Periodic solutions for nonlinear Volterra integrodifferential equations in Banach spaces

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Abstract. In this paper we examine periodic integrodifferential equations in Banach spaces. When the cone is regular, we prove two existence theorems for the extremal solutions in the order interval determined by an upper and a lower solution. Both theorems use only the order structure of the problem and no compactness condition is assumed. In the last section we ask the cone to be only normal but we impose a compactness condition using the ball measure of noncompactness. We obtain the extremal solutions for both the Cauchy and periodic problems in a constructive way, using a monotone iterative technique.

Keywords: extremal solutions, monotone map, regular cone, normal cone, quasi-monotone map, reproducing cone, dual cone, differential inequality, monotone iterative technique

Classification: 45J05

1. Introduction

In [5] Chen-Zhuang studied integrodifferential equations of the Volterra type in a Banach space. They established the existence of solutions for the initial value problem using a continuous vector field (continuous in all three variables), a compactness condition involving the Hausdorff measure of noncompactness and the method of upper and lower solutions. In this paper we extend the work of Chen-Zhuang [5] in several ways. By strengthening the condition on the cone of the underlying Banach space (from normal to regular), we are able to drop the compactness condition and allow the vector field to have discontinuities in all variables, in sharp contrast to the situation assumed in [5]. It seems to us, that our setting is more natural, since after all one of the goals of the method of upper and lower solutions, is to exploit the monotonicity structure of the problem in order to relax the restrictive and often difficult to verify compactness conditions, stated in terms of some measure of noncompactness. Of course our approach requires an order structure on the underlying Banach space, which is also assumed in Chen-Zhuang [5]. When the cone of the space has a nonempty interior, we are able to weaken further our hypotheses on the vector field and assume only quasi-monotonicity in the x-variable of the function $f(t, x, y)$. Finally at the end of the paper, we return to the case of the normal cone (considered by Chen-Zhuang [5]) and using a compactness condition, which is less restrictive than the one employed in [5] and a monotone iterative procedure, we are able to establish in a constructive way the existence of extremal solutions and periodic solutions in the order interval determined by an upper and a lower solution. At the same time, our work extends the earlier results of Lakshmikantham [12], where $X = R$ and the vector field is continuous in all variables and satisfies an one sided Lipschitz condition. In addition we extend corresponding results for ordinary differential equations in R^n of Lakshmikantham-Leela [14]. Finally we should point out that multivalued integrodifferential equations were recently considered by Papageorgiou [16] and Avgerinos-Papageorgiou [2].

2. Preliminaries

Let X be a Banach space. By a "cone" on X we understand a closed, convex set $K \subseteq X$ such that $\lambda K \subseteq K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. To avoid trivialities we will always assume that $K \neq \{\emptyset\}$. Given a cone K, we define a partial ordering $\leq$ with respect to K, by $x \leq y$ iff $y - x \in K$. A cone K is said to be "normal" if there exists $M > 0$ such that for all $0 \leq x \leq y$ we have $\|x\| \leq M\|y\|$ (i.e. the norm of X is semimonotone). A cone K is said to be regular if every increasing and order bounded sequence has a limit; i.e. if $\{x_n\}_{n\geq 1} \subseteq X$ and $y \in X$ satisfy $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. A regular cone K is always normal, but the converse is not in general true. For example let $X = C[0,1]$. Then the usual positive cone $K = \{x \in C[0,1] : x(t) \geq 0 \text{ for all } t \in [0,1]\}$ has nonempty interior, is normal but clearly is not regular. The dual cone $K^* \subseteq X^*$ is defined by $K^* = \{x^* \in X^* : (x, x^*) \geq 0 \text{ for all } x \in K\}$, where by $(.,.)$ we denote the duality brackets for the pair (X, X^*). Note that we call K^* a cone although the condition $K^* \cap (-K^*) = \{0\}$ may not be satisfied. However if $X = K - K$ (i.e. K is generating), then $K^* \cap (-K^*) = \{0\}$. For details we refer to the book of Guo-Lakshmikantham [8].

Let $T = [0, b]$ and X a Banach space partially ordered by a normal cone K. We consider the following Volterra integrodifferential periodic problem defined on $T \times X$:

$$\left\{ \begin{array}{c} x'(t) = f(t, x(t), V(x)(t)) \text{ a.e. on } T \\ x(0) = x(b). \end{array} \right\} \tag{1}$$

Here $V(.)$ is the usual Volterra integral operator defined by $V(x)(t) = \int_0^t G(t,s)x(s)\,ds$ for all $t \in T$. In what follows by $AC^{1,1}(T, X)$ we will denote the space of all absolutely continuous functions $x : T \to X$, whose derivative exists a.e. and is an $L^1(T, X)$ function. Recall (see Barbu [4, Theorem 2.2, p. 19]) that $AC^{1,1}(T, X) = W^{1,1}(T, X)$ and if X has the Radon-Nikodym property (RNP), then every absolutely continuous function is differentiable a.e. and $x(t) = x(t_0) + \int_{t_0}^t x'(s)\,ds$ for all $0 \leq t_0 \leq t \leq b$ (see Diestel-Uhl [7, p. 217]).

