Connected transversals to subnormal subgroups

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Abstract. Subnormal subgroups possessing connected transversals are briefly discussed.

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In [8] J.D.H. Smith introduced the notion of a stably nilpotent quasigroup, showing that a quasigroup Q is stably nilpotent if and only if the inner permutation groups of Q are subnormal in the multiplication group of Q. Generalizing this for abstract groups, we come by groups which are, in a certain sense, relatively nilpotent with respect to a subgroup. The present short note collects some basic information on such groups.

1. Preliminaries

1.1. Let H be a subgroup of a group G. Than $L_G(H)$ denotes the core and $N_G(H)$ the normalizer of H in G. Further, $N_{G,0}(H) = H$ and $N_{G,n+1}(H) = N_G(N_{G,n}(H))$ for every $n \geq 0$.

The subgroup H is said to be subnormal of depth at most $n \geq 0$ in G if there are subgroups $H_0, H_1, \dots, H_n$ of G such that $H_0 = H$, and $H_n = G$ and H_i is normal in H_{i+1} for every $0 \leq i \leq n-1$.

1.2. Let G be a group. For $n \geq 0$, $Z_n(G)$ denotes the nth member of the usual central series. That is, $Z_0(G) = 1$, and $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$.

Now, let H be a subgroup of G. We define two series of normal subgroups of G: $Z_{H,0}(G) = Z^*_{H,0}(G) = L_G(H)$, $Z_{H,n}(G) \subseteq Z^*_{H,n+1}(G)$ and $Z^*_{H,n+1}(G)/Z_{H,n}(G) = Z(G/Z_{H,n}(G))$, $Z_{H,n+1}(G) = L_G(H \cdot Z^*_{H,n+1}(G))$.

1.3 Remark. (i) A subgroup H is subnormal of depth at most $n \geq 0$ in a group G, provided that $N_{G,n}(H) = G$. The converse is not true in general (see, e.g., 4.1).

(ii) If G is a finite group, then subnormal subgroups form a sublattice in the lattice of all subgroups of G (see, e.g., [6, Theorem 6.5]). This is not true in general ([7, §13.1, p. 375]), albeit subnormal subgroups of arbitrary (i.e., even infinite) groups are closed under finite intersections.

2. Technical results

2.1 Lemma. *Let H be a subgroup of a group G. Then:*

(i) $L_G(H) = Z_{H,0}(G) \subseteq Z_{H,1}(G) \subseteq Z_{H,2}(G) \subseteq \dots$;
(ii) $L_G(H) = Z^*_{H,0}(G) \subseteq Z^*_{H,1}(G) \subseteq Z^*_{H,2}(G) \subseteq \dots$;
(iii) $Z_{H,n}(G) \subseteq Z^*_{H,n+1}(G) \subseteq Z_{H,n+1}(G) \subseteq Z^*_{H,n+2}(G) \subseteq \dots$ *for every* $n \geq 0$;
(iv) $Z_{H,n}(G) \subseteq L_G(N_{G,n}(H))$ *for every* $n \geq 0$.

Proof: The first three assertions are clear from definition 1.2, (iv) is clear for $n = 0$, and we shall proceed further by induction.

Let $f : G \to \overline{G} = G/Z_{H,n}(G)$, $g : G \to \widetilde{G} = G/L_G(N_{G,n}(H))$ and $h : \overline{G} \to \widetilde{G}$ denote the natural projections, $g = hf$. Then $Z^*_{H,n+1}(G) = f^{-1}(Z(\overline{G})) \subseteq g^{-1}(Z(\widetilde{G})) = K$, $HK \subseteq N_{G,n}(H)K \subseteq N_G(N_{G,n}(H)) = N_{G,n+1}(H)$ and $Z_{H,n+1}(G) = L_G(H \cdot Z^*_{H,n+1}(G)) \subseteq L_G(HK) \subseteq L_G(N_{G,n+1}(H))$. □

2.2 Lemma. *Let $H \subseteq K \subseteq G$ be subgroups of a group G. Then $Z_{H,n}(G) \subseteq Z_{K,n}(G)$ and $Z^*_{H,n}(G) \subseteq Z^*_{K,n}(G)$ for every $n \geq 0$.*

