-1)) + \\ &+ 2^{-i-1} \sum_{1 \le j < \omega} (-1)^j 2^{-j} f_i(x(2^i(2j-1))). \end{aligned} \tag{11}$$

Let $c_0(X)$ denote the Banach space of all X-valued sequences converging to zero. It follows that $c_0(X) = \{x \in C([1,\omega], X) : x(\omega) = 0\}$. The spaces $c(X)$ and $c_0(X)$ are isomorphic, an isomorphism $H : c(X) \to c_0(X)$ being given by the formula

$$H(x) = (0 | x(\omega), x(1) - x(\omega), x(2) - x(\omega), \dots) \tag{12}$$

for $x = (x(\omega) | x(1), x(2), \dots) \in c(X)$ (see [20, p. 55]). Its conjugate H^* will be an isomorphism of $c_0(X)^*$ onto $c(X)^*$. The conjugate $c_0(X)^*$ of $c_0(X)$ can be identified with the space

$$W := \{f \in l^1([1,\omega], X^*) : f = (f_i : 1 \le i \le \omega), f_\omega = 0\},$$

or equivalently

$$W = \{f \in l^1([1,\omega], X^*) : f = (0 | f_1, f_2, \dots)\}, \tag{13}$$

normed by $\|f\| = \sum_{1 \le i < \omega} \|f_i\|$. The duality between $c_0(X)$ and W is given by the formula

$$f(y) = \sum_{1 \le i < \omega} f_i(y(i)), \tag{14}$$

for $f = (0 | f_1, f_2, \dots) \in W$ and $y = (0 | y(1), y(2), \dots) \in c_0(X)$. Since for $x = (x(w) | x(1), x(2), \dots) \in c(X)$ and $f = (0 | f_1, f_2, \dots) \in W$ we have

$$H^* f(x) = f(Hx) = f((0 | x(\omega), x(1) - x(\omega), x(2) - x(\omega), \dots))$$

it follows that

$$H^* f = (f_1 - \sum_{2 \le j < \omega} f_j | f_2, f_3, \dots). \tag{15}$$

Denote by B_c and B_{c_0} the closed unit balls of $c(X)$ and $c_0(X)$ respectively, and put

$$V = (HA)^{-1}(B_{c_0}). \tag{16}$$

Since A and H are isomorphisms, it follows that V is a bounded symmetric closed convex body in $c(X)$. We shall show that the set V is antiproximinal in $c(X)$. To this end, by Lemma 2, it suffices to show that

$$\mathcal{S}(V) \cap \mathcal{S}(B_c) = \{0\}. \tag{17}$$

Since, by (16), $B_{c_0} = HA(V)$ we have

$$\mathcal{S}(B_{c_0}) = \mathcal{S}(HA(V)). \tag{18}$$

By Lemma 4, $\mathcal{S}(V) = \{(HA)^* f : f \in \mathcal{S}(HA(V))\}$ and therefore

$$\mathcal{S}(V) = \{(HA)^* f : f \in \mathcal{S}(B_{c_0})\}. \tag{19}$$

It follows that the relation (17) will be a consequence of the implication

$$f \in \mathcal{S}(B_{c_0}) \setminus \{0\} \ \Rightarrow (HA)^* f \notin \mathcal{S}(B_c). \tag{20}$$

In order to prove (20) observe that $f = (0|f_1, f_2, \dots) \in c_0(X)^*$, $f \neq 0$, supports the unit ball B_{c_0} of $c_0(X)$ if and only if there exists $n \in [1, \omega[$ such that $f_i = 0$ for $i > n$ and $f_i \in \mathcal{S}(B_X)$, for $1 \le i \le n$, where B_X denotes the closed unit ball of the space X.

Now let $f = (0|f_1, \dots, f_n, 0, \dots)$, $f_n \neq 0$, be a support functional of B_{c_0} and let us show that $(HA)^* f \notin \mathcal{S}(B_c)$.

First suppose $n = 1$, i.e. $f = (0|f_1, 0, \dots)$ with $f_1 \in \mathcal{S}(B_X)$, $f_1 \neq 0$. By (15), $H^* f = (f_1|0, \dots)$ so that, denoting $g = A^* H^* f = (HA)^* f$, formula (10) gives

$$g(x) = f_1(x(\omega)) + \sum_{1 \le j < \omega} (-1)^j 2^{-j-2} f_1(x(2j-1))$$

for all $x \in c(X)$. For $j = 2k$ and $j = 2k-1$, $1 \le k < \omega$, one obtains $g_{4k-1} = 2^{-2k-2} f_1$ and $g_{4k-3} = -2^{2k-3} f_1$, respectively, so that, by Lemma 3(b), $g \notin \mathcal{S}(B_c)$.

If $n \ge 2$ then

$$h := H^* f = (f_1 - \sum_{2 \le i \le n} f_i | f_2, \dots, f_n, 0, \dots).$$

Taking into account formula (11) it follows that $g = A^* h$ verifies $g_{2^{n-1}(4k-3)} = -2^{-2k+1-n} f_n$ and $g_{2^{n-1}(4k-1)} = 2^{-2k-n} f_n$ for all $k \in [1, \omega[$. Appealing again to Lemma 3(b) it follows that $g = A^* H^* f \notin \mathcal{S}(B_c)$.

Theorem 1 is completely proved. $\square$

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