Definition 2.1. *A function $\varphi \in AC^{1,1}(T,X)$ is said to be an "upper solution" of* (1) *if*

$$\varphi'(t) \geq f(t, \varphi(t), V(\varphi)(t)) \text{ a.e. on } T, \ \varphi(0) \geq \varphi(b).$$

A function $\psi \in AC^{1,1}(T,X)$ is said to be a "lower solution" of (1) *if*

$$\psi'(t) \leq f(t, \psi(t), V(\psi)(t)) \text{ a.e. on } T, \ \psi(0) \leq \psi(b).$$

Now let us introduce our hypotheses on the data of (1).

$\mathbf{H_0}$: There exists an upper solution $\varphi(.)$ and a lower solution $\psi(.)$ of (1) such that $\psi(t) \leq \varphi(t)$ for all $t \in T$.

$\mathbf{H(f)_1}$: $f : T \times X \times X \to X$ is a function s.t.

(i) for every $x, y \in C(T,X)$ $t \to f(t, x(t), y(t))$ is strongly measurable;

(ii) there exist $M, N \in L^1(T)$ such that for almost all $t \in T$ $(x,y) \to f(t,x,y)+ M(t)x + N(t)y$ is nondecreasing on $[\psi(t), \varphi(t)]\times [V(\psi)(t), V(\varphi)(t)]$; and

(iii) $f(., \psi(.), V(\psi)(.)),\ f(., \varphi(.), V(\varphi)(.)) \in L^1(T,X)$.

Remark 2.1. *Concerning hypothesis* $\mathrm{H(f)_1(i)}$ *which is somewhat implicit, let us indicate some classical situations for which it holds. If $f(t,x,y)$ is a Caratheodory function (i.e. is strongly measurable in t and continuous in $(x,y) \in X \times X$), then* $\mathrm{H(f)_1(i)}$ *is satisfied. The validity of* $\mathrm{H(f)_1(i)}$ *is also guaranteed if X is separable and $f(t,x,y)$ is $(\mathcal{L} \times B(X) \times B(X))$ measurable, with $\mathcal{L}$ being the Lebesgue σ-field of T and $B(X)$ the Borel σ-field of X.*

H(G): $G \in C(\Delta, R_+)$ with $\Delta = \{(t,s) \in T \times T : 0 \leq s \leq t \leq b\}$.

Our existence results in Section 3, will be based on the following fixed point theorem, essentially due to Amann [1]; see also Heikkila-Lakshmikantham-Sun [10, Theorem 3.1].

Theorem 2.1. *If V is a subset of an ordered Banach space Y, $[\psi, \varphi]$ is a nonempty order interval in V (i.e. $[\psi, \varphi] = \{y \in V : \psi \leq y \leq \varphi\}$) and $L(.) : [\psi, \varphi] \to [\psi, \varphi]$ is a nondecreasing map such that for every nondecreasing sequence $\{y_n\}_{n \geq 1}$ the sequence $\{L(y_n)\}_{n\geq 1}$ converges in V, then $L(.)$ has the least fixed point x_* and the greatest fixed point x^* in $[\psi, \varphi]$.*

Remark 2.2. *The fixed points x_* and x^* are called "extremal fixed points" of L in $[\psi, \varphi]$.*

3. Existence theorems for regular cones

In this section we present two existence theorems for problem (1), under the hypothesis that the ordering cone K of the Banach space X is regular.

Theorem 3.1. *If K is regular and hypotheses* H_0, $\mathrm{H(f)}_1$ *and* H(G) *hold, then problem* (1) *has the extremal solutions* x_* *and* x^* *in the order interval* $[\psi, \varphi] = \{y \in C(T, X) : \psi(t) \leq y(t) \leq \varphi(t)$ *for all* $t \in T\}$; *i.e.* (1) *has the greatest solution* x^* *and the smallest solution* x_* *in* $[\psi, \varphi]$, *in the sense that if* $x \in AC^{1,1}(T, X)$ *is any solution of* (1) *in* $[\psi, \varphi]$, *then* $x_*(t) \leq x(t) \leq x^*(t)$ *for all* $t \in T$.

Proof: Consider the ordered Banach space $C(T, X) \times X$ with positive cone $K_1 = C(T, K) \times K$ (here $C(T, K) = \{x \in C(T, X) : x(t) \in K$ for all $t \in T\}$). In $C(T, X) \times X$ we consider the order interval $V \times V_0$, where $V = [\psi, \varphi] = \{y \in C(T, X) : \psi(t) \leq y(t) \leq \varphi(t)$ for all t$\in T\}$ and $V_0 = [\psi(0), \varphi(0)] = \{u \in X : \psi(0) \leq u \leq \varphi(0)\}$. Given $(y, y_0) \in V \times V_0$, consider the following initial value problem:

$$(2) \qquad \left\{ \begin{array}{c} x'(t) + M(t)x(t) + N(t)V(x)(t) = f(t, y(t), V(y)(t)) + \\ + M(t)y(t) + N(t)V(y)(t) \text{ a.e. on } T \\ x(0) = y_0. \end{array} \right\}$$

Problem (2) has a solution (see for example Papageorgiou [15]) and the solution is clearly unique. Denote this solution by $L(y, y_0) \in C(T, X)$.

Claim #1: $L(V \times V_0) \subseteq V$.