Proof: By induction on n (see the proof of 2.1 (iv)). □

2.3 Lemma. *Let H be a subgroup of a group G. Then $Z_n(G) \subseteq Z^*_{H,n}(G) \subseteq Z_{H,n}(G)$ for every $n \geq 0$.*

Proof: Clearly, $Z_n(G) \subseteq Z^*_{1,n}(G) \subseteq Z_{1,n}(G)$ and we can use 2.2. □

2.4 Lemma. *Let H be a subgroup of a group G. Then:*

(i) $Z_{H,0}(G) = G$ *iff* $H = G$;
(ii) $Z_{H,1}(G) = G$ *iff* $G' \subseteq H$;
(iii) $Z_{H,n}(G) = G$ *for* $n \geq 0$ *iff* $G = H \cdot Z^*_{H,n}(G)$;
(iv) *if G is nilpotent of class at most $n \geq 0$, then $Z_{H,n}(G) = G$;*
(v) *if $Z_{H,n}(G) = G$ for $n \geq 0$, then $N_{G,n}(H) = G$ (and hence H is subnormal of depth at most n in G).*

Proof: The first assertions are easy, (iv) follows from 2.3, and (v) follows from 2.1 (iv). □

2.5 Lemma. *Let H be a subgroup of a group G such the $L_G(H) = 1$. Then:*

(i) $Z^*_{H,1}(G) = Z(G)$ *and* $Z_{H,1}(G) = L_G(HZ(G))$;
(ii) $Z_{H,1}(G) = G$ *iff G is abelian;*
(iii) $Z_{H,2}(G) = G$ *iff* $G' \subseteq HZ(G)$.

Proof: Obvious. □

2.6 Lemma. *Let H be a subgroup of a group G. Then:*

(i) $HZ_{H,n}(G) = HZ^*_{H,n}(G)$ *for every* $n \geq 0$*;*

(ii) *if K is a subgroup conjugate to H, then* $Z_{H,n}(G) = Z_{K,n}(G)$ *and* $Z^*_{H,n} = Z^*_{K,n}(G)$ *for every* $n \geq 0$.

Proof: The assertions follow easily from definition 1.2. □

2.7 Proposition. *Let H be a subgroup of a group G. The following conditions are equivalent for* $n \geq 1$*:*

(i) $Z^*_{H,n}(G) = G$*;*

(ii) $Z_{H,n}(G) = G$*;*

(iii) $HZ_{H,n}(G) = G$*;*

(iv) $HZ^*_{H,n}(G) = G$*;*

(v) $G' \subseteq Z_{H,n-1}(G)$*;*

(vi) $G' \subseteq HZ_{H,n-1}(G)$*;*

(vii) $G' \subseteq HZ^*_{H,n-1}(G)$.

Proof: (i) implies (ii) by 2.1 (iii); (ii) implies (iii) and (v) implies (vi) trivially; (iii) implies (iv) and (vi) implies (vii) by 2.6 (i).

We now show (iv) implies (v). Put $N = Z_{H,n-1}(G)$. We have $\overline{G} = G/N = HZ^*_{H,n}(G)/N = \overline{H}Z(\overline{G})$, and hence $(\overline{G})' \subseteq \overline{H}$, $G' \subseteq HN = HZ^*_{H,n-1}(G)$ and $N = L_G(HZ^*_{H,n-1}(G)) = HZ^*_{H,n-1}(G)$. Consequently $G' \subseteq N$. Finally, we show (vii) implies (i). Since $G' \subseteq HZ^*_{H,n-1}(G)$, we have $Z_{H,n-1}(G) = HZ_{H,n-1}(G), G' \subseteq Z_{H,n-1}(G)$ and $Z^*_{H,n}(G) = G$ (see 1.2). □

2.8. Let H be a subgroup of a group G, $n \geq 0$, $N = Z_{H,n}(G)$, $N^* = Z^*_{H,n}(G)$, $\overline{G} = G/N$, and $\overline{H} = HN/N \subseteq \overline{G}$.

