A note on lattice renormings

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Abstract. It is shown that every strongly lattice norm on $c_0(\Gamma)$ can be approximated by C^∞ smooth norms. We also show that there is no lattice and Gâteaux differentiable norm on $C_0[0,\omega_1]$.

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It has been recently shown in [1] and [2] that every equivalent norm on the classical separable Banach spaces c_0 or ℓ_p, p even, (as well as on many other spaces) can be uniformly approximated on bounded sets by a sequence of C^∞-Fréchet smooth norms.

Although the method of construction requires some technical conditions on the space to be satisfied (in particular the existence of a Schauder basis), it seems to suggest that perhaps the following statement should be valid:

Suppose X is a separable Banach space that admits an equivalent C^k-Fréchet smooth norm. Then every equivalent norm on X can be approximated uniformly on bounded sets by a sequence of C^k-Fréchet smooth norms.

On the other hand, we do not know of any example of a nonseparable Banach space where a similar statement would be valid for $k \geq 2$.

In the present note we give a partial solution to this problem for the space $c_0(\Gamma)$ and $k=\infty$. More precisely we show that on $c_0(\Gamma)$, Γ uncountable, every equivalent strongly lattice norm can be approximated by a sequence of C^∞-Fréchet smooth norms.

In the second part of our paper, we show that there exists no lattice Gâteaux differentiable norm on $C_0([0,\omega_1])$, the space of continuous functions on the ordinal segment $[0,\omega_1]$ that vanish at ω_1 (where ω_1 is the first uncountable ordinal and $[0,\omega_1]$ is in its normal topology as in [4]). More information on the space $C_0([0,\omega_1])$ can be found e.g. [3, p. 259]. Proposition 2 of this paper is of interest when compared with some results of Haydon [5]–[6]. In [5], a lattice norm on $C_0[0,\omega_1]\oplus c_0[0,\omega_1]$ is constructed, which is C^∞-Fréchet differentiable and locally dependent on finitely many coordinates when restricted to a rather large open

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subset of $C_0[0,\omega_1] \oplus c_0[0,\omega_1]$. This norm is then used to obtain C^∞-Fréchet smooth (necessarily non-lattice) renormings of $C_0[0,\omega_1]$.

The notation and terminology we use are mostly standard, as in [3].

By a strongly lattice norm on $c_0(\Gamma)$ we mean an equivalent norm $\|\cdot\|$ such that $\|\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma\| \geq \|\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma\|$ whenever $\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma$, $\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$ are such that for every $\gamma \in \Gamma$ $|y_\gamma| \geq |x_\gamma|$ is satisfied.

Theorem 1. *Every equivalent strongly lattice norm on $c_0(\Gamma)$ can be approximated (uniformly on bounded sets) by C^∞-Fréchet smooth norms.*

Proof: Denote the given strongly lattice norm by $\|\cdot\|$. We first introduce an auxiliary function f_Δ. For arbitrary $1 > \Delta > 0$ and $\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$ denote by

$$f_\Delta\Big(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Big) = \sup\Big\{\|\sum_{\gamma\in\Gamma} y_\gamma e_\gamma\|,$$

$$\text{where } \ y_\gamma = x_\gamma \ \text{ if } \ |x_\gamma| > \Delta \ \text{ and } \ |y_\gamma| \leq \Delta \ \text{ if } \ |x_\gamma| \leq \Delta\Big\}.$$

Clearly, $f_\Delta(\cdot) \geq \|\cdot\|$ on $c_0(\Gamma)$.

In fact, $f_\Delta(\cdot)$ is a Lipschitz function on $(c_0(\Gamma), \|\cdot\|_\infty)$ with the Lipschitz constant less than or equal to the Lipschitz constant of $\|\cdot\|$ (on $(c_0(\Gamma), \|\cdot\|_\infty)$).

It is standard to check the following elementary properties of $f_\Delta(\cdot)$:

(i) $f_\Delta(\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma) = f_\Delta(\sum\limits_{\gamma\in\{\alpha\in\Gamma, |x_\alpha|>\Delta\}} x_\gamma e_\gamma)$. In other words, the value of $f_\Delta(x)$ depends only on those coordinates of x that are in absolute value larger than Δ.