To this end, let $(y, y_0) \in V \times V_0$ and $x = L(y, y_0)$. We have:

$$(3) \qquad \left\{ \begin{array}{c} -x'(t) - M(t)x(t) - N(t)V(x)(t) = -f(t, y(t), V(y)(t)) - \\ -M(t)y(t) - N(t)V(y)(t) \text{ a.e. on } T \\ x(0) = y_0. \end{array} \right\}$$

Also since $\psi(.)$ is by hypothesis a lower solution, we have:

$$(4) \qquad \left\{ \begin{array}{c} \psi'(t) \leq f(t, \psi(t), V(\psi)(t)) \text{ a.e. on } T \\ \psi(0) \leq \psi(b). \end{array} \right\}$$

Adding (3) and (4) above and letting $u = \psi - x \in AC^{1,1}(T, X)$ we obtain:

$$(3.1) \qquad \left\{ \begin{array}{c} u'(t) \leq f(t, \psi(t), V(\psi)(t)) - f(t, y(t), V(y)(t)) - \\ -M(t)y(t) - N(t)V(y)(t) + M(t)x(t) + N(t)V(x)(t) \text{ a.e. on } T \\ u(0) \leq 0. \end{array} \right\}$$

hence

$$(3.2) \qquad \left\{ \begin{array}{c} u'(t) \leq f(t, \psi(t), V(\psi)(t)) + M(t)\psi(t) + N(t)V(\psi)(t) \\ -f(t, y(t), V(y)(t)) - M(t)y(t) - N(t)V(y)(t) \\ -M(t)u(t) - N(t)V(u)(t) \text{ a.e. on } T \\ u(0) \leq 0. \end{array} \right\}$$

From hypothesis $\mathrm{H(f)}_1$(ii), it follows that

$$\left\{\begin{array}{c} u'(t) \leq -M(t)u(t) - N(t)\displaystyle\int_0^t G(t,s)u(s)\,ds \text{ a.e. on T} \\ u(0) \leq 0. \end{array}\right\}$$

Given $x^* \in K^*$, let $u(x^*)(t) = (x^*, u(t))$. Then we have:

$$\left\{\begin{array}{c} u'(x^*)(t) \leq -M(t)u(x^*)(t) - N(t)\displaystyle\int_0^t G(t,s)u(x^*)(s)\,ds \text{ a.e. on T} \\ u(x^*)(0) \leq 0. \end{array}\right\}$$

Using a classical differential inequality (see for example Hale [9, Theorem 6.1. p. 31]), we deduce that $u(x^*)(t) \leq 0$ for all $t \in T$. Since $x^* \in K^*$ was arbitrary, it follows that $u(t) \leq 0$ in X (i.e. $u(t) \in -K$ for all $t \in T$). So $\psi(t) \leq x(t)$ for all $t \in T$. In a similar way, we can show that $x(t) \leq \varphi(t)$ for all $t \in T$. Therefore $L(V \times V_0) \subseteq V$ as claimed.

Claim #2: $L(.,.)$ is increasing on $V \times V_0$.

Let (y_1, y_0^1), $(y_2, y_0^2) \in V \times V_0$ and assume that $y_1 \leq y_2$ in $C(T,X)$ (for the cone $C(T,K)$) and $y_0^1 \leq y_0^2$ in K (i.e. $(y_1, y_0^1) \leq (y_2, y_0^2)$ in $C(T,X) \times X$ for the cone $K_1 = C(T,K) \times K$). Set $x_1 = L(y_1, y_0^1)$ and $x_2 = L(y_2, y_0^2)$. We have:

$$\left\{\begin{array}{c} x_1'(t) + M(t)x_1(t) + N(t)V(x_1)(t) = \\ f(t, y_1(t), V(y_1)(t)) + M(t)y_1(t) + N(t)V(y_1)(t) \text{ a.e. on } T. \end{array}\right\} \tag{5}$$

and

$$\left\{\begin{array}{c} -x_2'(t) - M(t)x_2(t) - N(t)V(x_2)(t) = \\ -f(t, y_2(t), V(y_2)(t)) - M(t)y_2(t) - N(t)V(y_2)(t) \text{ a.e. on } T. \end{array}\right\} \tag{6}$$

Adding (5) and (6) above and setting $w = x_1 - x_2$ we obtain:

$$\begin{aligned} &w'(t) + M(t)w(t) + N(t)V(w)(t) = f(t, y_1(t), V(y_1)(t)) + M(t)y_1(t) \\ &\quad + N(t)V(y_1)(t) - f(t, y_2(t), V(y_2)(t)) - M(t)y_2(t) - N(t)V(y_2)(t) \text{ a.e. on } T. \end{aligned}$$

Using hypothesis $\mathrm{H(f)}_1$(ii) (recall that $y_1 \leq y_2$ in $C(T,X)$ and $y_0^1 \leq y_0^2$ in X), we have

$$w'(t) \leq -M(t)w(t) - N(t)V(w)(t) \text{ a.e. on } T,\ w(0) \leq 0. \tag{7}$$

From (7), arguing as in Claim #1, we deduce that $w(t) \leq 0$ for all $t \in T$ and so $x_1(t) \leq x_2(t)$ for all $t \in T$. Thus $L(.,.)$ is increasing on $V \times V_0$ as claimed.

