Binormality of Banach spaces

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Abstract. We study binormality, a separation property of spaces endowed with two topologies known in the real analysis as the Luzin-Menchoff property. The main object of our interest are Banach spaces with their norm and weak topologies. We show that every separable Banach space is binormal and the space ℓ^∞ is not binormal.

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Luzin-Menchoff property of the pair of the Euclidean and the density topologies, or of the Euclidean topology and the fine topology of the potential theory, respectively, is a useful tool in the real analysis (see e.g. [LMZ]). Several years ago, L. Zajíček posed the question, whether the pairs of the norm and weak topologies in Banach spaces have the corresponding "Luzin-Menchoff property" called binormality in [LMZ] and introduced as pairwise normality in [K]. He also pointed out that the situation in Banach spaces is somewhat opposite to that of real analysis because the finer topology in Banach spaces is the metrizable one. We show that all separable Banach spaces are binormal but the space ℓ^∞ is not binormal. Up to now, we are not able to decide what is the answer for many other nonseparable Banach spaces, e.g. for nonseparable Hilbert spaces.

Let us begin with the definition of binormality which is a property of a space endowed with two topologies ("bitopological space") related naturally to the normality. Thus it is perhaps natural to call such a space binormal.

Definition. Let X be a nonempty set and σ, τ be two topologies on X. We say that (X, σ, τ) is *binormal*, if for every disjoint pair S, T of subsets of X such that S is closed in σ and T is closed in τ, there is a disjoint pair of sets V and U such that $S \subset V$, $T \subset U$, V is open in τ, and U is open in σ.

We say that a *Banach space is binormal* if it is binormal with respect to its norm and weak topologies.

By a slight modification of the standard proof that regular topological spaces with the Lindelöf property are normal, we show that every separable Banach space is binormal with respect to the norm and weak topologies.

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Theorem 1. *Let (X,τ) be a locally convex space and let σ be its weak topology. Let (X,τ) be Lindelöf. Then X is binormal with respect to σ and τ.*

Moreover, if (X,τ) is a locally convex space and σ the corresponding weak topology, S and T are as in Definition, S is Lindelöf with respect to τ and T is Lindelöf with respect to σ, then S and T can be separated by V and U as in Definition.

Proof: First we notice that it is enough to prove the second part of the assertion. If S is an σ-closed set, then it is also τ-closed and so (S,τ) is Lindelöf. If T is a τ-closed set, then (T,τ) is Lindelöf. Obviously, the weaker topology σ is also Lindelöf on T.

We suppose that S is σ-closed and T is τ-closed with $S\cap T=\emptyset$. Notice that every locally convex topology is completely regular. We are going to use the regularity of both, the weak topology σ and the topology τ.

By regularity of σ we find, for every $t\in T$ a weakly open U_t such that $t\in U_t$ and the weak closure $\overline{U_t}^{\sigma}$ of U_t does not intersect S.

Similarly, since τ is regular and locally convex, we find, for every $s\in S$, a τ-open convex set V_s such that $s\in V_s$ and the τ-closure $\overline{V_s}^{\tau}$ does not intersect T.

Now, we use the property of Lindelöf of (S,τ) and (T,σ). We choose $\{V_m \mid m\in\mathbb{N}\}\subset\{V_s \mid s\in S\}$ such that $S\subset \bigcup_{m\in\mathbb{N}}\{V_m \mid m\in\mathbb{N}\}$. Similarly, we choose $\{U_n \mid n\in\mathbb{N}\}\subset\{U_t \mid t\in T\}$ such that $T\subset\{U_n \mid n\in\mathbb{N}\}$.

Put

$$V_m^* = V_m \setminus \bigcup\{\overline{U_n}^{\tau} \mid n\le m\} \text{ and}$$

$$U_n^* = U_n \setminus \bigcup\{\overline{V_m}^{\sigma} \mid m\le n\}.$$

We put $V=\bigcup_{m\in\mathbb{N}} V_m^*$ and $U=\bigcup_{n\in\mathbb{N}} U_n^*$. Since τ is finer than σ, $\overline{U_n}^{\tau}\subset\overline{U_n}^{\sigma}$, and so $S\subset V$. Since V_m's are convex, we have $\overline{V_m}^{\sigma}=\overline{V_m}^{\tau}$ and thus $T\subset U$. The construction ensures that $U\cap V=\emptyset$, U is σ-open and V is τ-open. □

To show that ℓ^∞ is not binormal, we use the following reformulation of Lemma 3.2 from [JNR].

