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A note on the structure of quadratic Julia sets

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Abstract. In a series of papers, Bandt and the author have given a symbolic and topological description of locally connected quadratic Julia sets by use of special closed equivalence relations on the circle called Julia equivalences. These equivalence relations reflect the landing behaviour of external rays in the case of local connectivity, and do not apply completely if a Julia set is connected but fails to be locally connected.

However, rational external rays land also in the general case. The present note shows that for a quadratic map which does not possess an irrational indifferent periodic orbit and has a connected Julia set the following holds: The equivalence relation induced by the landing behaviour of rational external rays forms the rational part of a Julia equivalence.

Keywords: quadratic Julia set, Julia equivalence, external ray
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1. Introduction

By the *filled-in Julia set* J_c^0 of a quadratic map p_c, defined on the Riemann sphere by $p_c(z) = z^2 + c$, one understands the set of all points whose orbit remains bounded. In the present note, we are especially interested in the *Julia set* J_c defined to be the boundary of J_c^0 and supporting the most interesting behaviour of the map p_c. For the background from complex dynamics, we refer to the standard reference [8] and to [4], [5], [6], [20], [31], [24].

If J_c is connected, the dynamics of p_c on J_c is strongly related to the topological dynamical system $(T, h, ')$: T denotes the unit-circle, which we identify with the interval $[0, 1[$ by $\beta \longleftrightarrow e^{2\pi\beta i}$; $\beta \in [0, 1[$. Further, h is the angle-doubling map, defined by $h(\beta) = 2\pi\beta \bmod 1$ for $\beta \in T$, and $'$ the rotation by 180^0, given by $\beta' = (\beta + \frac{1}{2}) \bmod 1$ for $\beta \in T$.

This arises as an immediate consequence of Douady and Hubbard's fundamental results on the conformal representation of the complement of J_c^0 in the connected case:

There is a unique conformal map Φ_c from the complement of J_c^0 onto the complement of the unit disk in the Riemann sphere which conjugates p_c and the usual quadratic map p_0. The map G_c with $G_c(z) = \mathrm{Re}(\log \Phi_c(z)) = \log |\Phi_c(z)|$ — the so called *Green's function* of J_c^0 — assigns each point in the complement of J_c^0 a potential.

The fieldlines R_c^β of the potential, which are the curves consisting of all points whose image with respect to Φ_c has argument equal to $2\pi\beta$, are called *external* *rays* and play the key role for the description of the connected Julia sets. An external ray R_c^β *lands* at a point z of J_c if $\lim_{r\to 1}\Phi^{-1}(re^{2\pi\beta i})$ exists and is equal to z. Then z is called a *landing point* of R_c^β and β an *external angle* of z. It is important for the subsequent considerations that $R_c^{h(\beta)}$ and $R_c^{\beta'}$ land at the points $p_c(z)$ and $-z$ if R_c^β for $\beta \in T$ lands at z.

If J_c is locally connected, then by Carathéodory's theorem the inverse of Φ_c continuously extends to the unit-circle and each external ray lands. Thus J_c can be considered as the topological factor of T with respect to the equivalence relation $\approx$ identifying two points $\beta_1, \beta_2 \in T$ iff the external rays $R_c^{\beta_1}$ and $R_c^{\beta_1}$ land at the same point of J_c. As shown in [2], [12] (compare [15]), $\approx$ satisfies the following conditions:

1. it is closed (as a subset of $T \times T$) and the image of each equivalence class with respect to h and with respect to $'$ forms an equivalence class;
2. if two chords with end points $\beta_1, \beta_2 \in T$ and end points $\beta_3, \beta_4 \in T$ have a non-empty intersection and if $\beta_1 \approx \beta_2$ and $\beta_3 \approx \beta_4$, then $\beta_1 \approx \beta_3$;
3. each equivalence class is finite.

In general, we call an equivalence relation with 1., 2. and 3. *Julia equivalence.* Our concept and the study of Julia equivalences in [2], [3], [12], [13], [14], [15] is based on Thurston's invariant lamination concept, which was developed in his unpublished but widely circulated paper [32]. For the statements listed subsequently, we refer to [2], [3], [12], [13], and in particular to [15], the detailed but unfortunately in German written presentation of the subject.

To each $\alpha \in T$, there corresponds a unique Julia equivalence $\approx_\alpha$ satisfying the following property: There exists a point γ such that, with respect to h, one preimage of α and one preimage of γ are $\approx$-equivalent and have maximal distance among all pairs of $\approx$-equivalent points. (By the distance it is meant the inner distance on the circle T.) Each Julia equivalence is equal to $\approx_\alpha$ for some $\alpha \in T$, and depending on α it can be described in a symbolic manner. For given points $\alpha, \beta \in T$, we call the sequence

$$I^\alpha(\beta) = s_1 s_2 s_3 \dots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{for} \quad h^{i-1}(\beta) \in \,]\frac{\alpha}{2}, \frac{\alpha+1}{2}[\\ 1 & \text{for} \quad h^{i-1}(\beta) \in \,]\frac{\alpha+1}{2}, \frac{\alpha}{2}[\\ * & \text{for} \quad h^{i-1}(\beta) \in \{\frac{\alpha}{2}, \frac{\alpha+1}{2}\} \end{cases}$$

the *itinerary* of β with respect to α. The sequence $\hat{\alpha} = I^\alpha(\alpha)$ is said to be the *kneading sequence* of α. In view of the result which shall be proved here, we only recall the description of $\approx_\alpha$ for $\alpha \in T$ with non-periodic kneading sequence. (For the other cases we refer to [3], [15]). If $\hat{\alpha}$ is non-periodic, then for all $\beta_1, \beta_2 \in T$ the following holds:

