Continuity of the uniform rotundity modulus relative to linear subspaces

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Abstract. We prove the continuity of the rotundity modulus relative to linear subspaces of normed spaces. As a consequence we reduce the study of uniform rotundity relative to linear subspaces to the study of the same property relative to closed linear subspaces of Banach spaces.

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The notion of uniform rotundity in a normed space relies on the geometric condition that the mid-point of a variable chord of the unit sphere of the space cannot approach the sphere unless the length of the chord goes to zero.

This paper deals with a weaker type of rotundity, introduced by H. Fakhouri [2], called uniform rotundity relative to a linear subspace. Geometrically this differs from uniform rotundity in that it requires that the direction of the variable chord belongs to the subspace. Specifically, the normed linear space X is said to be uniformly rotund relative to its linear subspace Y if the *uniform rotundity modulus relative to* Y

$$\delta(Y,\epsilon)=\inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in B, x-y\in Y, \|x-y\|\geq\epsilon\right\} \tag{1}$$

is strictly positive when $0<\epsilon\leq 2$, where B denotes the closed unit ball of X. If $Y=X$, then $\delta(X,\epsilon)$ is Clarkson's uniform rotundity modulus ([1]). When $Y=\langle z\rangle$, the one-dimensional linear subspace spanned by $z\neq 0$, $\delta(Y,\epsilon)=\delta(\to z,\epsilon)$ is Garkavi's uniform directional rotundity modulus ([5]).

Let S be the unit sphere of X and $S_Y=S\cap Y$. Also let

$$\mathcal{S}=\{S_Y : Y \text{ is a linear subspace of } X\}$$

and h be the Hausdorff semi-metric on $\mathcal{S}$:

$$h(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}. \tag{2}$$

We note in Lemma 1 that the uniform rotundity modulus relative to Y is uniquely determined by the elements in S. Theorem 2 proves that $\delta(Y,\epsilon)$ is a continuous function on the product space $\mathcal{S} \times [0,2)$. This result completes the one in [2], where it is proved that for a fixed $0 \leq \epsilon < 2$, $\delta(Y,\epsilon)$ is an upper semicontinuous function on $\mathcal{S}$, provided that this function is strictly positive. For one-dimensional subspaces we recover the continuity of the directional uniform rotundity modulus, a result obtained in [7]. Another consequence is that X is uniformly rotund relative to Y if and only if the completion of X is uniformly rotund relative to the adherence of Y.

Continuity

The relative uniform rotundity modulus admits various equivalent definitions which enables us to pick the most convenient for each occasion. They are collected in the following lemma whose proof is essentially contained in [3, Lemma 1].

Lemma 1. *Let Y be a non-null linear subspace of X and let $0 \leq \epsilon \leq 2$. Then*

(i) $\delta(Y,\epsilon) = \inf\{1 - \|x + (\epsilon/2)z\| : x \in B, x + \epsilon z \in S, z \in S_Y\}$.

(ii) *If* $\dim X \geq 2$, *then*

$$\begin{aligned}\delta(Y,\epsilon) &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S, x - y \in Y, \|x - y\| \geq \epsilon\right\}\\ &= \inf\{1 - \|x + (\epsilon/2)z\| : x \in S, x + \epsilon z \in S, z \in S_Y\}.\end{aligned}$$

It is easy to verify now that Lemma 1.e.8 in [8, p. 66] also proves that $\delta(Y,\epsilon)/\epsilon$ is an increasing monotonic function on $0 < \epsilon \leq 2$ and that [6, p. 54] or [9, p. 23] show that

$$\delta(Y,\epsilon_1) - \delta(Y,\epsilon_2) \leq (\epsilon_1 - \epsilon_2)/(2 - \epsilon_2), \quad 0 \leq \epsilon_2 < \epsilon_1 \leq 2. \tag{3}$$

Thus, we obtain that $\delta(Y,\epsilon) \leq \epsilon/2$ for $0 < \epsilon \leq 2$ and that $\delta(Y,\cdot)$ is a continuous function on $0 \leq \epsilon < 2$. However, as B. Turett's following example shows, this function is not necessarily continuous at $\epsilon = 2$.

Example 1 (B. Turett)**.** Let X be the linear space of bounded real sequences endowed with the norm

$$\|x\| = \left((1/2)\|x\|_\infty^2 + \sum_{i=1}^{\infty} |x_i|^2/2^i\right)^{1/2},$$

where $x = (x_i)$ and $\|x\|_\infty = \sup|x_i|$.