(i) $HN = HN^*$, $N = L_G(HN^*) = L_G(HN)$ and this implies that $L_{\overline{G}}(\overline{H}) = 1$ and $\overline{H} \cong H/H \cap N$.

(ii) $Z^*_{H,n+1}(G)/N = Z(\overline{G}) = Z^*_{\overline{H},1}(\overline{G}), Z_{H,n+1}(G) = L_G(H \cdot Z^*_{H,n+1}(G))$ and $Z_{H,n+1}(G)/N = L_{\overline{G}}(\overline{H}Z(\overline{G})) = Z_{\overline{H},1}(\overline{G})$.

(iii) $Z^*_{H,n+m}(G)/N = Z^*_{\overline{H},m}(\overline{G})$ and $Z_{H,n+m}(G)/N = Z_{\overline{H},m}(\overline{G})$ for every $m \geq 1$.

2.9. Let H be a subgroup of a group G. Put $H_n = H \cap Z_{H,n}(G)$ for every $n \geq 0$. Then $L_G(H) = H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$ and H_n is normal in G.

2.10 Lemma. *Let H be a subgroup of a group G such that* $L_G(H) = 1$ *and let* $\alpha = [G : HZ(G)]$. *Then:*

(i) $Z_{H,1}(G) = L_G(HZ(G))$ *can be embedded into the Cartesian product of* α *copies of* $Z(G)$*;*

(ii) $Z_{H,1}(G)$ *is an abelian group;*

(iii) H_1 *(see 2.9) can be embedded into the Cartesian product of* $\alpha - 1$ *copies of* $Z(G)$ $(\alpha - 1 = \alpha$ *for* α *infinite).*

Proof: Put $N = Z_{H,1}(G)$. For every $x \in G$, $N = N^x = L_G(H^x \cdot Z(G))$, $H^x \cap Z(G) \subseteq L_G(H^x) = L_G(H) = 1$, $H^x \cdot Z(G)$ is the direct product of H^x and $Z(G)$ and consequently the restriction f_x of the natural projection $H^x \cdot Z(G) \to Z(G)$ to N is a homomorphism of N onto $Z(G)$ (we have $Z(G) \subseteq N$).

Now, let A be a right transversal to $HZ(G)$ in G such that $1 \in A$. Define a homomorphism $f : N \to \prod_{\alpha} Z(G)$ by $f(u) = \prod_{a \in A} f_a(u)$, $u \in N$. If $u \in Ker(f)$, then $aua^{-1} \in H$ for every $a \in A$. Consequently, $u \in H$ and if $x \in G$, $x = zva$, $a \in A$, $v \in H$, $z \in Z(G)$, then $xux^{-1} = zvaua^{-1}v^{-1}z^{-1} = vaua^{-1}v^{-1} \in H$. Thus $u \in L_G(H) = 1$ and we have proved that f is injective. Finally, for $g = \prod_{a \neq 1} f_a$ we get $Ker(g) \cap H = 1$, and hence $g|H_1$ is injective. $\square$

2.11 Proposition. *Let H be a subgroup of a group G and let $\alpha_n = [G : H \cdot Z_{H,n+1}(G)]$ for every $n \geq 0$. Then $Z_{H,n+1}(G)/Z_{H,n}(G)$ is an abelian group which can be embedded into the Cartesian product of α_n copies of $Z(G/Z_{H,n}(G)) = Z^*_{H,n+1}(G)/Z_{H,n}(G)$.*

Proof: The result follows by an easy combination of 2.10 and 2.8 (i),(ii). $\square$

2.12 Corollary. *Let H be a subgroup of a group G such that $Z_{H,n}(G) = G$ for some $n \geq 0$. If H is soluble of derived length $m \geq 0$, then G is also soluble and its derived length is at most $n + m$.*

2.13 Lemma. *Let H be a subgroup of a group G such that $Z_{H,2}(G) = G$. Then $H \subseteq L_G(H)$.*

Proof: By 2.10, $H/L_G(H)$ is abelian. $\square$

2.14 Proposition. *Let H be a subgroup of a finite group G such that $[G : H]$ is a power of a prime p and $L_G(H)$ is a p-group. Then $G = Z_{H,n}(G)$ for some $n \geq 0$ iff G is a p-group.*