$\beta_1 \approx_\alpha \beta_2$ iff either $I^\alpha(\beta_1) = I^\alpha(\beta_2)$ or $I^\alpha(\beta_1) = \mathbf{w}u\hat{\alpha}$ and $I^\alpha(\beta_2) = \mathbf{w}v\hat{\alpha}$ for some 0-1-word $\mathbf{w}$ and some $u, v \in \{0, 1, *\}$.

Before saying a little more about the relation of locally connected Julia sets and the Julia equivalences, let us recall some definitions concerning a periodic orbit $\{z = p_c^m(z), p_c(z), p_c^2(z), \ldots, p_c^{m-1}(z)\}$ for a quadratic map p_c: The value of the derivative of p_c^m in all points of the orbit coincides and is called the *multiplier* of the orbit. The orbit is said to be *attractive* (respectively *repelling*, *indifferent*) if its multiplier has absolute value greater than 1 (respectively less than 1, equal to 1). Moreover, depending on whether the argument of the multiplier (relative to 2π) is rational or irrational, one distinguishes *rational indifferent* and *irrational indifferent* periodic orbits.

If p_c possesses an attractive or rational indifferent orbit, then J_c is locally connected (see [8], [6]) and there exists a periodic $\alpha \in T$ such that two external rays $R_c^{\beta_1}$, $R_c^{\beta_1}$ land at the same point iff $\beta_1 \approx_\alpha \beta_2$. (For more information on the α, see [2], [15].)

If p_c has an irrational indifferent periodic orbit or all periodic orbits for p_c are repelling, then $c \in J_c$. Assuming that then J_c is locally connected, the identification defined by the landing behaviour of external rays is given by $\approx_\alpha$ for each external angle $\alpha \in T$ of c. In the case with an irrationally indifferent orbit, the point α is unique, and only in this case it is non-periodic with periodic kneading sequence (and J_c^0 contains Siegel disks).

Roughly speaking, the topological 'theory' of locally connected quadratic Julia sets forms the intersection of the 'theory' of (connected) quadratic Julia sets and the 'theory' of Julia equivalences. On the one side, not each Julia equivalence can be realized by a locally connected Julia set (see Section 4 in [13]), and on the other side, if the Julia set J_c of a quadratic map p_c is connected but fails to be locally connected, it cannot be a topological factor of T.

In the latter, at least the external rays R_c^β with rational β land at a point of J_c (see [20]), and the question arises, whether there exists a Julia equivalence $\approx_\alpha$ such that at least for rational $\beta_1, \beta_2 \in T$ the external rays $R_c^{\beta_1}$, $R_c^{\beta_2}$ have a common landing point if $\beta_1 \approx_\alpha \beta_2$. By the following Theorem, we shall give a positive answer to this question in case that p_c doesn't possess an irrational indifferent orbit. The irrational indifferent case is concerned with in [16].

Theorem. *For $c \in \mathbb{C}$, let the Julia set J_c be connected but not locally connected, and assume that p_c has no irrational indifferent periodic orbit. Then there exists a point $\alpha \in T$ such that c forms an accumulation point of the external ray R_c^α and such that for rational points $\beta_1, \beta_2 \in T$ the following holds: $\beta_1 \approx_\alpha \beta_2$ iff the external rays $R_c^{\beta_1}$ und $R_c^{\beta_2}$ land at the same point.*

Moreover, α is not preperiodic and has a non-periodic kneading sequence, and two rational points $\beta_1, \beta_2 \in T$ form external angles of the same point iff $I^\alpha(\beta_1) = I^\alpha(\beta_2)$.

2. Renormalization and Yoccoz's result

Often, on a part of the complex plane, a holomorphic map behaviours like a polynomial one. By their concept of a *polynomial-like map*, Douady and Hubbard have given an exact mathematical description of this phenomenon (see [9] and compare [24], [4], [6], [30]). For our purposes, we only need the concept for the quadratic case. We are following the representation of the subject in McMullen's book [24].