Now let $\{(y_n, y_0^n)\}_{n\geq 1} \subseteq V \times V_0$ be an increasing sequence and set $x_n = L(y_n, y_0^n)$. From Claims #1 and #2 it follows that $\{x_n\}_{n\geq 1}$ is an increasing sequence in V. Since by hypothesis K is regular, it follows that for every $t \in T$, there exists $x(t) \in [\psi(t), \varphi(t)]$ such that $x_n(t) \to x(t)$ in X as $n \to \infty$. Also because of hypothesis $\mathrm{H(f)}_1$(ii) and since $x_n \in V$, $n \geq 1$, we have for a.a. $t \in T$

$$\begin{aligned}
&x_n'(t) = f(t, y_n(t), V(y_n)(t)) + M(t)y_n(t) + N(t)V(y_n)(t)\\
&- M(t)x_n(t) - N(t)V(x_n)(t)\\
&\leq f(t, \varphi(t), V(\varphi)(t)) + M(t)\varphi(t) + N(t)V(\varphi)(t)\\
&- M(t)\psi(t) - N(t)V(\psi)(t) = h(t)
\end{aligned}$$

and

$$\begin{aligned}
x_n'(t) \geq f(t, \psi(t), V(\psi)(t)) + M(t)\psi(t) + N(t)V(\psi)(t)\\
-M(t)\varphi(t) - N(t)V(\varphi)(t) = \widehat{h}(t).
\end{aligned}$$

Therefore for every $n \geq 1$, $\widehat{h}(t) \leq x_n'(t) \leq h(t)$ a.e. on T and by virtue of hypothesis H(f)(iii) we know that $\widehat{h}, h \in L^1(T, X)$. Then $0 \leq x_n'(t) - \widehat{h}(t) \leq h(t) - \widehat{h}(t)$ a.e. on T and since K is regular, thus normal in particular, there exists $\gamma > 0$ such that

$$\|x_n'(t) - \widehat{h}(t)\| \leq \gamma \|h(t) - \widehat{h}(t)\| \text{ a.e. on } T$$

and so

$$\|x_n'(t)\| \leq \|\widehat{h}(t)\| + \gamma\|h(t) - \widehat{h}(t)\| = \xi(t) \text{ a.e. on } T$$

with $\xi \in L^1(T)$. Thus for every $n \geq 1$ and every $0 \leq s \leq t \leq b$, we have

$$\|x_n(t) - x_n(s)\| \leq \int_s^t \|x_n'(\tau)\| d\tau \leq \int_s^t \xi(\tau)\, d\tau,$$

from which it follows that $\{x_n\}_{n\geq 1} \subseteq C(T, X)$ is equicontinuous. Since we already know that $x_n(t) \to x(t)$ in X as $n \to \infty$, from the Arzela-Ascoli theorem it follows that $x_n \to x$ in $C(T, X)$ as $n \to \infty$, $x \in [\psi, \varphi] = V$. Now let $R : V \times V_0 \to V \times V_0$ be defined by $R = \widehat{e}_b \circ L$, where $\widehat{e}_b : V \to V \times V_0$ is given by $\widehat{e}_b(x) = (x, x(b))$. Using Theorem 1, we infer that R has the extremal fixed points in $V \times V_0$. Evidently these are the extremal solutions of (1) in the order interval $V = [\psi, \varphi]$. $\square$

Remark 3.1. *Note that if K is normal and the Banach space X is weakly complete, then K is regular. Indeed, let $\{y_n\}_{n\geq 1}$ be an increasing and order bounded sequence in X. Then for every $x^* \in K^*$ $\{(x^*, y_n)\}_{n\geq 1}$ is an increasing and bounded sequence in R, thus a Cauchy sequence in R. But from Krein's theorem we know that the normality of K implies that K^* is generating (i.e. $X^* = K^* - K^*$). So for all $x^* \in X^*$ $\{(x^*, y_n)\}_{n\geq 1}$ is a Cauchy sequence in*

R and so $\{y_n\}_{n\geq 1}$ is weakly Cauchy in X. Because X is by hypothesis weakly complete, we have that $y_n \xrightarrow{w} y$ in X as $n \to \infty$. By Mazur's lemma we can find $\{\lambda_n^k\}_{k=0}^{m_n} \subseteq [0,1]$, $\sum\limits_{k=0}^{m_n} \lambda_n^k = 1$ such that if $z_n = \sum\limits_{k=0}^{m_n} \lambda_n^k y_{n+k}$, then $z_n \to y$ in X as $n \to \infty$. Therefore for $i \geq n+m_n$ we have $z_n \leq y_i \leq y$ and so $0 \leq y-y_i \leq y-z_n$. Using the normality of K we can find $\gamma > 0$ such that $\|y-y_i\| \leq \gamma\|y-z_n\|$ and so we conclude that $y_n \to y$ in X as $n \to \infty$.

We can weaken further our hypotheses on $f(t,x,y)$, if we assume that $\operatorname{int} K \neq \emptyset$ (i.e. K is a solid cone). First a definition:

Definition 3.1. *A map $g : X \to X$ is said to be "quasi-monotone with respect to K" if "$x \leq y$, $x^* \in K^*$ and $(x^*, y-x) = 0$ imply $(x^*, g(x)-g(y)) \leq 0$".*

Remark 3.2. *If $X = R^N$, then Coppel* [6] *calls a quasi-monotone map "function of type K" (after Kamke who was the first to introduce and use this class of functions). Note that any scalar function is trivially quasi-monotone. A vector function $g = (g_1, g_2)$ of two variables (x,y) is quasi-monotone iff g_1 is an increasing function of x_2 and g_2 is an increasing function of x_1.*

The new hypotheses on the vector field $f(t,x,y)$ are the following:

H(f)$_2$: $f : T \times X \times X \to X$ is a function such that

- (i) for every $y \in C(T,X)$ such that $V(\psi)(t) \leq y(t) \leq V(\varphi)(t)$ for all $t \in T$, $t \to f(t,x,V(y)(t))$ is strongly measurable;
- (ii) for almost all $t \in T$ and all $x,y \in X$, $f(t,.,y)$ is quasi-monotone and increasing and $f(t,x,.)$ is increasing;
- (iii) $\|f(t,x,y) - f(t,x',y)\| \leq k(t)\|x-x'\|$ for almost all $t \in T$, all $y \in [V(\psi)(t), V(\varphi)(t)]$, with $k \in L^1(T)$; and
- (iv) $f(.,0,V(\psi)(.))$, $f(.,0,V(\varphi)(.)) \in L^1(T,X)$.