Let $z = (1, 0, \dots)$ and $z^n = (\underbrace{0, \dots, 1}_{n}, 0, \dots)$. Define

$$x^n = \sqrt{2^n/(2^n+1)}(-z + z^n), \quad y^n = \sqrt{2^n/(2^n+1)}(z + z^n).$$

Then $\|x^n\| = \|y^n\| = 1$, $\lim_{n\to\infty} \|x^n - y^n\| = 2$ and $\|(x^n + y^n)/2\| > 1/\sqrt{2}$. Therefore

$$\lim_{\epsilon\to 2} \delta(\to z, \epsilon) \leq 1 - (1/\sqrt{2}) < 1 = \delta(\to z, 2),$$

where the last equality is due to X being rotund.

Theorem 2. *The function $\delta\colon \mathcal{S}\times[0,2)\to\mathbb{R}_+$ is continuous.*

Proof: Fix $0<\epsilon<2$. First we prove that $\delta(\cdot,\epsilon)$ is continuous in $\mathcal{S}$. Let Y and Y' be linear subspaces of X. We claim that if $h(S_Y,S_{Y'})\le\eta$ then

$$\delta(Y,\epsilon/(1+2\eta))\le\delta(Y',\epsilon)+(2+\epsilon/2)\eta. \tag{4}$$

Indeed, let $x\in B$, $z'\in S_{Y'}$ be such that $x+\epsilon z'\in S$. Then there exists $z\in S_Y$ with $\|z-z'\|\le\eta$ and we have $x/(1+2\eta)$, $(x+\epsilon z)/(1+2\eta)\in B$. Therefore

$$\begin{aligned}\frac{(\|x+(\epsilon/2)z'\|-(\epsilon/2)\|z'-z\|)}{1+2\eta}&\le\frac{(\|x+(\epsilon/2)z'-(\epsilon/2)z'+(\epsilon/2)z\|)}{1+2\eta}\\&\le\left\|\frac{x+(\epsilon/2)z}{1+2\eta}\right\|\le 1-\delta(Y,\epsilon/(1+2\eta)).\end{aligned}$$

Since $\|z-z'\|\le\eta$, it follows that

$$\begin{aligned}\|x+(\epsilon/2)z'\|&\le\big(1-\delta\big(Y,\epsilon/(1+2\eta)\big)\big)(1+2\eta)+(\epsilon/2)\eta\\&\le 1-\delta(Y,\epsilon/(1+2\eta))+(2+(\epsilon/2))\eta.\end{aligned}$$

Therefore

$$\delta(Y,\epsilon/(1+2\eta))-(2+(\epsilon/2))\eta\le 1-\|x+(\epsilon/2)z'\|.$$

Taking the infimum over $z'\in S_{Y'}$, we have (4).

Using (3) we obtain that for every $\mu>0$ there exists $\eta>0$ such that

$$\epsilon(1+2\eta)<2,\quad\left(2+\frac{\epsilon}{2}(1+2\eta)\right)\eta<\frac{\mu}{2}, \tag{5}$$

and

$$\delta(Y,\epsilon(1+2\eta))-\delta\left(Y,\frac{\epsilon}{1+2\eta}\right)\le\frac{\epsilon\left((1+2\eta)-\dfrac{1}{1+2\eta}\right)}{2-\dfrac{\epsilon}{1+2\eta}}<\frac{\mu}{2}. \tag{6}$$

Interchanging Y and Y', formula (4) at $\epsilon(1+2\eta)$ gives

$$\delta(Y',\epsilon)\le\delta\big(Y,\epsilon(1+2\eta)\big)+\left(2+\frac{\epsilon}{2}(1+2\eta)\right)\eta. \tag{7}$$