Definition 1 (quadratic-like map)**.** *Let f be a proper holomorphic map between simply connected domains U and V in $\mathbb{C}$. (Proper means that the preimage of each compact set is compact.) Then f is said to be* quadratic-like *if f has degree 2 and the closure of U forms a compact subset of V.*

If f is a quadratic-like map, then the set $J^0(f) = \bigcap_{j=1}^{\infty} f^{-j}(V)$ is called the filled-in Julia set *and its boundary $J(f)$ the* Julia set *of f.*

Of course, a quadratic map is quadratic-like in a neighbourhood of its filled-in Julia set, and the two definitions of a (filled-in) Julia set are consistent. The relation of polynomial-like maps and polynomial maps is established by Douady and Hubbard's Straightening Theorem. For our purposes, we only need the following partial statement of this Theorem:

Each quadratic-like map f having a connected Julia set is *hybrid-equivalent* to a unique quadratic map $p_c; c \in \mathbb{C}$. This means the existence of a quasiconformal conjugacy ϕ from a neighbourhood of $J^0(f)$ onto a neighbourhood of J_c^0 with $\overline{\partial}\phi = 0$ on $J^0(f)$. (In fact, we only need that ϕ is a topological conjugacy preserving the orientation and, obviously, such conjugacy transforms $J^0(f)$ into J_c^0.)

One reason for the occurrence of self-similarity in quadratic iteration theory is that often a high iterate of a given quadratic map has quadratic-like behaviour on a part of the complex plane. If J_c is connected or, equivalently, if 0 is contained in J_c^0, and the map p_c^n for $n \in \mathbb{N}$ is quadratic-like anywhere, then it is quadratic-like in a neighbourhood of 0. One comes to that what is called a *renormalizable* quadratic map (see [24], [25], [21], [11]). Furthermore, we want to follow [24].

Definition 2 (renormalizable quadratic maps)**.** *Let p_c for $c \in \mathbb{C}$ be a quadratic map and let $n \in \mathbb{N}\setminus\{1\}$. Then p_c^n is said to be* renormalizable *if there exist simply connected domains U and V in $\mathbb{C}$ such that p_c^n between U and V is quadratic-like and $p_c^{jn}(0) \in U$ for all $j \in \mathbb{N}_0$. The pair (U, V) is called* renormalization *of p_c^n.*

If p_c for $c \in \mathbb{C}$ is a quadratic map and p_c^n is renormalizable, then the corresponding (filled-in) Julia set does not depend on the renormalization (U, V). This justifies to use the notion $J^0(p_c^n)$ $(J(p_c^n))$ for the (filled-in) Julia set corresponding to the renormalization of p_c^n. Moreover, by the second property in the definition, $J(p_c^n)$ is connected, hence there exists a unique $\tilde{c}^n \in \mathbb{C}$ such that $p_{\tilde{c}^n}$ and p_c^n are hybrid-equivalent.

By $p_c^j(J^0(p_c^n)); j = 0, 1, \ldots, n-1$ we have n topological copies of $J^0(p_c^n)$, which are invariant with respect to p_c^n and which we want to call the *small filled-in Julia sets.*

Each of these sets contains a unique fixed point with respect to p_c^n which is mapped to a fixed point with external angle 0 by each conjugacy establishing the hybrid-equivalence to a quadratic map. This fixed point is said to be the *BETA fixed point.* There exists at most one further fixed point with respect to p_c^n which is called the *ALPHA fixed point.* (Different from the usual conventions, we write 'ALPHA fixed point' and 'BETA fixed point' since the notions α, β already have been used for the points in T.)

Two different fixed small filled-in Julia sets intersect in at most one point. This one is a repelling fixed point (see [24, Theorem 7.3]), and independent of the choice of the small filled-in Julia sets, in each case it is a BETA fixed point or in each case it is an ALPHA fixed point.

Definition 3. *If p_c^n for $c \in \mathbb{C}$ and $n \in \mathbb{N}$ is renormalizable, then p_c^n is called* simply renormalizable *if two small filled-in Julia sets do not cut in an ALPHA fixed point.*

Moreover, p_c for $c \in \mathbb{C}$ is said to be infinitely renormalizable *if p_c^n is renormalizable for infinitely many $n \in \mathbb{N}$.*

In view of the proof of our result, we need two important facts on infinitely renormalizable quadratic maps p_c. The first one says that there exist infinitely many $n \in \mathbb{N}$ such that p_c^n is simply renormalizable (see [24, Theorem 8.4]), and the second one is the following celebrated result on the local connectivity of quadratic Julia sets by Yoccoz.

Yoccoz's result. *Let p_c be a quadratic map which does not possess an indifferent periodic orbit. If J_c fails to be locally connected, then p_c is infinitely renormalizable.*

3. Proof of the result

Preparations for the proof. To prepare the proof of our result, let us list some statements which can be found in our papers [2], [3], [13], [15] in the main or are well known. Subsequently, we consider dynamical properties of a point in T only with respect to h and so will indicate this not any more.

Let us start saying a little more one the rational points in T. A point $\beta \in T$ is rational iff it is periodic or preperiodic. (In our terminology, preperiodic means to have a finite orbit, but to be non-periodic.)

It is easy to see that in the first case the reduced fraction corresponding to β has an odd denominator and that in the second case it has an even denominator. We have mentioned that for $p_c; c \in \mathbb{C}$ with connected Julia set and a rational $\beta \in T$ the external ray R_c^β lands at a point z of J_c. In fact, it is known that z belongs to a repelling or rationally indifferent periodic orbit if β is periodic and z is preperiodic else (see [20]).