(ii) $f_\Delta(\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma) \leq f_\Delta(\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma)$
whenever we have $\|y_\gamma\| \geq \|x_\gamma\|$ for every $\gamma \in \Gamma$.

The property (ii) is a "strongly lattice" property of $f_\Delta(\cdot)$ and follows directly from the strongly lattice property of $\|\cdot\|$.

We now proceed with our construction of approximating C^∞-norm.

Given $\varepsilon > 0$, from the equivalence of $\|\cdot\|$ and $\|\cdot\|_\infty$ it follows that there exists $1 > \Delta > 0$ such that

$$\|\cdot\| \leq f_\Delta(\cdot) \leq \|\cdot\| + \varepsilon$$

for every $x \in c_0(\Gamma)$.

Put $F_\Delta(x) = f_\Delta^2(x)$.

Then $F_\Delta(\cdot)$ shares properties (i), (ii) and satisfies:

$$\|\cdot\|^2 \leq F_\Delta(\cdot) \leq (\|\cdot\| + \varepsilon)^2 = \|\cdot\|^2 + 2\varepsilon\|\cdot\| + \varepsilon^2.$$

Thus the convex function $C_\Delta(\cdot)$ defined by:

$$C_\Delta(x) = \inf\left\{\sum_{i=1}^n \lambda_i F_\Delta(x_i),\ \ x = \sum_{i=1}^n \lambda_i x_i,\ \sum_{i=1}^n \lambda_i = 1,\ \lambda_i > 0\right\}$$

also satisfies $\|\cdot\|^2 \le C_\Delta(\cdot) \le (\|\cdot\|+\varepsilon)^2$, because $\|\cdot\|^2$ is convex and $C_\Delta(\cdot) \le F_\Delta 9(\cdot)$. It is straightforward to show that also the strongly lattice property for $C_\Delta(\cdot)$ is preserved, i.e. $C_\Delta(x) \ge C_\Delta(y)$ for $x, y \in c_0(\Gamma)$, such that for every $\gamma \in \Gamma$ either $\|y_\gamma\| \ge \|x_\gamma\|$. We will now show that for $1 > \varepsilon > 0$ we have

$$C_\Delta(x) = \inf\left\{\sum_{i=1}^n \lambda_i F_\Delta(x_i),\ \ x = \sum_{i=1}^n \lambda_i x_i,\ \sum_{i=1}^n \lambda_i = 1,\ \lambda_i > 0 \text{ and } \|x_i\| \le 100\right\}$$

for every $x \in c_0(\Gamma)$ with $\|x\| \le 2$.

To this end, it is enough to find for every $\{x_i\}_{i=1}^n$, $\lambda_i > 0$, $\sum\limits_{i=1}^n \lambda_i = 1$, $x = \sum\limits_{i=1}^n \lambda_i x_i$ another system $\{y_i\}_{i=1}^m$, $\lambda_i' > 0$, $\sum\limits_{i=1}^m \lambda_i' = 1$, $x = \sum\limits_{i=1}^m \lambda_i' y_i$, where $\|y_i\| \le 100$ and such that

$$\sum_{i=1}^m \lambda_i' F_\Delta(y_i) \le \sum_{i=1}^n \lambda_i F_\Delta(x_i).$$

Suppose without loss of generality that $\|x_i\| \le 100$ for $1 \le i \le j$ and $\|x_i\| > 100$ for $j < i \le n$. We may assume that $j \ge 1$, since otherwise $F_\Delta(x_i) \ge 100^2$ for every $1 \le i \le n$, and then $F_\Delta(x) \le 3^2 < 100^2$ would give us a better estimate.

Put

$$v_1 = \frac{\sum\limits_{i=1}^j \lambda_i x_i}{\sum\limits_{i=1}^j \lambda_i},\quad v_2 = \frac{\sum\limits_{i=j+1}^n \lambda_i x_i}{\sum\limits_{i=j+1}^n \lambda_i},$$

$$\xi_1 = \sum_{i=1}^j \lambda_i,\quad \xi_2 = 1 - \xi_1.$$

Clearly, $x = \xi_1 v_1 + \xi_2 v_2$.

We may assume that $F_\Delta(v_1) \ge \frac{1}{\xi_1} \sum\limits_{i=1}^j \lambda_i F_\Delta(x_i)$ and

$F_\Delta(v_2) \ge \frac{1}{\xi_2} \sum\limits_{i=j+1}^n \lambda_i F_\Delta(x_i)$.