In the proof of our second existence theorem we will need the following comparison principle due to Redheffer-Walter [17, Theorem 6].

Proposition 3.2. *If $\operatorname{int} K \neq \emptyset$ and $g : T \times X \to X$ is a function such that*

- (i) *$\|g(t,x) - g(t,y)\| \leq k(t)\|x-y\|$ for almost all $t \in T$, all x,y in X with $k \in L^1(T)$; and*
- (ii) *for almost all $t \in T$ $g(t,.)$ is quasi-monotone increasing,*

then if $x,y \in AC^{1,1}(T,X)$ satisfy $x'(t) - g(t,x(t)) \leq y'(t) - g(t,y(t))$ a.e. on T and $x(0) \leq y(0)$, we have $x(t) \leq y(t)$ for all $t \in T$.

Using the above comparison principle, we can prove the following existence theorem for problem (1):

Theorem 3.3. *If K is regular, $\operatorname{int} K \neq \emptyset$ and hypotheses H_0, $\mathrm{H(f)}_2$ and $\mathrm{H(G)}$ hold, then problem* (1) *has its extremal solutions in the order interval $[\psi,\varphi]$.*

Proof: Let $V = [\psi,\varphi] = \{y \in C(T,X) : \psi(t) \leq \varphi(t)$ for all $t \in T\}$ and $V_0 = [\psi(0),\varphi(0)] = \{y \in X : \psi(0) \leq y \leq \varphi(0)\}$. Given $(y,y_0) \in V \times V_0$, consider the following initial value problem:

$$\left\{\begin{array}{c} x'(t) = f(t, x(t), V(y)(t)) \text{ a.e. on } T \\ x(0) = y_0. \end{array}\right\} \tag{8}$$

Because of hypothesis $H(f)_2$, problem (8) has a unique solution $x = L(y, y_0)$ (see for example Lakshmikantham-Leela [14]).

Claim #1: $L(V \times V_0) \subseteq V$.

Indeed let $(y, y_0) \in V \times V_0$ and let $x = L(y, y_0)$. We have

$$-x'(t) = -f(t, x(t), V(y)(t)) \text{ a.e. on } T,\ x(0) = y_0 \geq \psi(0) \tag{9}$$

and

$$\psi'(t) \leq f(t, \psi(t), V(\psi)(t)) \text{ a.e. on } T,\ \psi(0) \leq \psi(b) \tag{10}$$

Adding (9) and (10) we obtain that

$$\begin{aligned} &\psi'(t) - x'(t) \leq f(t, \psi(t), V(\psi)(t)) - f(t, x(t), V(y)(t)) \\ &= f(t, \psi(t), V(\psi)(t)) - f(t, x(t), V(\psi)(t)) \\ &+ f(t, x(t), V(\psi)(t)) - f(t, x(t), V(y)(t)) \\ &\leq f(t, \psi(t), V(\psi)(t)) - f(t, x(t), V(\psi)(t)) \text{ a.e. on } T \end{aligned} \tag{11}$$

(see hypothesis $H(f)_2$(ii)). Let $g(t, z) = f(t, z, V(\psi)(t))$. By virtue of hypothesis $H(f)_2$(i) $t \to g(t, z)$ is strongly measurable, while from hypothesis $H(f)_2$(ii) and (iii) it follows that $z \to g(t, z)$ is Lipschitz continuous and quasi-monotone nondecreasing. Moreover from (11) we have that

$$\psi'(t) - g(t, \psi(t)) \leq x'(t) - g(t, x(t)) \text{ a.e. on } T,\ \psi(0) \leq x(0).$$

Applying Proposition 3.2 we obtain that $\psi(t) \leq x(t)$ for all $t \in T$. In a similar manner, we can also show that $x(t) \leq \varphi(t)$ for all $t \in T$. Therefore $L(V \times V_0) \subseteq V$ as claimed.

Claim #2: $L(.,.)$ is increasing on $V \times V_0$.

Let (y_1, y_0^1), $(y_2, y_0^2) \in V \times V_0$ and assume that $y_1 \leq y_2$ and $y_0^1 \leq y_0^2$. Let $x_1 = L(y_1, y_0^1)$ and $x_2 = L(y_2, y_0^2)$. We have:

$$x_1'(t) = f(t, x_1(t), V(y_1)(t)) \text{ a.e. on } T,\ x_1(0) = y_0^1, \tag{12}$$

$$-x_2'(t) = -f(t, x_2(t), V(y_2)(t)) \text{ a.e. on } T,\ x_2(0) = y_0^2. \tag{13}$$

Adding (12) and (13) and after some simple algebra as before, we obtain:

$$\left\{\begin{array}{c} x_1'(t) - x_2'(t) \leq f(t, x_1(t), V(y_1)(t)) - f(t, x_1(t), V(y_2)(t)) \text{ a.e. on } T \\ x_1(0) \leq x_2(0). \end{array}\right\} \tag{14}$$

Setting $g(t, z) = f(t, z, V(y_1)(t))$ and using Proposition 3.2, from (14) we infer that $x_1(t) \leq x_2(t)$ for all $t \in T$. So $L(.,.)$ is increasing on $V \times V_0$.