I. In the following, we will deal with chords having their end points at the circle. Let us make some arrangements concerning these chords. First of all, a chord with end points β_1 and β_2 is denoted by $\beta_1\beta_2$. As the *length* of a chord we take the inner distance of its end points in T, where the circumference of the circle is measured by 1. Also chords of length 0 consisting of only one point are allowed.

We shall say that two chords S_1, S_2 *crosses* each other if they are different and have a common interior point. A chord S divides the disk into two open parts. Two subsets of the disk are said to be *separated* by S if they lie in different parts.

Finally, a point of T is called *between* $\beta_1, \beta_2 \in T$ if the distance of β_1 and β_2 is different from $\frac{1}{2}$ and the point is contained in the smaller open interval with end points β_1, β_2.

We have mentioned that our Julia equivalences are based on Thurston's theory of invariant laminations. In fact, the Julia equivalences are constructed from special invariant laminations, and it is efficient to use the construction here.

If X is a subset of T, then the application of a map f to the convex hull of X, i.e. the set $\{\sum_{i=1}^{k} a_i x_i | k \in \mathbb{N}, x_i \in X, a_i \in \mathbb{R}^+, \sum_{i=1}^{k} a_i = 1\}$, is defined to be the convex hull of $f(X)$ in the following. A (quadratic) *invariant lamination* $\mathcal{L}$ is a set of mutually non-crossing chords whose union $\bigcup \mathcal{L}$ is closed, such that for all $S \in \mathcal{L}$ the following holds: $h(S), S' \in \mathcal{L}$, and there exists a chord $\overline{S}$ with $h(\overline{S}) = S$.

The complement of $\bigcup \mathcal{L}$ in the disk divides into connectedness components. The closure of such component is said to be a *gap* of $\mathcal{L}$. A gap is convex, and it is called *polygonal* if its intersection with T is finite.

Here we only consider invariant laminations related to $\approx_\alpha$ for points $\alpha \in T$ which are not preperiodic and have a non-periodic kneading sequence. In this case we use the following notations:

If $\mathbf{w} = w_1 w_2 \dots w_j$ is a word consisting of symbols 0 and 1 — a 0-1-*word* — and β is a point in T, then $l^\alpha_{\mathbf{w}}(\beta)$ denotes the unique point whose j-th iterate is equal to β and whose itinerary starts with $\mathbf{w}$, when this point exists. (For the empty word $\mathbf{w}$, let $l^\alpha_{\mathbf{w}}(\beta) = \beta$.)

Moreover, let $\mathcal{B}^\alpha_* = \{l^\alpha_{\mathbf{w}}(\frac{\alpha}{2}\frac{\alpha+1}{2}) \mid \mathbf{w} \text{ is a 0-1-word}\}$, and let $\mathcal{B}^\alpha$ be the closure of $\mathcal{B}^\alpha_*$, i.e. the union of $\mathcal{B}^\alpha_*$ with the set of all accumulation chords (including degenerate one-point chords). We denote the latter set by $\partial\mathcal{B}^\alpha$.

Both $\mathcal{B}^\alpha$ and $\partial\mathcal{B}^\alpha$ form invariant laminations, and $\approx_\alpha$ is the equivalence relation which identifies two points β and γ iff there exists a sequence $\beta_0 = \beta, \beta_1, \dots, \beta_{j-1}, \beta_j = \gamma$ such that all chords $\beta_{i-1}\beta_i$ for $i = 1, 2, \dots, j$ belong to $\mathcal{B}^\alpha$ or, equivalently, belong to $\partial\mathcal{B}^\alpha$. There is no difference between the equivalence relation generated by $\mathcal{B}^\alpha$ to that generated by $\partial\mathcal{B}^\alpha$ since all gaps for $\mathcal{B}^\alpha$ are polygonal if the kneading sequence of α is non-periodic (see Section 7 in [2], where the notion $\mathcal{S}^\alpha$ is used instead of $\mathcal{B}^\alpha$). Let us say more about the structure of $\mathcal{B}^\alpha$.

Lemma 1 (The structure of $\mathcal{B}^\alpha$)**.** *For a point $\alpha \in T$, which is not preperiodic and has a non-periodic kneading sequence, one of the following two cases is satisfied:*

1. *$[\alpha]_{\approx_\alpha}$ consists of one point, $\partial\mathcal{B}^\alpha = \mathcal{B}^\alpha$, and $\frac{\alpha}{2}\frac{\alpha+1}{2}$ is the longest chord of $\partial\mathcal{B}^\alpha$;*
2. *$[\alpha]_{\approx_\alpha}$ consists of α and a point $\gamma \neq \alpha$, such that $\frac{\alpha}{2}\frac{\gamma+1}{2}$ and $\frac{\alpha+1}{2}\frac{\gamma}{2}$ are the longest chord of $\partial\mathcal{B}^\alpha$. Moreover, $\partial\mathcal{B}^\alpha$ is the closure of $\{l^\alpha_{\mathbf{w}}(\frac{\alpha}{2}\frac{\gamma+1}{2}) \mid \mathbf{w}$ is a 0-1-word$\} \cup \{l^\alpha_{\mathbf{w}}(\frac{\alpha+1}{2}\frac{\gamma}{2}) \mid \mathbf{w}$ is a 0-1-word$\}$.*

Proof: With exception of the last part in 2., the above statements are verified in Section 7 of [2]. We want to give an outline of the corresponding arguments, what is necessary for understanding the rest of the present proof.