Indeed, if for example $F_\Delta(v_1) < \frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i)$, we obtain that $x = \xi_1 v_1 + \sum_{i=j+1}^{n}\lambda_i x_i$, $\xi_1 + \sum_{i=j+1}^{n}\lambda_i = 1$, $\xi_1 \geq 0$, $\lambda_i \geq 0$ and

$$\xi_1 F_\Delta(v_1) + \sum_{i=j+1}^{n} F_\Delta(x_i) < \sum_{i=1}^{n}\lambda_i F_\Delta(x_i)$$

gives us even a better estimate of $C_\Delta(x)$.

By assumption, $F_\Delta(x_i) \geq 100^2$ for $j+1 \leq i \leq n$. Thus $\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i) \geq 100^2$. The trivial estimate for $C_\Delta(x)$ is $F_\Delta(x) \leq 3^2 = 9$. Thus $\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) \leq 9$ (otherwise the trivial estimate would give us a smaller value than $\sum_{i=1}^{n}\lambda_i F_\Delta(x_i) = \xi_1\big(\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i)\big) + \xi_2\big(\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i)\big)$.

Consequently, $\|v_1\|^2 \leq C_\Delta(v_1) \leq 9$ and we have $\|v_1\| \leq 3$. Similarly, $\big(\|v_2\| + \varepsilon\big)^2 \geq F_\Delta(v_2) \geq 100^2$ and we have $\|v_2\| \geq 99$.

Thus there exists $v_3 \in c_0(\Gamma)$, $\|v_3\| = 50$, $v_3 = \alpha_1 v_1 + \alpha_2 v_2$ where $\alpha_1 + \alpha_2 = 1$, $\alpha_i \geq 0$. Since $v_3 - \alpha_1 v_1 = \alpha_2 v_2$, we have $47 \leq \alpha_2\|v_2\|$. Thus

$$\alpha_1\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \alpha_2\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i) \geq \alpha_2\|v_2\|^2 \geq 47\|v_2\| \geq 47\cdot 99.$$

Moreover the trivial estimate gives us

$$F_\Delta(v_3) \leq \big(\|v_3\| + \varepsilon\big)^2 \leq 51^2 < 47\cdot 99.$$

Therefore

$$F_\Delta(v_3) \leq \alpha_1\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \alpha_2\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i),$$

$$\frac{\xi_2}{\alpha_2}F_\Delta(v_3) \leq \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i),$$

$$\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) - \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \frac{\xi_2}{\alpha_2}F_\Delta(v_3) \leq \sum_{i=1}^{n}\lambda_i F_\Delta(x_i),$$

$$\sum_{i=1}^{j}\big(1 - \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\big)\lambda_i F_\Delta(x_i) + \frac{\xi_2}{\alpha_2}F_\Delta(v_3) \leq \sum_{i=1}^{n}\lambda_i F_\Delta(x_i).$$

However,

$$\left(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\right)\sum_{i=1}^{j}\lambda_i x_i+\frac{\xi_2}{\alpha_2}v_3=\xi_1 v_1+\xi_2\left(\frac{v_3}{\alpha_2}-\frac{\alpha_1}{\alpha_2}v_1\right)=\xi_1 v_1+\xi_2 v_2=x.$$

It is easy to verify that $\sum\limits_{i=1}^{j}\left(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\right)\lambda_i+\frac{\xi_2}{\alpha_2}=1$. It follows that $\alpha_2>\xi_2$, since $\|v_3\|=50$ while $\|x\|\le 2$. Therefore $(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1})\lambda_i\ge 0$ for every $1\le i\le j$.

Thus the system $\{x_i\}_{i=1}^{j}\cup\{v_3\}$, $\{\left(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\right)\lambda_i\}_{i=1}^{j}\cup\{\frac{\xi_2}{\alpha_2}\}$ gives us a smaller estimate of $C_\Delta(x)$ than the original one $\{x_i\}_{i=1}^{n}$, $\{\lambda_i\}$. Clearly, all $\|x_i\|\le 100$, $1\le i\le j$, $\|v_3\|\le 100$.