The rest of the proof is almost identical with that of Theorem 3.1, with only some trivial changes. $\square$

Remark 3.3. (a) *If $f(t,x,y)$ satisfies the one sided Lipschitz condition*

$$f(t,x,y)-f(t,x',y')\geq -M(t)(x-x')-N(t)(y-y')$$

for almost all $t\in T$, all $\psi(t)\leq x'\leq x\leq \varphi(t)$, all $V(\psi)(t)\leq y'\leq y\leq V(\varphi)(t)$ and with $M,N\in L^1(T)$, then hypothesis H(f)$_1$(ii) *is satisfied. This is the situation assumed in Chen-Zhuang* [5], *where moreover M,N are taken to be positive constants. Recall that in* [5], *$f\in C(T\times X\times X,X)$.*

(b) *A careful reading of the proofs of Theorems* 3.1 *and* 3.3, *reveals that our approach can also handle the case of a nonlinear Volterra integral operator $V(x)(t)=\int_0^t G(t,s,x(s))\,ds$, provided we assume the following about $G(t,s,x)$:*

"$G\in C(\Delta\times X,X)$ and for every $(t,s)\in\Delta=\{(t,s):0\leq s\leq t\leq b\}$ $G(t,s,.)$ is nondecreasing".

This way we generalize the work of Pachpatte [15], where $X=R^N$, $f(t,x,.)$ is nondecreasing and in the x-variable satisfies the one sided Lipschitz condition $f(t,x,y)-f(t,x',y)\geq -M(t)(x-x')$ for all $\psi(t)\leq x'\leq x\leq\varphi(t)$ and with $M>0$. Pachpatte also assumes that $f\in C(T\times X\times X,X)$.

4. Monotone iterative techniques

In this section we drop the regularity hypothesis on the cone K, at the expense of introducing a compactness condition involving the Hausdorff (ball) measure of noncompactness and we develop a monotone iterative method generating the extremal solutions in $[\psi,\varphi]$ for both the Cauchy and the periodic problems.

So let $T=[0,b]$ and let X be an ordered Banach space with a normal ordering cone K. We start with the Cauchy (initial value) problem:

$$\left\{\begin{matrix} x'(t)=f(t,x(t),V(x)(t)) \text{ a.e. on } T \\ x(0)=x_0. \end{matrix}\right\} \tag{15}$$

The hypotheses on the vector field $f(t,x,y)$ are the following:

H(f)$_3$: $f:T\times X\times X\to X$ is a function such that

(i) for all $x,y\in C(T,X)$, $t\to f(t,x(t),y(t))$ is strongly measurable and $(x,y)\to f(t,x,y)$ is continuous;

(ii) there exist functions $M,N\in L^1(T)$ such that for almost all $t\in T$ the function $(x,y)\to f(t,x,y)+M(t)x+N(t)y$ is nondecreasing on $[\psi(t),\varphi(t)]\times[V(\psi)(t),V(\varphi)(t)]$;

(iii) $\beta(f(t,B_1,B_2))\leq k(t)(\beta(B_1)+\beta(B_2))$ a.e. on T for all $B_1,B_2\subseteq X$ nonempty and bounded, with $k\in L^1(T)$ (here $\beta(.)$ is the ball measure of noncompactness); and

(iv) $f(.,\psi(.),V(\psi)(.))$, $f(.,\varphi(.),V(\varphi)(.))\in L^1(T,X)$.

Remark 4.1. *Recall that for every $B \subseteq X$ nonempty and bounded, $\beta(B) = \inf[r > 0; B$ can be covered by finitely many balls of radius $r]$. For the properties of $\beta(.)$ see Banas-Goebel* [3].

Theorem 4.1. *If hypotheses* H_0, $\mathrm{H(f)}_3$ *and* H(G) *hold, then problem* (15) *has its extremal solutions in the order interval* $[\psi, \varphi]$ *and these solutions can be obtained by a monotone iterative process.*

PROOF: As in the proof of Theorem 3.1, given $y \in V = [\psi, \varphi]$, we consider the Cauchy problem:

$$\left\{ \begin{array}{c} x'(t) = f(t, y(t), V(y)(t)) - M(t)(x(t) - y(t)) \\ -N(t)(V(x-y)(t)) \text{ a.e. on } T \\ x(0) = x_0. \end{array} \right\} \tag{16}$$

Let $L_1(y)$ be the unique solution of (16). From the proof of Theorem 3.1 we know that $L_1(V) \subseteq V$ and $L_1(.)$ is nondecreasing on V. Now let $x_0 = \psi$ and define $x_n = L_1(x_{n-1})$ for $n \geq 1$. Then $\{x_n\}_{n\geq 1}$ is a nondecreasing sequence in $[\psi, \varphi]$ and from the proof of Theorem 3.1 we also know that $\{x_n\}_{n\geq 1} \subseteq C(T, X)$ is equicontinuous. So if $u(t) = \beta(\{x_n(t)\}_{n\geq 1})$, $t \in T$, then $u(.) \in C(T)$. Also, by hypothesis $\mathrm{H(f)}_3$(i) for every $n \geq 1$ $t \to f(t, x_{n-1}(t), V(x_{n-1})(t))$ is strongly measurable. So by the Pettis measurability theorem (see Diestel-Uhl [7, Theorem 2, p. 42]), we may assume in what follows, without loss of generality, that X is separable. For every $n \geq 1$ and every $t \in T$ we have:

$$\begin{aligned} x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s), V(x_{n-1})(s))\, ds \\ - \int_0^t M(s)(x_n(s) - x_{n-1}(s))\, ds - \int_0^t N(s) \int_0^t G(s,\tau)(x_n(\tau) - x_{n-1}(\tau))\, d\tau\, ds. \end{aligned}$$