If $[\alpha]_{\approx_\alpha}$ is a single set, $\frac{\alpha}{2}\frac{\alpha+1}{2}$ cannot be isolated in $\mathcal{B}^\alpha_*$ since all gaps of $\mathcal{B}^\alpha$ are polygonal. Thus by continuity argument it follows that all elements of $\mathcal{B}^\alpha_*$ are contained in $\partial\mathcal{B}^\alpha$.

If the equivalence class $[\alpha]_{\approx_\alpha}$ contains more than one point, then it must consist of two points, where the point different from α is denoted by γ here. The gap of $\partial\mathcal{B}^\alpha$ whose intersection with T is symmetric with respect to $'$ must forms a 'rectangle' spanned by the angles $\frac{\alpha}{2}$, $\frac{\alpha+1}{2}$, $\frac{\gamma}{2}$ and $\frac{\gamma+1}{2}$. Moreover, $\frac{\alpha}{2}\frac{\gamma+1}{2}$ and $\frac{\gamma}{2}\frac{\alpha+1}{2}$ are the longest chord of $\partial\mathcal{B}^\alpha$. Their length d is at least $\frac{1}{3}$.

Since the length of the preimage of a chord with length a is equal to $\frac{a}{2}$ or $\frac{1-a}{2}$ and since $\frac{\alpha}{2}\frac{\gamma}{2}$ has length $\frac{1}{2} - d$, one easily sees that $l^\alpha_{\mathbf{w}}(\frac{\alpha}{2}\frac{\gamma}{2})$ has length $2^{-j}(\frac{1}{2} - d)$ when $\mathbf{w}$ is a 0-1-word of length j. This yield the following: If $(\mathbf{w}_i)_{i=1}^\infty$ is a sequence of 0-1-words such that $(l^\alpha_{\mathbf{w}_i}(\frac{\alpha}{2}\frac{\alpha+1}{2}))_{i=1}^\infty$ converges to a chord S, then also $(l^\alpha_{\mathbf{w}_i}(\frac{\alpha}{2}\frac{\gamma+1}{2}))_{i=1}^\infty$ converges to S. This completes the proof of Lemma 1. $\square$

All gaps in $\mathcal{B}^\alpha$ are polygonal. So, if $\{\beta_1, \beta_2, \ldots, \beta_k = \beta_0\}$ is an equivalence class of $\approx_\alpha$ whose elements are given in an anticlockwise cyclic order, then the chords $\beta_{i-1}\beta_i$ for $i = 1, 2, \ldots, k$ must belong to $\partial\mathcal{B}^\alpha$. Moreover, Thurston has shown that the 'angles' of a periodic polygonal gap of an invariant lamination lie at a common orbit (see [32, proof of II.5.3], compare Proposition 5.3 in [2]). Let us summarize:

Lemma 2 (Equivalence classes containing periodic or preperiodic points)**.**
Assume that $\alpha \in T$ is not preperiodic and has a non-periodic kneading sequence. Further, let $\beta \in T$ be periodic respectively preperiodic. Then each point in $[\beta]_{\approx_\alpha}$ is periodic respectively preperiodic. (In the first case, all periods are equal.) Moreover, if β is periodic and $\beta_1, \beta_2, \ldots, \beta_k = \beta_0 = \beta$ are the points of the equivalence class $[\beta]_{\approx_\alpha}$, given in an anticlockwise cyclic order, then all chords $\beta_{i-1}\beta_i$ for $i = 1, 2, \ldots, k$ are contained in $\partial\mathcal{B}^\alpha$. $\square$

II. As shown in [2], [3], for each point $\alpha \in T \setminus \{0\}$ there exists a unique periodic point $\overline{\alpha}$ different from α and satisfying $\approx_\alpha = \approx_{\overline{\alpha}}$. In each case, α and $\overline{\alpha}$ have the same period. The set of all chords $\alpha\overline{\alpha}$ for periodic $\alpha \neq 0$ plays an important role for the description of the Mandelbrot set (compare [18], [32], [7], [3]). For the proof of our result, we only need the following statements:

Lemma 3. *If $\delta \neq 0$ is periodic, then $\overline{\delta}$ is equal to the unique point $\eta \neq \delta$ with the property that the chords $h^i(\delta)h^i(\eta)$ for $i \in \mathbb{N} \cup \{0\}$ do not cross $\frac{\delta}{2}\frac{\delta+1}{2}$ and $\frac{\eta}{2}\frac{\eta+1}{2}$.*

Moreover, if $\alpha\gamma$ is an accumulation chord of $\mathcal{B}_* = \{\delta\overline{\delta} \mid \delta$ is periodic and $\delta \neq 0\}$ (such that α, γ are not preperiodic), then the following holds:

(i) *the kneading sequences of α and γ coincide and are non-periodic,*
(ii) *$\approx_\alpha = \approx_\gamma$, and*
(iii) *if $\delta\overline{\delta} \in \mathcal{B}_*$ separates 0 and $\alpha\gamma$, then $l^\alpha_{\mathbf{w}}(\delta\overline{\delta})$ is contained in the invariant lamination $\mathcal{B}^\alpha$ for each 0-1-word $\mathbf{w}$.*

Proof: The first statement is an immediate consequence of Lemma 3 in [3]. So let $\alpha\gamma$ be an accumulation chord of $\mathcal{B}_*$.