Since $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent norms on $c_0(\Gamma)$, it follows from our previous considerations that there exists a constant k such that

$$C_\Delta(x)=\inf\left\{\sum_{i=1}^{j}\lambda_i F_\Delta(x_i),\ \ x=\sum_{i=1}^{j}\lambda_i x_i,\ \sum_{i=1}^{j}\lambda_i=1,\ \lambda_i>0 \text{ and } \|x_i\|_\infty\le k\right\}$$

for every $\|x\|\le 2$.

We proceed by proving that there exists $\delta>0$ such that

$$C_\Delta\Big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\Big)=C_\Delta\Big(\sum_{\gamma\in\{\alpha,|x_\alpha|>\delta\}}x_\gamma e_\gamma\Big)$$

for every $x=\sum\limits_{\gamma\in\Gamma}x_\gamma e_\gamma\in c_0$ such that $\|x\|\le 2$.

In fact, we will show that choosing $\delta<\frac{\Delta^2}{2k+2+\Delta}$ is sufficient.

Since C_Δ is upper semi-continuous (as the infimum of a family of continuous functions - F_Δ is continuous as the square of a Lipschitz function f_Δ), and, moreover, from the strongly lattice property of C_Δ it is enough to prove that

$$C_\Delta\Big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\Big)=C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}}x_\gamma e_\gamma\Big),$$

whenever $|x_{\gamma_0}|\le\delta$.

We will proceed as follows. Given $x=\sum\limits_{\gamma\in\Gamma}x_\gamma e_\gamma$, for arbitrary $\{y_i\}_{i=1}^{n}\subset c_0(\Gamma)$, $\{\lambda_i\}_{i=1}^{n}$, $\lambda_i>0$, $\sum\limits_{i=1}^{n}\lambda_i=1$, $\|y_i\|_\infty\le k$ such that $\sum\limits_{i=1}^{n}\lambda_i y_i=\sum\limits_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}}x_\gamma e_\gamma$, we will construct $\{x_i\}_{i=1}^{n}\subset c_0(\Gamma)$ such that $(x_i)_\gamma=(y_i)_\gamma$ for $1\le i\le n$, $\gamma\ne\gamma_0$, $\sum\limits_{i=1}^{n}\lambda_i x_i=x$ and in addition

$$\sum_{i=1}^{n}\lambda_i F_\Delta(x_i)\le\sum_{i=1}^{n}\lambda_i F_\Delta(y_i).$$

Consequently,

$$C_\Delta\Big(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Big) \le C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} x_\gamma e_\gamma\Big).$$

This implies our claim, since $C_\Delta(\cdot)$ shares the strongly lattice property, so the opposite inequality is satisfied.

Without loss of generality assume that, $\delta \ge x_{\gamma_0} > 0$ and

$$\begin{aligned}
k \ge (y_i)_{\gamma_0} &> \Delta && \text{for } 1 \le i \le j_1,\\
\Delta \ge (y_i)_{\gamma_0} &\ge 0 && \text{for } j_1 < i \le j_2,\\
0 > (y_i)_{\gamma_0} &\ge -\Delta && \text{for } j_2 < i \le j_3,\\
-\Delta > (y_i)_{\gamma_0} &\ge -k && \text{for } j_3 < i \le n.
\end{aligned}$$

Put $s_1 = \sum_{i=1}^{j_1} \lambda_i$, $s_2 = \sum_{i=j_1+1}^{j_2} \lambda_i$, $s_3 = \sum_{i=j_2+1}^{j_3} \lambda_i$, $s_4 = \sum_{i=j_3+1}^{n} \lambda_i$.

If $(s_3+s_4)\Delta \ge \delta$, then

$$\sum_{i=1}^{j_2} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i \Delta \ge \sum_{i=j_2+1}^{n} \lambda_i \Delta \ge (s_3+s_4)\Delta \ge \delta.$$

Therefore for every $j_2 < i \le n$ we can find numbers $\tilde{y}_i$, such that $\Delta \ge \tilde{y}_i \ge (y_i)_{\gamma_0}$ and

$$\sum_{i=1}^{j_2} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i \tilde{y}_i = x_{\gamma_0}.$$

We define $x_i = y_i$ for $1 \le i \le j_2$, and $x_i = \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} (y_i)_\gamma e_\gamma + \tilde{y}_i e_{\gamma_0}$ for $j_2 < i \le n$. It follows that

$$F_\Delta(x_i) = F_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} (y_i)_\gamma e_\gamma\Big) \le F_\Delta(y_i).$$

Thus $\sum_{i=1}^{n} \lambda_i F_\Delta(x_i) \le \sum_{i=1}^{n} \lambda_i F_\Delta(y_i)$ and the claim is established.