Proof: If G is a p-group, then G is nilpotent and our result follows from 2.3. Now assume that $Z_{H,n}(G) = G$. We shall proceed by induction on card(G). Further, considering the factor $G/L_G(H)$, we can restrict ourselves to the case $L_G(H) = 1$. Then $H \cap Z(G) = 1$, $[HZ(G) : H] = \text{card}(Z(G))$, and hence $Z(G)$ is a p-group. From this, $N = Z_{H,1}(G)$ is a p-group by 2.10 (i). Since $N \neq 1$ (otherwise $G = 1$), G/N is a p-group by induction. $\square$

2.15. Let H be a subgroup of a group G such that $G/Z_{H,n}(G)$ is a two element group for some $n \geq 0$.

(i) If $n = 0$, then $G/L_G(H)$ is a two element group, which means that H is normal and of index 2 in G.

(ii) Assume that $n \geq 1$. Clearly, $Z_{H,n+1}(G) = Z^*_{H,n+1}(G) = G$ and $G' \subseteq Z_{H,n}(G) = H \cdot Z^*_{H,n}(G)$. Put $N = Z_{H,n-1}(G)$, $\overline{G} = G/N$ and $\overline{G} = HN/N = HZ^*_{H,n-1}(G)/L_G(HZ^*_{H,n-1}(G))$. We have $L_{\overline{G}}(\overline{H})$=1, $Z(\overline{G}) =$

$Z^*_{H,n}(G)/N$, $(\overline{G})' \subseteq Z_{H,n}(G)/N = \overline{H} \cdot Z(\overline{G})$ and $\overline{G}/\overline{H}Z(\overline{G}) \cong G/Z_{H,n}(G)$, so that $\overline{G}/\overline{H}Z(\overline{G})$ is a two element group.

(iii) Assume that $n = 1$ and that $L_G(H) = 1$ (cf. (ii)). Then $Z_{H,2}(G) = Z^*_{H,2}(G)$ and $G' \subseteq Z_{H,1}(G) = HZ(G)$. Take $w \in G - HZ(G)$ and put $W = Z(G) \cup wZ(G)$. Then $w^2 = uz$ for suitable $u \in H$, $z \in Z$ and $w^{-1}uw = w^{-1}w^2z^{-1}w = u$. This implies that $u \in L_G(H) = 1$, so that $w^2 \in Z(G)$ and we see that W is an abelian subgroup of $G, W \cap H = 1$ and $G = HW$.

3. Connected transversals to subnormal subgroups

3.1. In this section, let H be a subgroup of a group G such that there exist H-connected transversals A, B to H in G (i.e., A, B are left transversals and $[A, B] \subseteq H$).

3.2 Lemma.

(i) $HZ_{H,n}(G) = HZ^*_{H,n}(G) = N_{G,n}(H)$ *for every* $n \geq 0$.

(ii) $Z_{H,n}(G) = L_G(N_{G,n}(H))$ *for every* $n \geq 0$.

Proof: This is clear for $n = 0$ and we shall proceed by induction on n.

Put $N = Z_{H,n}(G)$ and consider the factors $\overline{G} = G/N$ and $\overline{H} = HN/N$. Then $L_{\overline{G}}(\overline{H}) = 1$, and so $N_{\overline{G}}(\overline{H}) = \overline{H}Z(\overline{G})$ by [3, Proposition 2.7]. This implies that $N_G(HN) = HZ^*_{H,n+1}(G)$. However, $HN = N_{G,n}(H)$ by the induction and we have $N_{G,n+1}(H) = HZ^*_{H,n}(G) = HZ_{H,n}(G)$ (2.6 (ii)). The rest is clear. □

3.3 Proposition. *The following conditions are equivalent for* $n \geq 1$*:*

(i) $Z_{H,n}(G) = G$;

(ii) $HZ_{H,n-1}(G)$ *is normal in* G;

(iii) $H \subseteq Z_{H,n-1}(G)$;

(iv) $H_{n-1} = H$ *(see* 2.9*)*;

(v) H *is subnormal of depth at most* n *in* G;

(vi) $N_{G,n}(H) = G$;

(vii) $N_G(H)$ *is subnormal of depth at most* $n - 1$ *in* G.