Then (ii) and $\hat{\alpha} = \hat{\gamma}$ follow from Theorem 1(a) and Theorem 2 in [3]. (That what in the present paper is $\mathcal{B}_*$, is denoted by $\mathcal{S}_*$ in [3].) Moreover, by Lemma 3 in [3] and Theorem 1 in [3], two different nonperiodic points with periodic kneading sequence cannot be end points of an accumulation chord of $\mathcal{B}_*$, and by Theorem 1 in [13] an accumulation point of $\mathcal{B}_*$ has a non-periodic kneading sequence. This shows (i).

Finally, if $\delta\overline{\delta}$ is given as in (iii), then by Theorem 2(b) in [3] we have $\delta \approx_\alpha \overline{\delta}$, and by the Corollary in [3], no element of the orbits of $\delta, \overline{\delta}$ lies between $\frac{\alpha}{2}$ and $\frac{\gamma+1}{2}$ or between $\frac{\gamma}{2}$ and $\frac{\alpha+1}{2}$. Thurston's argument mentioned above Lemma 2 leads to the statement that $\delta\overline{\delta} \in \mathcal{B}^\alpha$, and the rest is obvious. $\square$

III. The last Lemma in preparation of the proof of our result is based on the fact that each repelling periodic point in a connected Julia set has at least one periodic external angle (see [20]). If $\delta \in T$ is periodic, the we denote its periodic preimage by $\dot{\delta}$ and its preperiodic one by $\ddot{\delta}$.

Lemma 4. *Let $c \in \mathbb{C}$ and let p_c^n be simply renormalizable for $n > 1$. Further, let z be the* ALPHA *fixed point in $J^0(p_c^n)$.*

Then there exists a unique periodic point α such that $R_c^{\dot{\alpha}}$ and $R_c^{\dot{\overline{\alpha}}}$ land at z (and $R_c^{\ddot{\alpha}}$ and $R_c^{\ddot{\overline{\alpha}}}$ land at $-z$).

Proof: Since p_c^n is simply renormalizable, the period of z is equal to n. At first we note that z has more than one periodic external angle. To show this, let ϕ be a map establishing the hybrid-equivalence of p_c^n and $p_{\tilde{c}^n}$ between neighbourhoods of $J^0(p_c^n)$ and $J^0_{p_{\tilde{c}^n}}$. (Compare the notion below Definition 2.) If R_c^η lands at z, then also $R_c^{h^n(\eta)} = p_c^n(R_c^\eta)$ lands at z, but $h^n(\eta)$ must be different from η.

Otherwise, by the action of ϕ there would exist a path δ in the complement of $J^0_{p_{\tilde{c}^n}}$ with the properties that $p_{\tilde{c}^n}(z)$ would be accessibly by δ, and that δ and $p_{\tilde{c}^n}(\delta)$ would have a common end segment. Then, by Lindelöf's Theorem (see

Theorem 6.3 and Corollary 6.4 in [24]), 0 would be an external angle of $p_{\tilde{c}^n}(z)$, which contradicts the assumption that z is an ALPHA fixed point.

Let now β_1, β_2 be periodic points in T such that they form external angles of a common point in the orbit of z and have maximal distance with this property. Then one easily sees that the distance of β_1 and β_2 is not less than $\frac{1}{3}$, and that the chords $h^i(\beta_1)h^i(\beta_2)$; $i \in \mathbb{N} \cup \{0\}$ do not cross $\beta_1\beta_2$ and $\beta_1'\beta_2'$.

Let d be the length of $\beta_1\beta_2$. Then, following an argument of Thurston (see [32]), one obtains that no of the chords $h^i(\beta_1)h^i(\beta_2)$; $i \in \mathbb{N}$ can be shorter than $\frac{1}{2} - d$ and so no point in the orbits of β_1 and β_2 lies between β_1 and β_2' or between β_2 and β_1'. In fact, assuming that i_0 would be the first number such that $h^{i_0}(\beta_1)h^{i_0}(\beta_2)$ has length $d_0 < \frac{1}{2} - d$. Then $h^{i_0}(\beta_1)h^{i_0}(\beta_2)$ would have length $\frac{1-d_0}{2} > d$, which is impossible.

Since $J^0(p_c^n)$ is the only small filled-in Julia set which contains 0, the points β_1, β_2 must be external angles of z and so β_1', β_2' external angles of $-z$. The above statement follows from Lemma 3 now. $\square$

The main part of the proof. Let $c \in \mathbb{C}$ be given, such that J_c is connected but fails to be locally connected and p_c does not possess an irrational indifferent periodic orbit. Further, let $\approx$ be the equivalence relation on the rational points in T which is defined to identify two points β_1, β_2 if the external rays $R_c^{\beta_1}$, $R_c^{\beta_2}$ land at the same point.