If $(s_3+s_4)\Delta < \delta$, we obtain $0 = \big(\sum_{i=1}^{n} \lambda_i (y_i)\big)_{\gamma_0} \ge s_1\Delta - (s_3+s_4)k$. Therefore $s_1 \le \frac{\delta k}{\Delta^2}$. Thus $s_2 = 1 - s_1 - s_3 - s_4 \ge 1 - \frac{\delta(k+1)}{\Delta^2}$. We can find numbers $\tilde{y}_i$ for $j_1 < i \le j_2$, such that $(y_i)_{\gamma_0} \le \tilde{y}_i \le \Delta$ and

$$\sum_{i=1}^{j_1} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_1+1}^{j_2} \lambda_i \tilde{y}_i = x_{\gamma_0}.$$

Indeed, $\left|\sum\limits_{i=j_2+1}^{n}\lambda_i(y_i)_{\gamma_0}\right| \le (s_3+s_4)k \le \frac{\delta k}{\Delta}$. Consequently, $s_2\Delta - \frac{\delta k}{\Delta} \ge \Delta - \frac{\delta(k+1)}{\Delta} - \frac{\delta k}{\Delta} > \delta$ by our choice of δ.

Putting $(x_i)_\gamma = \tilde{y}_i$ for $j_1 < i \le j_2$, $\gamma = \gamma_0$ and $(x_i)_\gamma = (y_i)_\gamma$ for any other choices of i and γ, we obtain again

$$\sum_{i=1}^{n}\lambda_i F_\Delta(x_i) = \sum_{i=1}^{n}\lambda_i F_\Delta(y_i).$$

Hence we proved that $C_\Delta(\cdot)$ is a convex function on $c_0(\Gamma)$, $\|\cdot\|^2 \le C_\Delta(\cdot) \le \big(\|\cdot\|+\varepsilon\big)^2$ and, for $\|x\|\le 2$, $C_\Delta(x)$ depends only on those coordinates x_γ of x for which $|x_\gamma| \ge \delta$. More precisely,

$$C_\Delta\big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\big) = C_\Delta\big(\sum_{\gamma\in\Gamma_1}x_\gamma e_\gamma\big),$$

where $\Gamma_1 = \{\gamma\in\Gamma,\ \ |x_\gamma|\ge\delta\}$.

We will now construct a C^∞-Fréchet smooth convex function on the set $\{x\in c_0(\Gamma),\ \|x\| < 2\}$, which uniformly approximates $C_\Delta(\cdot)$. To this end, choose a C^∞-smooth bump function $b(t)$ on $\mathbb{R}$, $0 \le b(t) = b(-t)$, $\operatorname{supp} b \subset [-\frac{\delta}{4},\frac{\delta}{4}]$, $\int\limits_{-\infty}^{\infty} b(t)\,dt = 1$.

It is elementary to check that from the symmetry condition on b and the convexity of f it follows that

$$f(r) \le \int_{-\infty}^{\infty} f(t)b(r-t)\,dt$$

for arbitrary convex continuous function defined on $\mathbb{R}$.

It is standard to check that for arbitrary $\gamma_0\in\Gamma$, the function

$$C_\Delta^{\gamma_0}\big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\big) = \int_{-\infty}^{\infty} C_\Delta\big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}}x_\gamma e_\gamma + te_{\gamma_0}\big)b(x_{\gamma_0}-t)\,dt$$

is convex and $C_\Delta^{\gamma_0}(\cdot) \ge C_\Delta(\cdot)$.

Put $\Pi = \big\{\pi = \{\gamma_1,\dots,\gamma_n\},\ n\in\mathbb{N},\ \gamma_i\in\Gamma\big\}$ to be the set of all finite subsets of Γ. For $\pi = \{\gamma_1,\dots,\gamma_n\}\in\Pi$ define

$$C_\Delta^{\pi}\big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\big) =$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} C_\Delta\big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\pi}}x_\gamma e_\gamma + \sum_{i=1}^{n}t_i e_{\gamma_i}\big)b(x_{\gamma_1}-t_1)\dots b(x_{\gamma_n}-t_n)\,dt_1\dots dt_n.$$

For every $\pi \in \Pi$, C^π_Δ is a convex function satisfying $C^{\pi_2}_\Delta(\cdot) \geq C^{\pi_1}_\Delta(\cdot)$ whenever $\pi_1 \subset \pi_2$.