Using the properties of $\beta(.)$ and Lemma 2.2 of Kisielewisz [11], we obtain:

$$\begin{aligned} u(t) \leq \int_0^t k(s)(u(s) + \xi \int_0^s u(\tau)\, d\tau)\, ds \\ + \int_0^t 2M(s)u(s)\, ds + \int_0^t 2\xi N(s) \int_0^s u(\tau)\, d\tau\, ds \end{aligned}$$

where $\xi = \sup[G(t,s) : (t,s) \in \Delta]$. Therefore

$$u(t) \leq \int_0^t (k(s) + 2M(s))u(s)\, ds + \int_0^t (\xi k(s) + 2\xi N(s)) \int_0^s u(\tau)\, d\tau\, ds, \quad t \in T.$$

Invoking Theorem 1 of Pachpatte [15] we conclude that $u(t) = 0$ for all $t \in T$. So for every $t \in T$ $\overline{\{x_n(t)\}}_{n\geq 1}$ is compact in X. Thus from the Arzela-Ascoli theorem it follows that $\{x_n\}_{n\geq 1} \subseteq C(T,X)$ is relatively compact and since we know that this sequence is monotone, we deduce that $x_n \to x_*$ in $C(T,X)$ as $n \to \infty$. Observe that by virtue of hypotheses $\mathrm{H(f)}_3$(i) and (iii) and the dominated convergence theorem, in the limit as $n \to \infty$, we have

$$x_*(t) = x_0 + \int_0^t f(s, x_*(s), V(x_*)(s))\, ds \quad \text{for all} \quad t \in T$$

and so

$$x_*'(t) = f(s, x_*(s), V(x_*)(s)), \quad x_*(0) = x_0,$$

i.e. x_* is a solution of (15). We claim that x_* is the least solution in the order interval $[\psi, \varphi]$. Indeed if $x \in [\psi, \varphi]$ is another solution of (15), then $L_1(x) = x$ and from the monotonicity of $L_1(.)$ we have that $x_1 = L_1(\psi) \leq L_1(x) = x$ and by a trivial induction $x_n \leq L_1(x) = x$ for all $n \geq 1$. Hence in the limit as $n \to \infty$ $x_* \leq x$.

Similarly if $z_0 = \varphi$ and $z_n = L_1(z_{n-1})$, $n \geq 1$, we obtain a nonincreasing sequence $\{z_n\}_{n\geq 1} \subseteq V = [\psi, \varphi]$. Arguing as above we can show that $z_n \to x^*$ in $C(T,X)$ as $n \to \infty$ and x^* is the greatest solution of (15) in $[\psi, \varphi]$. $\square$

We will prove an analogous result for the periodic problem. We will need the following stronger hypotheses:

$\mathbf{H(f)_4}$: $f : T \times X \times X \to X$ is a continuous function such that

(i) there exist $M, N > 0$ such that $(x,y) \to f(t,x,y) + Mx - Ny$ is nondecreasing on $[\psi(t), \varphi(t)] \times [V(\psi)(t), V(\varphi)(t)]$; and

(ii) $\beta(f(t, B_1, B_2)) \leq k(t)(\beta(B_1) + \beta(B_2))$ a.e. on T for all $B_1, B_2 \subseteq X$ nonempty and bounded and with $k \in L^1(T)$.

$\mathbf{H_1}$: hypothesis H_0 is satisfied with $\psi, \varphi \in C^1(T,X)$.

Remark 4.2. *Because of hypotheses* $\mathrm{H(f)}_4$*, a solution* $x(.)$ *of* (1) *belongs in* $C^1(T,X)$.

Theorem 4.2. *If hypotheses* $\mathrm{H(f)}_4$, H_1, $\mathrm{H(G)}$ *hold,* $[\psi(0), \varphi(0)]$ *is weakly compact in* X *and* $\frac{N\xi b}{M} < 1$ *where* $\xi = \sup[G(t,s) : (t,s) \in \Delta]$*, then problem* (1) *has its extremal solutions in the order interval* $[\psi, \varphi]$ *and these solutions can be obtained by a monotone iterative process.*

PROOF: Given $y \in V = [\psi, \varphi] = \{y \in C(T,X) : \psi(t) \leq y(t) \leq \varphi(t) \text{ for all } t \in T\}$ consider the following periodic problem:

$$\left\{ \begin{array}{c} x'(t) + Mx(t) - NV(x)(t) = f(t, y(t), V(y)(t)) \\ +My(t) - NV(y)(t) \text{ for all } t \in T \\ x(0) = x(b). \end{array} \right\} \tag{17}$$

From Theorem 3.1 we know that problem (17) has at least one solution $x \in C^1(T,X)$, $x \in V = [\psi, \varphi]$. We claim that this solution is unique. To see this let $x_1, x_2 \in C^1(T,X)$ be two solutions of (17) in $[\psi, \varphi]$. Then if $u = x_1 - x_2$ we have:

$$\left\{ \begin{matrix} u'(t) = -Mu(t) + NV(u)(t) \text{ for all } t \in T \\ u(0) = u(b). \end{matrix} \right\}$$