By Yoccoz's resultat, p_c is infinitely renormalizable, and we find an increasing sequence $(n_i)_{i\in\mathbb{N}}$ satisfying the following properties:

1. $p_c^{n_i}$ is simply renormalizable;
2. $J_c^0(p_c^{n_{i+1}})$ does not contain a point with period less than or equal to n_i.

Consequently, $\bigcap_{i\in\mathbb{N}} J_c^0(p_c^{n_i})$ contains no periodic, thus no preperiodic point. The statements listed and the fact that all periodic orbits of p_c are repelling can be found in [24] (see Theorems 8.1, 8.4 and 7.8).

Now in each case let z_i; $i \in \mathbb{N}$ be the ALPHA fixed point in $J^0(p_c^{n_i})$. This one exists by the following reason: Since all periodic orbits for p_c are repelling, also p_d with $d = \tilde{c}^{n_i}$ possesses only repelling periodic orbits. (Compare the notion below Definition 2.) The two fixed points of the quadratic map p_d coincide iff $d = \frac{1}{4}$, but then the corresponding double fixed point is rationally indifferent.

Further, for each $n \in \mathbb{N}$ let $\alpha_i \in T$ be the periodic point such that $R_c^{\dot{\alpha}_i}$ and $R_c^{\dot{\overline{\alpha_i}}}$ land at z_i, which is uniquely defined by Lemma 4.

By 2., in each case the chords $\dot{\alpha}_{i+1}\dot{\overline{\alpha_{i+1}}}, \ddot{\alpha}_{i+1}\ddot{\overline{\alpha_{i+1}}}$ separate the chords $\dot{\alpha}_i\dot{\overline{\alpha_i}}$ and $\ddot{\alpha}_i\ddot{\overline{\alpha_i}}$. Moreover, by $(\alpha_i)_{i\in\mathbb{N}}$ we have a sequence of periodic points with the property that, for each $i \in \mathbb{N}$, the chord $\alpha_i\overline{\alpha_i}$ separates the point 0 and the chord $\alpha_{i+1}\overline{\alpha_{i+1}}$. Subsequently, we want to assume that, in dependence on i, the period of α_i monotonically increases.

Let us consider the points $\alpha = \lim_{n\to\infty} \alpha_n$ and $\gamma = \lim_{n\to\infty} \gamma_n$. Since the intersection of the sets $J(p_c^{n_i})$ does not contain a periodic or preperiodic point, α and γ cannot be periodic or preperiodic. Moreover, by Lemma 3 the kneading sequence of α und γ is non-periodic, and one has $\approx_\alpha = \approx_\gamma$. Of course, it is possible that α and γ coincide.

By Lemma 3, $\mathcal{A} = \{l^\alpha_{\mathbf{w}}(\alpha_i\overline{\alpha_i}) \mid \mathbf{w} \in \{0,1\}^*, i \in \mathbb{N}\}$ is a subset of $\mathcal{B}^\alpha$. Moreover, $\partial\mathcal{B}^\alpha$ is contained in the closure of $\mathcal{A}$. This follows from Lemma 1 and the fact that, for each 0-1-word $\mathbf{w}$, the (possibly equal) chords $l^\alpha_{\mathbf{w}}(\frac{\alpha}{2}\frac{\gamma+1}{2}), l^\alpha_{\mathbf{w}}(\frac{\gamma}{2}\frac{\alpha+1}{2})$ form accumulation chords of the set $\{l^\alpha_{\mathbf{w}0}(\alpha_i\overline{\alpha_i}) \mid \mathbf{w}$ is a 0-1-word$\} \cup \{l^\alpha_{\mathbf{w}1}(\alpha_i\overline{\alpha_i}) \mid \mathbf{w}$ is a 0-1-word$\}$.

Let us show that the end points of each chord in $\mathcal{A}$ are equivalent with respect to $\approx$. But first note that a chord connecting two $\approx$-equivalent rational points β_1, β_2 cannot have a point common with $\frac{\alpha}{2}\frac{\alpha+1}{2}$.

Indeed, since $\frac{\alpha}{2}\frac{\gamma+1}{2}$ and $\frac{\gamma}{2}\frac{\alpha+1}{2}$ are (one-side) accumulation chords of the set $\{\dot{\alpha}_i\dot{\overline{\alpha_i}} | i \in \mathbb{N}\} \cup \{\ddot{\alpha}_i\ddot{\overline{\alpha_i}} | i \in \mathbb{N}\}$, we can assume that β_1 lies between $\frac{\alpha}{2}$ and $\frac{\gamma}{2}$, and β_2 between $\frac{\alpha+1}{2}$ and $\frac{\gamma+1}{2}$. If the external rays $R_c^{\beta_1}$ und $R_c^{\beta_2}$ would land commonly at a point x, then x would be periodic or preperiodic and would be contained in each small filled-in Julia set $J^0(p_c^{n_i})$. This is impossible.