Lemma. *Let B be the unit ball of ℓ^∞ and $B=\bigcup_{n=1}^{\infty} E_n$. Then there is an $n_0\in\mathbb{N}$ such that, for some $M\subset\mathbb{N}$ such that the set $\mathbb{N}\setminus M$ is infinite, and for some $x\in B$, the set $E_{n_0}\cap\{y \mid y\restriction M = x\restriction M\}$ is weakly dense in $B\cap\{y \mid y\restriction M=x\restriction M\}$.*

Proof: By Lemma 3.2 of [JNR] there is some $n_0\in\mathbb{N}$ such that E_{n_0} is not "nowhere dense on coordinate sets" which means that there is a set $M\subset\mathbb{N}$ such that $\mathbb{N}\setminus M$ is infinite and an $x\in B$ such that for every nonempty relatively weakly open subset U of $S=\{y\in B \mid y\restriction M = x\restriction M\}$ the set $E_{n_0}\cap U$ is nonempty. However, this means that $E_{n_0}\cap S$ is weakly dense in S which is the claim of Lemma. □

Theorem 2. *The space ℓ^∞ is not binormal.*

Proof: We choose in the unit ball B of ℓ^∞ elements $y_\alpha^{(n)}$, $\alpha < \mathfrak{c}$, $n \in \mathbb{N}$, in the following way. First, we find some well ordering of the set $R = \{x : A \subset \mathbb{N} \to \mathbb{R} \mid |\mathbb{N} \setminus A| = \infty, x \in B_{\ell^\infty(A)}\}$, where $B_{\ell^\infty(A)}$ is the unit ball of $\ell^\infty(A)$, say $R = \{x_\alpha : A_\alpha \to \mathbb{R} \mid \alpha < \mathfrak{c}\}$.

Let $y_\beta^{(n)} \in B$, $\beta < \alpha$ and $n \in \mathbb{N}$, for some $\alpha < \mathfrak{c}$ be already chosen. We find $y_\alpha^{(n)} \in B$ such that $y_\alpha^{(n)} \restriction A_\alpha = x_\alpha$, $y_\alpha^{(n)} \restriction (\mathbb{N} \setminus A_\alpha) \in \{0,1\}^{\mathbb{N} \setminus A_\alpha}$ and $y_\alpha^{(n)} \restriction (\mathbb{N} \setminus A_\alpha) \neq y_\beta^{(m)} \restriction (\mathbb{N} \setminus A_\alpha)$ for $\beta < \alpha$, $m \in \mathbb{N}$, and $y_\alpha^{(n)} \neq y_\alpha^{(m)}$ for $m \neq n$. Hence we have $\|y_\alpha^{(n)} - y_\beta^{(m)}\| \geq 1$ for $(\alpha, n) \neq (\beta, m)$ in $\mathfrak{c} \times \mathbb{N}$.

Now, we put $f(y_\alpha^{(n)}) = \frac{1}{n}$ and $f : \ell^\infty \to (0, \infty)$ be some continuous extension. Let $S = \{(y, 0) \mid y \in \ell^\infty\}$ and T be the graph of f. We suppose that S and T can be separated in $X = \ell^\infty \times \mathbb{R}$ by some V and U as in Definition, i.e. $\overline{V}^{\text{weak}} \cap T = \emptyset$.

Put

$$E_n = \{y \in B \mid (\exists \text{ open } W_y)\ y \in W_y \text{ and } W_y \times (-\frac{1}{n}, \frac{1}{n}) \subset V\}.$$

We see that $B = \bigcup_{n \in \mathbb{N}} E_n$. Put $S_\alpha = \{y \in B \mid y \restriction A_\alpha = x_\alpha\}$. Using Lemma, we get that there is an $\alpha < \mathfrak{c}$ and an $n_0 \in \mathbb{N}$ such that $S_\alpha \cap B = \overline{S_\alpha \cap B \cap E_{n_0}}^{\text{weak}}$. Thus $y_\alpha^{(n_0)} \in \overline{E_{n_0}}^{\text{weak}}$; therefore $(y_\alpha^{(n_0)}, \frac{1}{n_0})$ is in $\overline{V}^{\text{weak}}$. But $f(y_\alpha^{(n_0)}) = \frac{1}{n_0}$ and so $(y_\alpha^{(n_0)}, f(y_\alpha^{(n_0)})) \in \overline{V}^{\text{weak}} \cap T$ which is a contradiction.

We found the pair S, T which cannot be separated in $X = \ell^\infty \times \mathbb{R}$, and since X is isomorphic to ℓ^∞, we have proved that ℓ^∞ is not binormal. $\square$

Remark. As in the case of the Luzin-Menchoff property (cf. [LMZ]), we might derive a Urysohn-type lemma. Namely, in a binormal space X one can separate the sets S, T from Definition by a real function $f : X \to [0,1]$ such that, e.g., $f(S) \subset [0, \frac{1}{3}]$, $f(T) \subset [\frac{2}{3}, 1]$, and such that f is σ-lower semi-continuous and τ-upper semi-continuous (see [K]). Notice that we get the existence of a separating function f which is τ-continuous if τ is finer than σ. However, this is not interesting if the finer topology τ is a normal one which is the case of the pair of a weak and norm topologies on Banach spaces. On the other hand, this was a very important conclusion for the Luzin-Menchoff property because both, the density topology and the fine topology of the potential theory in $\mathbb{R}^n$, $n > 1$, are finer than the Euclidean one, but they are not normal.

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