Define $\tilde{C}_\Delta(x) = \sup\{C^\pi_\Delta(x), \;\; \pi \in \Pi\}$.

Suppose $x = \sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma$, $\|x\| \leq 2 - \frac{\delta}{2}$, $\Gamma_1 = \{\gamma \in \Gamma, \; |x_\gamma| \leq \frac{\delta}{4}\}$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Clearly $\Gamma_2 \in \Pi$. For every $y \in c_0(\Gamma)$ such that $\|y - x\|_\infty < \frac{\delta}{4}$, we have $|y_\gamma| \leq \frac{\delta}{2}$ for $\gamma \in \Gamma_1$. For such y the following formula is satisfied:

$$\tilde{C}_\Delta(y) = C^{\Gamma_2}_\Delta(y) =$$
$$= \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_2}} y_\gamma e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1) \dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_n,$$

where $\Gamma_2 = \{\gamma_1, \dots, \gamma_n\}$.

Indeed, for every $\Gamma_3 = \{\gamma_1, \dots, \gamma_m\}$, $\Gamma_2 \subset \Gamma_3$ we have

$$C^{\Gamma_3}_\Delta(y) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_3}} y_\gamma e_\gamma + \sum_{i=1}^{m} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1) \dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_m,$$

and thus

$$\begin{aligned} &C^{\Gamma_3}_\Delta(y) \\ &= \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_2}} y_\gamma e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1) \dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_n \\ &= C^{\Gamma_2}(y), \end{aligned}$$

because the function $\phi(t_1, \dots, t_m) = C_\Delta\Big(\sum\limits_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_3}} y_\gamma e_\gamma + \sum\limits_{i=1}^{m} t_i e_{\gamma_i}\Big)$ is for any given $t_1, \dots, t_n$ constant in variables $t_{n+1}, \dots, t_m$ satisfying $|t_{n+1} - y_{\gamma_{n+1}}| \leq \frac{\delta}{4}, \dots$, $|t_m - y_{\gamma_m}| \leq \frac{\delta}{4}$. The function $\tilde{C}_\Delta(\cdot)$ restricted to $B_{\|\cdot\|_\infty}(x, \frac{\delta}{4})$ thus depends only on the coordinates $\{y_{\gamma_1}, \dots, y_{\gamma_n}\}$ of y and is easily observed to be C^∞-Fréchet smooth. The trivial estimate gives us

$$\begin{aligned} \|x\|^2 \leq C_\Delta(x) \leq \tilde{C}_\Delta(x) &\leq \sup\{C_\Delta(x+v), \|v\|_\infty < \frac{\delta}{2}\} \\ &\leq \sup\{\big(\|x+v\| + \varepsilon\big)^2, \; \|v\|_\infty < \frac{\delta}{2}\}. \end{aligned}$$

By the standard argument of choosing ε and δ small enough, we obtain, via the implicit function theorem, that the C^∞-Fréchet smooth norm defined as the Minkowski functional of the set $\{x,\ \tilde{C}_\Delta(x) \le 1\}$ approximates arbitrary well (on bounded sets) the original norm $\|\cdot\|$.

We say that a norm $|||\cdot|||$ defined on a $C(K)$ space depends locally on finitely many coordinates if for every $f \in C(K)$ there exist a finite set $\{k_1, \dots, k_n\} \subset K, \varepsilon > 0$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$|||g||| = \phi\big(g(k_1), \dots, g(k_n)\big),$$

whenever $\|g - f\| < \varepsilon$. □

Proposition 2. *There exists no lattice and Gâteaux differentiable* (*not necessarily equivalent*) *norm* $C_0([0, \omega_1])$. *There exists no lattice* (*not necessarily equivalent*) *norm on* $C_0([0, \omega_1])$ *that depends locally on finitely many coordinates.*