Proof: (i) implies (ii) by 2.7 (ii),(vi) (in fact, $G' \subseteq HZ_{H,n-1}(G)$); (ii) implies (iii), since $Z_{H,n-1}(G) = L_G(HZ_{H,n-1}(G))$; (iii) implies (iv) trivially; (iv) implies (ii), since $Z_{H,n-1}(G) = L_G(HZ_{H,n-1}(G))$; (i) implies (vi) by 2.1 (iv); (vi) implies (vii) and (vii) implies (v) trivially; (vi) implies (i) by 3.2 (ii).

We now show (ii) implies (i). The existence of H-connected transversals easily yields that $G' \subseteq HZ_{H,n-1}(G)$ (consider the factor $G/Z_{H,n-1}(G)$), and the result follows from 2.7.

We proceed by induction on n to show (v) implies (vi). If $n = 1$, then H is normal in G and (vi) is clear. Let $n \geq 2$ and let $L_G(H) = 1$ (considering the factor $G/L_G(H)$, we can restrict ourselves to this case). There is a subgroup K of G such that H is a normal subgroup of K and K is subnormal of depth at most $n-1$ in G. Put $L = L_G(K)$, $\overline{G} = G/L$ and $\overline{K} = K/L$. Then $L_{\overline{G}}(\overline{K}) = 1$ and $\overline{K}$ is subnormal of depth at most $n-1$ in $\overline{G}$. Consequently, $N_{\overline{G},n-1}(\overline{K}) = \overline{G}$ and $N_{G,n-1}(K) = G$. On the other hand, $K \subseteq N_G(H) = HZ(G)$ ([3, Proposition 2.7]), and hence $N_G(H) = KZ(G)$ is normal in $N_G(K)$. We have proved that $N_G(H)$ is subnormal of depth at most $n-1$ in G. Using the induction hypothesis again (for $N_G(H)$), we get $N_{G,n}(H) = N_{G,n-1}(N_G(H)) = G$. □

3.4 Proposition. *Suppose that $G = \langle A, B\rangle$ and that $G/Z_{H,n}(G)$ is a two element group for some $n \geq 0$. Then $n = 0$ and H is a normal subgroup of index 2 in G.*

Proof: Assume on the contrary, $n \geq 1$. With respect to 2.15, we can in fact assume that $n = 1$ and $L_G(H) = 1$. Then $Z_{H,1}(G) = HZ(G)$ and $H \cap Z(G) = 1$. By [1, Lemma 1.4], $Z(G) \subseteq A \cap B$. Now, let $a \in A$ and $z \in Z(G)$. Then $az = bu$ for some $b \in A$ and $u \in H$. We have $u = b^{-1}az$ and $c^{-1}uc = c^{-1}b^{-1}cb \cdot b^{-1}c^{-1}ac \cdot z = c^{-1}b^{-1}cb \cdot b^{-1}az \cdot a^{-1}c^{-1}ac \in H$ for every $c \in B$. This shows that $u \in L_G(H) = 1$ and $az = b \in A$. Now, since $[G : HZ(G)] = 2$, it is clear that $A = Z(G) \cup aZ(G)$ for each $a \in A - Z(G)$. Quite similarly, $B = Z(G) \cup bZ(G)$ for each $b \in B - Z(G)$. In particular, both A and B are abelian subgroups of G (see 2.15 (iii)).