From (4), (5), (6), and (7) we have

$$\begin{aligned}
-\mu = -\frac{\mu}{2} - \frac{\mu}{2} &\le \delta\left(Y, \frac{\epsilon}{1+2\eta}\right) - \delta\big(Y, \epsilon(1+2\eta)\big) - \left(2 + \frac{\epsilon}{2}(1+2\eta)\right)\eta \\
&\le \delta\left(Y, \frac{\epsilon}{1+2\eta}\right) - \delta(Y', \epsilon) \\
&\le \delta(Y, \epsilon) - \delta(Y', \epsilon) \\
&\le \delta(Y, \epsilon) - \delta\left(Y, \frac{\epsilon}{1+2\eta}\right) + \left(2 + \frac{\epsilon}{2}\right)\eta \\
&\le \delta(Y, \epsilon) - \delta\left(Y, \frac{\epsilon}{1+2\eta}\right) + \left(2 + \frac{\epsilon}{2}(1+2\eta)\right)\eta \\
&\le \delta(Y, \epsilon(1+2\eta)) - \delta\left(Y, \frac{\epsilon}{1+2\eta}\right) + \left(2 + \frac{\epsilon}{2}(1+2\eta)\right)\eta \\
&\le \frac{\mu}{2} + \frac{\mu}{2} = \mu.
\end{aligned}$$

Therefore $\delta(\cdot, \epsilon)$ is continuous. To complete the proof note that

$$\begin{aligned}
|\delta(Y, \epsilon) - \delta(Y', \epsilon')| &\le |\delta(Y, \epsilon) - \delta(Y', \epsilon)| + |\delta(Y', \epsilon) - \delta(Y', \epsilon')| \\
&\le |\delta(Y, \epsilon) - \delta(Y', \epsilon)| + \frac{|\epsilon - \epsilon'|}{2 - \min(\epsilon, \epsilon')},
\end{aligned}$$

where the last inequality is a consequence of (3). □

The following example shows that the relative uniform rotundity modulus at $\epsilon = 2$, $\delta(\cdot, 2)$ may fail to be a continuous function.

We shall henceforth use $\delta_X(Y, \epsilon)$ instead of $\delta(Y, \epsilon)$ in order to emphasize in the subscript the space in which the modulus is defined.

Example 2. Let X_i be the linear space $\mathbb{R}^2$ endowed with the norm $\|(r,s)\|_i = (|r|^i + |s|^i)^{1/i}$, $i = 2, 3, \ldots$, and let $\ell_\infty(X_i)$ be the space of sequences (x_i) such that $x_i \in X_i$ and $(\|x_i\|_i)$ is bounded. This space is normed by $\|(x_i)\| = \sup_i(\|x_i\|_i)$. In [4] it is shown that

$$\delta_{\ell_\infty(X_i)}(\to z, \epsilon) = \inf_i \{\delta_{X_i}(\to z_i, \epsilon \|z_i\|_i)\}, \qquad 0 \le \epsilon \le 2\,.$$

Let

$$z = ((1,0), (1,0), \ldots) \text{ and } z^n = \left((1,0), \left(\frac{n}{n+1}, 0\right), \left(\frac{n}{n+1}, 0\right), \ldots\right).$$

Then $\lim_{n\to\infty} z^n = z$ in $\ell_\infty(X_i)$, $\delta_{\ell_\infty(X_i)}(\to z, 2) = 1$, and $\delta_{\ell_\infty(X_i)}(\to z^n, 2) = 0$ for every $n \in \mathbb{N}$.

Corollary 3. *Let $\widetilde{X}$ be the completion of X and $\overline{Y}$ the adherence of Y. Then X is uniformly rotund relative to Y if and only if $\widetilde{X}$ is uniformly rotund relative to $\overline{Y}$.*

PROOF: We prove that

$$\delta_X(Y,\epsilon) = \delta_X(\overline{Y},\epsilon) = \delta_{\widetilde{X}}(\overline{Y},\epsilon), \quad 0 \le \epsilon < 2\,.$$

The first equality is a direct consequence of Theorem 2. So we only must show that $\delta_X(\overline{Y},\epsilon) \le \delta_{\widetilde{X}}(\overline{Y},\epsilon)$, for every $0 \le \epsilon < 2$. Let $x \in B_{\widetilde{X}}$ and $z \in S_{\overline{Y}}$ be such that $x + \epsilon z \in S_{\widetilde{X}}$, and let $\{x_n\} \subset B_X$ be a sequence convergent to x. If $\gamma_n = \max(1, \|x_n + \epsilon z\|)$, then $x_n/\gamma_n, (x_n + \epsilon z)/\gamma_n \in B_X$, and

$$\left\| \frac{x_n}{\gamma_n} + \frac{\epsilon z}{2\gamma_n} \right\| \le 1 - \delta_X(\overline{Y}, \epsilon/\gamma_n)\,.$$

Thus the continuity of $\delta(\overline{Y},\cdot)$ at ϵ yields $\|x + (\epsilon/2)z\| \le 1 - \delta_X(\overline{Y},\epsilon)$. □

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