Proof: Assume that $\|\cdot\|$ is a given norm on $C_0([0, \omega_1])$. Let us first define, for a given non-limit ordinal $\alpha < \omega_1$, φ_α on $[\alpha, \omega_1)$ by

$$\begin{aligned} &\varphi_\alpha(\beta) = \|\chi_{[\alpha,\beta]}\| \ \text{ for } \beta \text{ a nonlimit ordinal,} \\ &\varphi_\alpha(\beta) = \sup\{\varphi_\alpha(\gamma),\ \gamma < \beta,\ \gamma \text{ nonlimit}\} \text{ for } \beta \text{ a limit ordinal.} \end{aligned}$$

The function φ_α is well defined since $\chi_{[\alpha,\beta]} \in C_0[0, \omega_1]$ whenever α, β are nonlimit ordinals. By the lattice condition on $\|\cdot\|$, φ_α is a nondecreasing function defined on $[0, \omega_1)$. Thus for some nonlimit $\beta_\alpha > \alpha$ we have

$$\varphi_\alpha(\beta_\alpha) = \varphi_\alpha(\gamma) \text{ for every } \gamma \in [\beta_\alpha, \omega_1].$$

Similarly, by the lattice assumption, whenever $\alpha_1 < \alpha_2$ are nonlimit ordinals, $\varphi_{\alpha_1}(\beta_{\alpha_1}) \le \varphi_{\alpha_2}(\beta_{\alpha_2})$. Therefore, there exists $\alpha_0 \in \omega_1$ such that

$$\varphi_{\alpha_0}(\beta_{\alpha_0}) \ge \varphi_\alpha(\beta) \text{ whenever } \beta \ge \alpha \ge \alpha_0.$$

Let us define, by induction, a sequence $\{\alpha_i\}_{i=0}^\infty$ as follows: α_0 comes from the above consideration, $\alpha_{i+1} = \beta_{\alpha_i} + 1$.

Choose a closed and open countable interval $[\alpha_0, \beta] \subset [0, \omega_1)$ such that $\beta \ge \alpha_i$ for every $i \in \mathbb{N}$. Clearly, $\chi_{[\alpha_0,\beta]} \in C_0([0, \omega_1])$ and

$$0 < \|\chi_{[\alpha_0,\beta]}\| = \|\chi_{[\alpha_i,\beta_{\alpha_i}]}\| \ \text{ for every } i \in \mathbb{N}.$$

Also,

$$\|\chi_{[\alpha_0,\beta]} + t\,\chi_{[\alpha_i,\beta_{\alpha_i}]}\| \ge \|(1+t)\,\chi_{[\alpha_i,\beta_{\alpha_i}]}\| = (1+t)\|\chi_{[\alpha_0,\beta]}\| \ \text{ for every } t \ge 0.$$

Thus, the directional derivative of $\|\cdot\|$ at $\chi_{[\alpha_0,\beta]}$ in direction of $v_i = \chi_{[\alpha_i,\beta_{\alpha_i}]}$ satisfies:

$$\frac{\partial\|\chi_{[\alpha_0,\beta]}\|}{\partial v_i} \geq \frac{\partial\|\chi_{[\alpha_i,\beta_{\alpha_i}]}\|}{\partial v_i} \geq \|\chi_{[\alpha_i,\beta_{\alpha_i}]}\| = \|\chi_{[\alpha_0,\beta]}\|.$$

However, assuming the existence of the Gâteaux derivative $\|\chi_{[\alpha_0,\beta]}\|'$, we estimate

$$\Big\|\|\chi_{[\alpha_0,\beta]}\|'\Big\|_1 \geq \frac{\langle\|\chi_{[\alpha_0,\beta]}\|', \sum\limits_{i=0}^{n} v_i\rangle}{\sum\limits_{i=0}^{n} v_i} = \frac{\sum\limits_{i=0}^{n} \frac{\partial\|\chi_{[\alpha_0,\beta]}\|}{\partial v_i}}{\|\chi_{[\alpha_0,\beta]}\|} \geq n$$

for all $n \in \mathbb{N}$. ($\|\sum\limits_{i=0}^{n} v_i\| = \|\chi_{[\alpha_0,\beta]}\|$ by the lattice property of $\|\cdot\|$.) This is a contradiction. $\square$

This proves the first half of Proposition 2. The proof for the second part requires only minor modifications.

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