Finally, let $a \in A$. Then $a^{-1}b \in H$ for some $b \in B$ and, for every $c \in B, c^{-1}a^{-1}bc = c^{-1}a^{-1}ca \cdot a^{-1}b \in H$. Thus $a^{-1}b \in L_G(H) = 1$ and $a = b \in B$. We have proved that $A = B$ and consequently $G = \langle A, B\rangle = A$ is an abelian group, $H = 1$, $Z_{H,1}(G) = G$ and $G/Z_{H,1}(G)$ is trivial, a contradiction. □

3.5 Lemma. *Suppose that $L_G(H) = 1$, H is not abelian, every proper factor group of H is cyclic and that $G = \langle A, B\rangle$. Then $Z_{H,n}(G) \neq G$ for every $n \geq 0$, i.e., H is not subnormal in G (see 3.3).*

Proof: Put $N = Z_{H,1}(G)$ $(= L_G(HZ(G))$, $\overline{G} = G/N$ and $\overline{H} = HN/N \cong H/H_1$, $H_1 = H \cap N$. If $H_1 \neq 1$, then $\overline{H}$ is cyclic, and so $\overline{A} = \overline{B}$ is an abelian subgroup of $\overline{G}$ by [1, Corollary 2.3]. However, this implies that $\overline{G} = \overline{A}$ is an abelian group, $\overline{H} = 1$, $H \subseteq N = HZ(G)$ and $H = H_1$ is abelian by 2.10 (iii), which is a contradiction.

We have proved that $H_1 = 1$, so that $N = H_1Z(G) = Z(G)$ and $\overline{H} \cong H$. Proceeding by induction, we get $Z_{H,m}(G) = Z_m(G)$ for every $m \geq 0$. Now, if $Z_{H,n}(G) = G$ for some $n \geq 0$, then G (and hence H) is nilpotent. But in such a case, $Z(H) \neq 1$, $H/Z(H)$ is cyclic and this implies that H is abelian a contradiction. □

3.6 Proposition. *Suppose that every proper factorgroup of H is cyclic, that H is subnormal in G and that $G = \langle A, B\rangle$. Then $G' \subseteq N_G(H)$ and H is subnormal depth at most 2 in G. Moreover, if H is not abelian, then $G' \subseteq H$ and H is normal in G.*

Proof: First, assume that $L_G(H) \neq 1$. Then $\overline{H} = H/L_G(H)$ is a cyclic subgroup of $\overline{G} = G/L_G(H)$ and $G' \subseteq H$ by [1, Theorem 2.2].

Next, let $L_G(H) = 1$. Then H is abelian by 3.5 and if H is cyclic, then we can use [1, Theorem 2.2] again to show that $H = 1$ and G is abelian. Finally, if H is not cyclic, than $H \cong Z_p^{(2)}$ for a prime p and the result follows from [5, Lemma 4.2]. □

3.7 Remark. According to [2], G is soluble, provided that G is finite and $H \cong S_3$. On the other hand, by 3.5, if $L_G(H) = 1$ and $G = \langle A, B\rangle$, then H is not subnormal in G.

3.8 Proposition. *Suppose that $L_G(H) = 1$ and G is nilpotent of class at most 2. Then $[A, B] = 1$ and A, B are isomorphic subgroups of G.*

Proof: $[A, B] \subseteq H \cap G' \subseteq H \cap Z(G) \subseteq L_G(H) = 1$. The rest follows from [4, Lemma 2.3]. □

4. Examples

4.1. Let G be the subgroup of S_6 (the symmetric group on {1,2, . . . ,6}) generated by the following permutations: (1 2), (3 4), (5 6), (1 3)(2 4), (1 3 5)(2 4 6). Further, let $K = \langle(1\ 2), (3\ 4), (5\ 6)\rangle \subseteq G$ and $H = \langle(1\ 2)\rangle \subseteq K$. Then H is normal in K, K is normal in G, $\mathrm{card}(G) = 48$, $K \cong Z_2^{(3)}$, $H \cong Z_2$, $L_G(H) = 1$ and H is subnormal of depth 2 in G. On the other hand, $N_G(H) = \langle K,(3\ 5)(4\ 6)\rangle$, $\mathrm{card}(N_G(H)) = 16$, $K = L_G(N_G(H))$, $N_{G,2}(H) = N_G(N_G(H)) = N_G(H)$, $G/K \cong S_3$ and $Z(G) = 1$. Now, $Z_{H,n}(G) \neq G$ for every $n \geq 0$ and there exist no H-connected transversals to H in G (see 2.4 (v) and 3.3).