Let a 0-1-word $\mathbf{w}$ and an $i \in \mathbb{N}$ be given, such that $l^\alpha_{\mathbf{w}}(\alpha_i) \approx l^\alpha_{\mathbf{w}}(\overline{\alpha_i})$, and let $\beta_1 = l^\alpha_{0\mathbf{w}}(\alpha_i)$, $\beta_2 = l^\alpha_{0\mathbf{w}}(\overline{\alpha_i})$. Then one obtains $\beta_1' = l^\alpha_{1\mathbf{w}}(\alpha_i)$ and $\beta_2' = l^\alpha_{1\mathbf{w}}(\overline{\alpha_i})$. Moreover, by the even shown one has the following: If among the external rays $R_c^{\beta_1}$, $R_c^{\beta_2}$, $R_c^{\beta_1'}$ und $R_c^{\beta_2'}$ two land at a common point, then these rays are $R_c^{\beta_1}$ and $R_c^{\beta_2}$ or $R_c^{\beta_1'}$ and $R_c^{\beta_2'}$.

Since p_c maps two to one, from this it follows $\beta_1 \approx \beta_2$ and $\beta_1' \approx \beta_2'$. Now, by induction on the length of a 0-1-word $\mathbf{w}$, one easily shows that the ends of a chord in $\mathcal{A}$ are $\approx$-equivalent.

Two rational points in T are equivalent with respect to $\approx_\alpha$ iff the chord connecting them, and its iterates, have no common point with $\frac{\alpha}{2}\frac{\alpha+1}{2}$. Thus the restriction of $\approx_\alpha$ to the rationals contains $\approx$, and it remains to show that each $\approx_\alpha$-equivalence class containing a rational point forms a subset of a $\approx$-equivalence class.

So let $\beta \in T$ be periodic with period m and let $\beta_1, \beta_2, \ldots, \beta_k = \beta_0 = \beta$ be the points of the equivalence class $[\beta]_{\approx_\alpha}$, given in an anticlockwise cyclic order. By Lemma 2, we find points $\gamma_1, \gamma_2, \ldots, \gamma_k = \gamma_0$ and $\delta_1, \delta_2, \ldots, \delta_k = \delta_0$ satisfying the following properties:

1. for all $i = 1, 2, \ldots, k$, the point β_i lies between γ_{i-1} and δ_i,
2. the chords $\gamma_i\delta_i$; $i = 1, 2, \ldots, k$ are elements of $\mathcal{A}$,
3. the first $m-1$ iterates of $\frac{\alpha}{2}$ and $\frac{\alpha+1}{2}$ are not contained in one of the intervals $[\gamma_{i-1}, \delta_i]$.

Finally, let U be the bounded simply connected domain which is separated from the rest of the complex plane by the equipotential of the niveau 1 (of G_c) and the external rays $R_c^{\gamma_i}$; $i = 1, 2, \dots, k$ and $R_c^{\delta_i}$; $i = 1, 2, \dots, k$ together with their landing points.

By construction, on the union of the intervals $[\gamma_{i-1}, \delta_i]$; $i = 1, 2, \dots, k$ the map h^m is injective. Moreover, since all points of $[\beta]_{\approx_\alpha}$ have period m, the set $\bigcup_{i=1}^k [\gamma_{i-1}, \delta_i]$ is contained in the interior of $h^m(\bigcup_{i=1}^k [\gamma_{i-1}, \delta_i])$.

Therefore, the closure of U forms a subset of $p_c^m(U)$, and p_c^m is injective on U. (Take into considerations that p_c doubles the potential.) Thus there exists a point $z \in J_c$ satisfying $\{z\} = \bigcap_{i\in\mathbb{N}} p_c^{-im}(U)$, where p_c^m is regarded as a map on U now. This is an immediate consequence of the Wolff-Denjoy Theorem, which says the following: If f is a conformal map on a domain V in $\mathbb{C}$ which is conformally equivalent to the open disk and contains the closure of $f(V)$, then $(f^n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets to a constant function (compare Theorem 3.2 in [20]).

All $p_c^{-im}(U)$; $i = 1, 2, \dots, k$ contain an end segment of each external ray $R_c^{\beta_i}$; $i = 1, 2, \dots, k$, hence z is the landing point of each external ray and the points β_i; $i = 1, 2, \dots, k$ are $\approx$-equivalent.

Obviously, c is an accumulation point of at least one of the external rays R_c^α, R_c^γ, but of no other one. Taking into consideration that each equivalence class of preperiodic points is iterated into an equivalence class of periodic points, one completes that $\approx$ and the restriction of $\approx_\alpha = \approx_\gamma$ to the set of rational points coincide.

The last statement in the Theorem is a property of the symbolic description of $\approx_\alpha$ in the Introduction: A rational point $\beta \in T$ cannot have an itinerary ending with $\hat{\alpha}$. Otherwise, by the symbolic description of $\approx_\alpha$, the point α would be $\approx_\alpha$-equivalent to a periodic or preperiodic point, which contradicts Lemma 2.

□

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