For every $x^* \in X^*$ let $u(x^*)(t) = (x^*, u(t))$. Then we have:

$$\left\{ \begin{matrix} u'(x^*)(t) = -Mu(x^*)(t) + NV(u(x^*))(t) \text{ for all } t \in T \\ u(x^*)(0) = u(x^*)(b). \end{matrix} \right\}$$

Let $x^* \in X^*$ be fixed but arbitrary. Suppose that there exists $t_0 \in T$ and $m > 0$ such that

$$u(x^*)(t_0) = m \text{ and } u(x^*)(t_0) \le m \text{ for all } t \in T.$$

If $t \in (0, b]$, then $u'(x^*)(t_0) \ge 0$ so we have:

$$\begin{aligned} 0 \le u'(x^*)(t_0) &= -Mm + N \int_0^{t_0} G(t_0, s) u(x^*)(s)\, ds \\ &\le -Mm + N\xi bm < 0, \end{aligned}$$

since by hypothesis $\frac{N\xi b}{M} < 1$. If $t_0 = 0$, then $u(x^*)(0) = u(x^*)(b) = m$ so

$$\begin{aligned} 0 \le u'(x^*)(b) &= -Mm + N \int_0^{b} G(b, s) u(x^*)(s)\, ds \\ &\le -Mm + N\xi bm < 0. \end{aligned}$$

So in both cases we have a contradiction, which of course means that $u(x^*)(t) \le 0$ for all $t \in T$. Since $x^* \in X^*$ was arbitrary, we conclude that $x_1(t) = x_2(t)$, i.e. the solution of (17) in $V = [\psi, \varphi]$ is unique.

Let $L : V \to V$ be the map which to each $y \in V = [\psi, \varphi]$ assigns the unique solution $L(y) \in V = [\psi, \varphi]$ of (17). We claim that $L(.)$ is nondecreasing on V. To this end let $y_1, y_2 \in V$, $y_1 \le y_2$ and let $x_1 = L(y_1)$, $x_2 = L(y_2)$. If $u = x_1 - x_2$, using hypothesis $\mathrm{H}(f)_4$(i), we see that

$$\left\{ \begin{matrix} u'(t) \le -Mu(t) + NV(u)(t) \text{ for all } t \in T \\ u(0) = u(b). \end{matrix} \right\}$$

If $x^* \in X^*$ and $u(x^*)(t) = (x^*, u(t))$, then we have:

$$\left\{ \begin{matrix} u'(x^*)(t) \le -Mu(x^*)(t) + NV(u(x^*))(t) \text{ for all } t \in T \\ u(x^*)(0) = u(x^*)(b). \end{matrix} \right\}$$

With the same argument that we used above to establish the uniqueness of the solution of (17), we get that $u(x^*)(t) \leq 0$ for all $t \in T$. Since $x^* \in X^*$ was arbitrary, we conclude that $u(t) \leq 0$, hence $x_1(t) \leq x_2(t)$ for all $t \in T$. This proves that $L(.)$ is nondecreasing as claimed.

Now let $x_0 = \psi$ and $x_n = L(x_{n-1})$, $n \geq 1$. Then $\{x_n\}_{n\geq 1} \subseteq C^1(T,X)$ is nondecreasing. Also note that for every $x^* \in K^*$, $\{(x^*, x_n(0))\}_{n\geq 1}$ is increasing and bounded in R, hence it is Cauchy in R. Since K^* is generating (K being normal), we infer that $\{(x^*, x_n(0))\}_{n\geq 1}$ is Cauchy in R for all $x^* \in X^*$. Because by hypothesis $V_0 = [\psi(0), \varphi(0)]$ is weakly compact in X, we deduce that $x_n(0) \xrightarrow{w} v_0 \in V_0$ as $n \to \infty$. Arguing as in the remark 3.1 via Mazur's lemma and the normality of K, we have that $x_n(0) \to v_0$ in X as $n \to \infty$. So $\beta(\{x_n(0)\}_{n\geq 1}) = 0$. Also as in the proof of Theorem 3.1, we can check that $\{x_n\}_{n\geq 1}$ is equicontinuous and so $u(t) = \beta(\{x_n(t)\}_{n\geq 1})$ belongs in $C(T,R)$. Then using Lemma 2.2 of Kisielewisz [11] and standard properties of $\beta(.)$, we obtain that

$$u(t) \leq M_1 \int_0^t u(s)\,ds + N_1 \int_0^t \int_0^s u(\tau)\,d\tau\,ds$$

for some $M_1, N_1 > 0$. Invoking Pachpatte's inequality [15], we conclude that $u(t) = 0$ for all $t \in T$. So by the Arzela-Ascoli theorem $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T,X)$. Since it is also monotone, we have that $x_n \to x_*$ in $C(T,X)$ as $n \to \infty$. It is easy to see that x_* solves problem (1) and as in the proof of Theorem 4.1, we can check that x_* is the least solution of (1) in $V = [\psi, \varphi]$.

Similarly, if we consider the nonincreasing sequence $z_0 = \varphi$ and $z_n = L(z_{n-1})$, $n \geq 1$, we can show that $z_n \to x^*$ in $C(T,X)$ as $n \to \infty$ and x^* is the greatest solution of (1) in $V = [\psi, \varphi]$. □

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