4.2. Let G be the subgroup of S_{18} generated by A = {id, (1 2)(3 10 15 4 9 16)(5 12 17 6 11 18)(7 8)(13 14), (1 3 11 7 9 17 13 15 5)(2 10 18)(4 12 14)(6 8 16), (1 4 11 14 3 12 7 10 17 2 9 18 13 16 5 8 15 6), (1 5 10 14 6 9 7 11 16 2 12 15 13 17 4 8 18 3), (1 6 10 7 12 16 13 18 4)(2 11 15)(3 8 17)(5 9 14), (1 7 13)(2 8 14)(3 9 15)(4 10 16)(5 11 17)(6 12 18), (1 8 13 2 7 14)(3 16 9 4 15 10)(5 18 11 6 17 12), (1 9 5 7 15 11 13 3 17)(2 16 12)(4 18 8)(6 14 10), (1 10 5 14 9 6 7 16 11 2 15 12 13 4 17 8 3 18), (1 11 4 14 12 3 7 17 10 2 18 9 13 5 16 8 6 15), (1 12 4 7 18 10 13 6 16)(2 17 9)(3 19 11)(5 15 8), (1 13 7)(2 14 8)(3 15 9)(4 16 10)(5 17 11)(6 18 12), (1 14 7 2 13 8)(3 4)(5 6)(9 10)(11 12)(15 16)(17 18), (1 15 17 7 3 5 13 9 11)(2 4 6)(8 10 12)(14 16 18), (1 16 17 14 15 18 7 4 5 2 3 6 13 10 11 8 9 12), (1 17 16 14 18 15 7 5 4 2 6 3 13 11 10 8 12 9), (1 18 16 7 6 4 13 12 10)(2 5 3)(8 11 9)(14 17 15)} and let H be the stabilizer of 1 in G. Then $L_G(H) = 1$, $\mathrm{card}(H) = 972 = 2^2 3^5$, H is not nilpotent, A is an H-selfconnected transversal to H in $G = \langle A\rangle$, $\mathrm{card}(G) = 17496 = 2^3 3^7$, and $Z_{H,3}(G) = G$ (cf. 2.13).

4.3. Let G be the subgroup of S_6 generated by A = {id, (1 2)(3 4)(5 6), (1 3 5)(2 4 6), (1 4 5 2 3 6), (1 5 4 2 6 3), (1 6 4)(2 5 3)} and let H be the stabilizer of 1 in G. Then $L_G(H) = 1$, $H \cong Z_2^{(2)}$, A is an H-selfconnected transversal to H in $G = \langle A\rangle$, $\mathrm{card}(G) = 24$, $Z_{H,2}(G) = G$, $\mathrm{card}(Z(G)) = 2$, G is not nilpotent, $\mathrm{card}(N_G(H)) = 8$, $N_G(H) = HZ(G) = Z_{H,1}(G) \cong Z_2^{(3)}$ and $G/Z_{H,1}(G) \cong Z_3$ (cf. 2.4 (iv) and 3.4).

4.4. Let G be the subgroup of S_6 generated by A $= \{$id, (1 2 3)(4 5 6), (1 3 2)(4 5 6), (1 4)(2 6 3 5), (1 5 3 6)(2 4), (1 6 2 5)(3 4)$\}$ and let H be the stabilizer of 1 in G. Then $L_G(H) = 1$, $H \cong S_3$ is soluble, A is an H-selfconnected transversal to H in $G = \langle A \rangle$, card$(G) = 36$, $G \neq Z_{H,n}(G)$ for every $n \leq 0$ and H is not subnormal in G (see 3.5).

4.5. Let G be the subgroup of S_4 generated by (1 2), (3 4), (1 3 2 4), (1 4 2 3), let H be the stabilizer of 1 in G and let A $= \{$id, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)$\}$. Then $L_G(H) = 1$, $H \cong Z_2$, A is an H-selfconnected transversal to H in G, $A \cong Z_2^{(2)}$ is a subgroup of G, G is a dihedral eight-element group, $Z_{H,1}(G) \cong Z_2^{(2)}$ and $G/Z_{H,1}(G) \cong Z_2$ (cf. 3.4).

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